

Addition Operation In Semigraphs*

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Abstract

The addition of graphs is a well-studied operation on graphs which results in a new graph with more number of vertices and edges. The operation of adding vertices to graphs is different from the addition of graphs but also defines a new graph with more number of vertices and edges. In this article, the above two operations are extended to a generalization of graphs called semigraphs. The article also deals with the rank of a special incidence matrix of semigraphs resulting by the application of the above two operations.

1 Introduction

It is convenient to be able to express ‘a given structure’ in terms of smaller and simpler structures, which is possible only when we know the building blocks of the given structure. In graph theory many operations are defined which give raise to new graphs from the given graphs and the properties of new graphs in terms of their generators are studied. In this article, we extend some operations existing in the theory of graphs to semigraphs, a generalization of graphs.

The incidence matrix of a semigraph, as defined by E. Sampathkumar [9], does not represent the semigraph uniquely. Deshpande et al. [9], came up with a new definition of the incidence matrix of a semigraph, which represents the semigraph uniquely, but does not reveal the fact that (v_1, v_2, \dots, v_n) and $(v_n, v_{n-1}, \dots, v_1)$ represent the same edge in a semigraph.

Authors in [10] make use of a property of binomial coefficient while defining the binomial incidence matrix, which not only represents a semigraph uniquely but also has the following property. The (i, j) entry of the matrix gives information about position of every vertex v_i incident on e_j from either end vertex on the edge e_j and also the size of the edge e_j .

2 Preliminaries

In this section, we give basics of semigraph [9] and readers are referred to [5] for all the elementary notations and definitions not described but used in this article.

Definition 1 *A semigraph G is a pair (V, E) where V is a nonempty set whose elements are called vertices of G , and E is a set of k -tuples of distinct vertices, called edges of G , for various $k \geq 2$, satisfying the following conditions.*

1. Any two edges of G can have at most one vertex in common.

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2. Two edges (a_1, a_2, \dots, a_p) and (b_1, b_2, \dots, b_q) are said to be equal if and only if

- $p = q$ and
- either $a_i = b_i$ for $1 \leq i \leq p$ or $a_i = b_{p-i+1}$ for $1 \leq i \leq p$.

Note 1 Let E be the set of vertices on the edge e . Then the size of E is called the size of the edge e and it is usually denoted by $|E|$.

Let $G = (V, E)$ be a semigraph and let $e = (u_1, u_2, \dots, u_k)$ be an edge of G . Then u_1 and u_k are called the end vertices and $u_i, 2 \leq i \leq k-1$, are called the mid vertices of e . Two vertices of G are adjacent if there is an edge containing both of them. An edge is said to be incident on every vertex on it. Two edges of G are adjacent if they have a vertex in common. Two vertices are consecutively adjacent if they are consecutive on the edge containing them. In a semigraph, an edge of size at least three is known as semiedge.

Like a graph, a semigraph G also has a geometric representation on plane. Vertices of G are represented either by dots or by small circles according to whether they are end vertices or mid vertices of the edge containing them and edges of G by curves passing through all the vertices on them. When a mid vertex v of an edge e_1 is an end vertex of another edge, say e_2 , then a small tangent is drawn to the circle representing vertex v where e_2 meets v . A semigraph G and its representation are given in Example 1.

Example 1 Let $G = (V, E)$ be a semigraph with $V(G) = \{u_1, u_2, \dots, u_8\}$ and

$$E(G) = \{e_1 = (u_1, u_2, u_3); e_2 = (u_3, u_4, u_5); e_3 = (u_2, u_8); e_4 = (u_5, u_6, u_7, u_8)\}.$$

It can be represented as shown in Figure 1, and the edges e_1, e_2 and e_4 are semiedges.

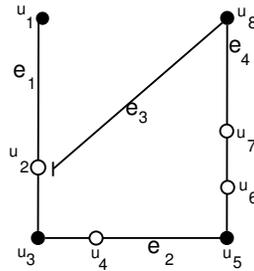


Figure 1: Semigraph G with 8 vertices and 4 edges

Definition 2 A subedge of an edge $e = (a_1, a_2, \dots, a_m)$ is a p -tuple $e' = (a_{i_1}, a_{i_2}, \dots, a_{i_p})$, induced by the vertices $a_{i_1}, a_{i_2}, \dots, a_{i_p}$, where $1 \leq i_1 < i_2 < \dots < i_p \leq m$ or $1 \leq i_p < i_{p-1} < \dots < i_1 \leq m$. A partial edge of e is a k -tuple $e'(a_i, a_k) = (a_i, a_{i+1}, \dots, a_{i+k-1})$, where $1 \leq i \leq m - k + 1$. Here two consecutive vertices in e' are also consecutive vertices in e . Note that an edge is a subedge (partial edge) of itself, but a subedge (partial edge) is not an edge of G .

Definition 3 A complete semigraph is a semigraph in which every two vertices are adjacent. In addition, if every vertex is an end vertex of some edge then it is called a strongly complete semigraph. For example, a semigraph which consists of a single edge of size $n \geq 3$ is complete but not strongly complete and is denoted by E_n^c . The strongly complete semigraph on n vertices with one edge of size $(n - 1)$ and all other edges of size two is denoted by T_{n-1}^1 .

Definition 4 A semigraph G is said to be a zig-zag semigraph if $V(G) = \{u_1, u_2, \dots, u_k, u_{k+2}, \dots, u_{2k-1}\}$ and

$$E(G) = \{(u_1, u_2, \dots, u_k), (u_1, u_{k+1}), (u_2, u_{k+1}), (u_2, u_{k+2}), \dots, (u_{k-1}, u_{2k-2}), (u_{k-1}, u_{2k-1}), (u_k, u_{2k-1})\},$$

denoted by Z_k^{k-1} .

Definition 5 A semigraph $G' = (V', E')$ is a subsemigraph of a semigraph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Definition 6 Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 , respectively. Their union denoted by $G_1 \cup G_2$ has vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. Their join (addition) denoted by $G_1 + G_2$ consists of $G = G_1 \cup G_2$ and all edges joining V_1 with V_2 .

Now we define the binomial incidence matrix of a semigraph.

Definition 7 ([10]) Let $G = (V, E)$ be a semigraph with $V(G) = \{u_1, u_2, \dots, u_n\}$ and $E(G) = \{e_1, e_2, \dots, e_m\}$. Let size of the edge e_j be $n_j + 1$, $1 \leq j \leq m$. The binomial incidence matrix of G , denoted by $\mathcal{B}(G)$, is a $n \times m$ matrix, whose rows are indexed by the vertex set and columns are indexed by the edge set of G . The column corresponding to e_j in the binomial incidence matrix consists of entries $0, {}^{n_j}C_0, \dots, {}^{n_j}C_{n_j}$, where nonzero entries correspond to the vertices on the edge. The entries ${}^{n_j}C_0$ and ${}^{n_j}C_{n_j}$ correspond to the end vertices of the edge e_j . The (i, j) entry of $\mathcal{B}(G)$ is given by

$$b_{ij} = \begin{cases} {}^{n_j}C_r & \text{if vertex } u_i \text{ and edge } e_j \text{ are incident and } u_i \text{ is the } r^{\text{th}} \text{ vertex} \\ & \text{from the end vertex of } e_j \text{ with entry } {}^{n_j}C_0, 0 \leq r \leq n_j \\ 0 & \text{if vertex } u_i \text{ and edge } e_j \text{ are not incident on each other.} \end{cases}$$

Example 2 The binomial incidence matrix $\mathcal{B}(G)$ of semigraph G as shown in Figure 1 is given by

$$\mathcal{B}(G) = \begin{matrix} & e_1 & e_2 & e_3 & e_4 \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{matrix} & \begin{pmatrix} {}^2C_0 & 0 & 0 & 0 \\ {}^2C_1 & 0 & {}^1C_0 & 0 \\ {}^2C_2 & {}^2C_0 & 0 & 0 \\ 0 & {}^2C_1 & 0 & 0 \\ 0 & {}^2C_2 & 0 & {}^3C_0 \\ 0 & 0 & 0 & {}^3C_1 \\ 0 & 0 & 0 & {}^3C_2 \\ 0 & 0 & {}^1C_1 & {}^3C_3 \end{pmatrix} \end{matrix}.$$

We refer the interested readers to the articles [2, 3, 4, 6, 7, 8] for the other important studies in semigraphs.

3 Number of Subedges and Number of Partial Edges in a Semigraph

The semigraph E_n^c , $n \geq 3$ seems to be the simplest semigraph which is not a graph.

Example 3 The semigraph E_3^c is as shown in Figure 2. Subedges of the semigraph G are (v_1, v_2, v_3) , (v_1, v_2) ,

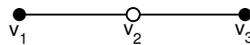


Figure 2: Semigraph $G = E_3^c$

(v_1, v_3) and (v_2, v_3) out of which (v_1, v_2, v_3) , (v_1, v_2) , and (v_2, v_3) , are partial edges.

It is quite interesting to count the number of subedges and the number of partial edges in a given semigraph and the task can be simplified by knowing the number of subedges and partial edges in an edge of size n .

Theorem 1 *The number of subedges in an edge of size n is given by $2^n - n - 1$ and that of partial edges is ${}^n C_2$.*

Proof. Let $e = (v_1, v_2, \dots, v_n)$ be an edge of size n . With v_1 as starting vertex, we have $(n - 1)$ partial edges namely $[(v_1, v_2), (v_1, v_2, v_3) \dots (v_1, v_2, \dots, v_n)]$. Similarly, with v_2 as starting vertex there are $(n - 2)$ partial edges.

Finally, with v_{n-1} as starting vertex, we have one partial edge. Hence, number of partial edges in e is,

$$1 + 2 + 3 + \dots + n - 1 = \frac{n(n-1)}{2}.$$

With v_1 as starting vertex, there are ${}^{n-1}C_1$ subedges of size 2 namely $[(v_1, v_2), (v_1, v_3) \dots (v_1, v_n)]$. From v_1 there are ${}^{n-1}C_2$ subedges of size 3 and so on. With v_1 as starting vertex there is one subedge of size n . Hence the total number of subedges in e with v_1 as starting vertex is,

$${}^{n-1}C_1 + {}^{n-1}C_2 + \dots + {}^{n-1}C_{n-1} = 2^{n-1} - 1.$$

Out of $2^{n-1} - 1$ subedges with v_1 as starting vertex, there are $(n - 1)$ partial edges.

Similarly, with v_2 as starting vertex, there are $2^{n-2} - 1$ subedges in e , out of which $(n - 2)$ are partial edges and so on. With v_{n-1} as starting vertex, there is only one subedge which is also a partial edge. Hence, total number of subedges in the edge e which is of size n is,

$$(2^{n-1} - 1) + (2^{n-2} - 1) + \dots + (2^1 - 1) = 2^n - n - 1.$$

■

Corollary 1 *The number of subedges of size 2, in an edge of size n is given by ${}^n C_2$ and that of partial edges of size 2, is $n - 1$.*

Proof. Let $e = (v_1, v_2, \dots, v_n)$ be an edge of size n . With v_1 as starting vertex, there are ${}^{n-1}C_1$ subedges of size 2, out of which (v_1, v_2) is a partial edge.

From v_2 , there are ${}^{n-2}C_1$ subedges of size 2, out of which (v_2, v_3) is a partial edge.

⋮

With v_{n-1} as starting vertex, there is only one subedge of size 2, which is also a partial edge. Hence, total subedges of size 2 in the edge e of size n is,

$$1 + 2 + \dots + {}^{n-2}C_1 + {}^{n-1}C_1 = {}^n C_2$$

Each vertex with its immediate next vertex forms a partial edge each of size 2. Those are $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)$. Hence total partial edges of size 2 are, $n - 1$. ■

4 Addition of a Complete Graph and Path Graph to Semigraph

In this section, we discuss about adding a complete graph and path graph to semigraphs. We have also explored the rank of the binomial incidence matrix of the resulting semigraphs.

Definition 8 $T_r^{K_{n-r}}$ is a complete semigraph on n vertices which contains a semiedge of size $r \geq 3$ and a complete graph K_{n-r} as subsemigraphs, all the vertices of K_{n-r} are made adjacent to all the vertices on the semiedge by edges of size two. Clearly, number of edges in $T_r^{K_{n-r}}$ is given by $1 + {}^{n-r}C_2 + (n - r)r$.

Example 4 The semigraph $T_3^{K_4}$ as shown in Figure 3 is a complete semigraph on 7 vertices which contains a semiedge of size 3 and a complete graph K_4 as subsemigraphs.

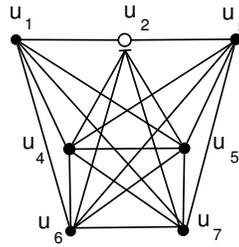


Figure 3: Semigraph $T_3^{K_4}$

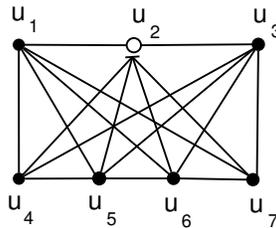


Figure 4: Semigraph $T_3^{P_4}$

Definition 9 $T_r^{P_{n-r}}$ is a semigraph on n vertices which contains a semiedge of size $r \geq 3$ and a path graph P_{n-r} as subsemigraphs, all the vertices of P_{n-r} are made adjacent to all the vertices on the semiedge by edges of size two. Clearly, number of edges in $T_r^{P_{n-r}}$ is given by $(n - r)r + 1$.

Example 5 The semigraph $T_3^{P_4}$ as shown in Figure 4 is a semigraph on 7 vertices which contains a semiedge of size 3 and a path graph P_4 as subsemigraphs.

Note 2 Clearly, $T_r^{K_{n-r}} = E_r^c + K_{n-r}$ and $T_r^{P_{n-r}} = E_r^c + P_{n-r}$, when '+' represents addition of graphs.

We have the following theorem.

Theorem 2 Let H be any connected graph on $(n - r)$ vertices and G be the semigraph given by $G = E_r^c + H$. Then, rank of $\mathcal{B}(G)$ is n .

Proof. Let v_1, v_2, \dots, v_r be vertices of E_r^c and $v_{r+1}, v_{r+2}, \dots, v_n$ be the vertices of H . The first column of $\mathcal{B}(G)$ has non zero entries ${}^{r-1}C_0, {}^{r-1}C_1, \dots, {}^{r-1}C_{r-1}$. Assume that they appear in that order in the first r rows.

Suppose, $X \in R^n$ be such that $X' \mathcal{B}(G) = 0$. Since, for every j with $1 \leq j \leq r$, the vertex v_j is adjacent to v_{r+i} , for all $i, 1 \leq i \leq n - r, x_j + x_{r+i} = 0$, i.e $x_j = -x_{r+i}, 1 \leq j \leq r$. We also have,

$$\begin{aligned} & {}^{r-1}C_0 x_1 + {}^{r-1}C_1 x_2 + \dots + {}^{r-1}C_{r-1} x_r = 0, \\ \text{i.e } & -x_{r+i} \{ {}^{r-1}C_0 + {}^{r-1}C_1 + \dots + {}^{r-1}C_{r-1} \} = 0 \\ \implies & x_{r+i} = 0, \quad \forall i, 1 \leq i \leq n - r. \\ \implies & x_j = 0, \quad \forall j, 1 \leq j \leq r. \end{aligned}$$

Thus X is the zero vector. The dimension of the left null space of $\mathcal{B}(G)$ is zero. Hence the result. ■

Corollary 2 Let $H = E_{n-r}^c$ and $G = E_r^c + H = E_r^c + E_{n-r}^c = T_r^H$ is a complete semigraph on n vertices with $r(n - r) + 2$ edges. Then rank $\mathcal{B}(T_r^H) = n$.

Proof. The proof follows from Theorem 2. ■

- Note 3**
1. If $H = K_{n-r}$ then $T_r^H = T_r^{K_{n-r}}$ which is a complete semigraph on n vertices having $1 + {}^{n-r}C_2 + (n-r)r$ edges.
 2. If H is the complete semigraph with k edges, then $T_r^H = E_r^c + H$ has n vertices and $1 + r(n-r) + k$ edges with the rank of $\mathcal{B}(T_r^H)$ equal to n .
 3. Both the semigraphs $T_r^{K_{n-r}}$ and $T_r^{P_{n-r}}$ have the same rank n .
 4. Generalizing Corollary 2, if r_1, r_2, \dots, r_k are positive integers greater than or equal to 2 with $r_1 + r_2 + \dots + r_k = n$, then $G = E_{r_1}^c + E_{r_2}^c + \dots + E_{r_k}^c$ is a semigraph obtained by joining every vertex of $E_{r_j}^c$ to every vertex which is not on $E_{r_j}^c$, $1 \leq j \leq k$. This results in a complete semigraph on n vertices and $(r_1 \times r_2 \times \dots \times r_k) + k$ edges with the rank of $\mathcal{B}(G)$ equal to n .

5 Rank of Binomial Incidence Matrix of a Semigraph after Addition of Vertices

In [1], Jean H. Bevis et al. have defined the addition of vertices to a given graph and discussed its effect on the rank of the resulting graph. They have defined the following operations on an undirected graph H .

1. Let H be an undirected graph, u be a vertex of H and v be a vertex not in $V(H)$. Then $H_1 = H \oplus_u v$ is the graph obtained by adding the vertex v and the edge between u and v , to H .
2. Let H be an undirected graph with $V(H) = \{u_1, u_2, \dots, u_n\}$, v be a vertex not in $V(H)$ and X be a 0-1 vector with n components. Then $H_2 = H \oplus_X v$ is the graph obtained by adding the vertex v and making it adjacent with u_i only if i^{th} component of X is 1, $1 \leq i \leq n$.

Motivated by the above, we have defined the following operations on semigraphs.
Let G be a semigraph with $V(G) = \{u_1, u_2, \dots, u_n\}$.

1. Let $v \notin V(G)$ and let X be a 0-1 vector with n components. Then $G \oplus_X v$ is the semigraph obtained by adding vertex v to G and making it adjacent to u_i by an edge of size 2, only if i^{th} component of X is 1, $1 \leq i \leq n$.

If $X = \mathbf{1}$, a vector in which all components are equal to 1, then $G \oplus_X v = G \oplus_{\mathbf{1}} v$ is the semigraph obtained by making v adjacent to all vertices of G by the edges (u_i, v) of size 2, $\forall i, 1 \leq i \leq n$.

Example 6 The complete semigraph $T_5^{K_1} = [E_5^c \oplus_{\mathbf{1}} u_6]$ is as shown in Figure 5.

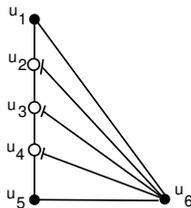


Figure 5: Complete semigraph $T_5^{K_1}$

Note 4 Let $\{u_1, u_2, \dots, u_n\}$ be the set of vertices of the complete semigraph E_n^c and $G = E_n^c \oplus_{\mathbf{1}} u_{n+1} = E_n^c + K_1 = T_n^{K_1}$. Then from Theorem 2, $\mathcal{B}(G)$ is a full rank matrix.

2. Let $u_{n+1}, u_{n+2}, \dots, u_{2n-1}$ be vertices such that $u_{n+i} \notin V(G), \forall i, 1 \leq i \leq n-1$. Then

$$\left[G \oplus'_{\{u_i, u_{i+1}\}} u_{n+i} \right]_{i=1}^{n-1}$$

is a semigraph obtained by adding the edges of size 2 joining u_{n+i} and $u_i, i, 1 \leq i \leq n-1$.

Example 7 The Zig-zag semigraph, $Z_6^5 = \left[E_6^c \oplus'_{\{u_i, u_{i+1}\}} u_{n+i} \right]_{i=1}^5$ is as shown in Figure 6.

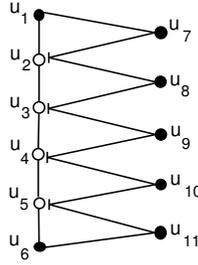


Figure 6: Zig-zag semigraph Z_6^5

Note 5 Let $G = E_n^c$ be a semigraph with vertex set $\{u_1, u_2, \dots, u_n\}$, and

$$G' = \left[G \oplus'_{\{u_i, u_{i+1}\}} u_{n+i} \right]_{i=1}^{n-1} = Z_n^{n-1}.$$

Then $\mathcal{B}(G')$ is a full rank matrix.

3. Let $u_{n+1}, u_{n+2}, \dots, u_{n+N}$ be vertices such that $u_{n+i} \notin V(G), \forall i, 1 \leq i \leq N$. Then $\left[G \oplus''_{\{u_1, u_n\}} u_{n+i} \right]_{i=1}^N$ is a semigraph obtained by making the vertices $u_{n+1}, u_{n+2}, \dots, u_{n+N}$ adjacent to both u_1 and u_n by edges of size 2.

Example 8 The semigraph, $G' = \left[E_5^c \oplus''_{\{u_1, u_5\}} u_{5+i} \right]_{i=1}^6$ is as shown in Figure 7.

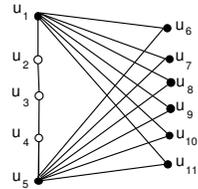


Figure 7: Semigraph $G' = \left[E_5^c \oplus''_{\{u_1, u_5\}} u_{5+i} \right]_{i=1}^6$

4. Let $u_{n+1}, u_{n+2}, \dots, u_{n+n}$ be vertices of $H = E_n^c$ such that $u_{n+i} \notin V(G), \forall i, 1 \leq i \leq n$. Then $\left[G \oplus'''_{u_i} u_{n+i} \right]_{i=1}^n$ is a semigraph obtained by making the vertices $u_{n+1}, u_{n+2}, \dots, u_{n+n}$ adjacent to the corresponding vertex of H by edges of size 2.

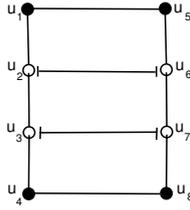


Figure 8: Semigraph $G' = \left[E_4^c \oplus_{u_i}''' u_{n+i} \right]_{i=1}^4$

Example 9 The semigraph, $G' = \left[E_4^c \oplus_{u_i}''' u_{n+i} \right]_{i=1}^4$ is as shown in Figure 8.

Theorem 3 Let $G = E_n^c$ with $V(G) = \{u_1, u_2, \dots, u_n\}$ and let $G' = \left[G \oplus_{\{u_1, u_n\}}'' u_{n+i} \right]_{i=1}^N$ where N is any positive integer. Then rank of $\mathcal{B}(G')$ is $N + 2$.

Proof. G' has $n + N$ vertices $u_1, u_2, \dots, u_n, u_{n+1}, u_{n+2}, \dots, u_{n+N}$ and $2N + 1$ edges given by, $e_1 = (u_1, u_2, \dots, u_n)$, $e_{i+1} = (u_1, u_{n+i})$ and $e'_{i+1} = (u_n, u_{n+i})$, $1 \leq i \leq N$.

Then, $\mathcal{B}(G')$ is of the form

$$\begin{array}{l}
 u_1 \\
 u_2 \\
 u_3 \\
 \vdots \\
 \vdots \\
 u_n \\
 u_{n+1} \\
 u_{n+2} \\
 \vdots \\
 \vdots \\
 u_{n+N}
 \end{array}
 \left(
 \begin{array}{cccc|cccc}
 & e_1 & e_2 & e_3 & \dots & e_{N+1} & e'_2 & e'_3 & \dots & e'_{N+1} \\
 & {}^{n-1}C_0 & 1 & 1 & \dots & 1 & & & & \\
 & {}^{n-1}C_1 & & O_{n-1 \times n-1} & & & & & O_{n-1 \times n-1} & \\
 & {}^{n-1}C_2 & & O_{n-1 \times n-1} & & & & & & \\
 & \vdots & & & & & & & & \\
 & \vdots & & & & & & & & \\
 & {}^{n-1}C_{n-1} & & & & & 1 & 1 & \dots & 1 \\
 & 0 & & & & & & & & \\
 & 0 & & & & & & & & \\
 & \vdots & & I_{N \times N} & & & & I_{N \times N} & & \\
 & \vdots & & & & & & & & \\
 & 0 & & & & & & & &
 \end{array}
 \right)$$

where $I_{N \times N}$ is the identity matrix of order N and $O_{n-1 \times n-1}$ is the zero matrices of order $n - 1$.

One can easily observe that rank of $\mathcal{B}(G') = N + 2$. ■

Corollary 3 Let $G = E_n^c$ be a semigraph with vertex set $\{u_1, u_2, \dots, u_n\}$, and let $G' = \left[G \oplus_{\{u_1, u_n\}}'' u_{n+i} \right]_{i=1}^N$, where $N = n - 1$ and $\det \mathcal{B}(G')$ is equal to zero.

Proof. When $N = n - 1$, the matrix $\mathcal{B}(G')$ becomes a square matrix of order $(2n - 1)$. From Theorem 3, rank of $\mathcal{B}(G') = (n - 1) + 2 = n + 1$. Hence G' is not a full rank matrix and $\det \mathcal{B}(G') = 0$. ■

Theorem 4 Let $G = E_n^c$ be a semigraph with vertex set $\{u_1, u_2, \dots, u_n\}$ and let $u_{n+1}, u_{n+2}, \dots, u_{n+n}$ be vertices of $H = E_n^c$. Let $G' = \left[G \oplus_{u_i}''' u_{n+i} \right]_{i=1}^n$. Then the rank of $\mathcal{B}(G')$ is $m - 1$ where m is the number of edges in G' .

Proof. The number of vertices and edges in G' is $2n$ and $n + 2$, respectively. Clearly, $2n > n + 2 \forall, n \geq 3$. The binomial incidence matrix of G' is of the form

$$\mathcal{B}(G') = \begin{matrix} & e_1 & e_2 & e'_1 & e'_2 & \dots & e'_{n-1} & e'_n \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ \cdot \\ u_{n-1} \\ u_n \\ u_{n+1} \\ u_{n+2} \\ u_{n+3} \\ \cdot \\ \cdot \\ u_{2n-1} \\ u_{2n} \end{matrix} & \begin{pmatrix} {}^{n-1}C_0 & 0 & 1 & 0 & \dots & 0 & 0 \\ {}^{n-1}C_1 & 0 & 0 & 1 & \dots & 0 & 0 \\ {}^{n-1}C_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ {}^{n-1}C_{n-1} & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & {}^{n-1}C_0 & 1 & 0 & \dots & 0 & 0 \\ 0 & {}^{n-1}C_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & {}^{n-1}C_2 & 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & {}^{n-1}C_{n-1} & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \end{matrix}$$

If $X = (x_1, x_2, \dots, x_{n+2})^T$ is a vector in the right null space $\mathcal{B}(G')$ then $\mathcal{B}X = 0$. This implies that,

$$\begin{aligned} {}^{n-1}C_s x_1 + x_{s+3} &= 0, & \forall s, 0 \leq s \leq n-1, \\ {}^{n-1}C_s x_2 + x_{s+3} &= 0, & \forall s, 0 \leq s \leq n-1, \\ \implies x_1 = x_2 &= -\frac{x_{s+3}}{{}^{n-1}C_s}, & \forall s, 0 \leq s \leq n-1. \end{aligned}$$

Hence, there exists only one vector in the right null space of $\mathcal{B}X = 0$ i.e $(1, 1, -{}^{n-1}C_0, \dots, -{}^{n-1}C_{n-1})$. Therefore, rank of $\mathcal{B}(G') = (n + 2) - 1 = m - 1$. ■

Theorem 5 Let $G = (V, E)$ a connected semigraph with exactly one semiedge. If G contains T_{n-1}^1 or Z_n^{n-1} as its subsemigraph then rank of $\mathcal{B}(G)$ is equal to $|V(G)|$.

Proof. Let G be a semigraph with p vertices and m edges among which only one edge is a semiedge and let T_{n-1}^1 be a subsemigraph of G . We note that, $n \leq p \leq m$.

Let $\mathcal{B}(G)$ be the binomial incidence matrix of G Then it is of the form,

$$\mathcal{B}(G) = \left[\begin{array}{c|c} \mathcal{B}(T_{n-1}^1)_{n \times n} & \\ \hline O_{p-n \times n} & F \end{array} \right],$$

where $\mathcal{B}(T_{n-1}^1)$ is a square matrix of order n with rows as parts of first n rows of $\mathcal{B}(G)$ and column corresponding to the edges e_1, e_2, \dots, e_n which are first n edges of G . And, $O_{p-n \times n}$ is a zero matrix of order $(p - n) \times n$ and F is $p \times (m - n)$ matrix with columns corresponding to edges $e_{n+1}, e_{n+2}, \dots, e_m$ which are of size two.

Suppose, $X \in R^p$ be such that $X' \mathcal{B}(G) = 0$. Then, $x_i + x_j = 0$ whenever the vertices u_i and u_j are adjacent by an edge of size two. Therefore, $x_i + x_n = 0$ that is $x_i = -x_n, \forall i, 1 \leq i \leq n-1$. We have from first column of $\mathcal{B}(G)$,

$$\begin{aligned} x_1 {}^{n-2}C_0 + x_2 {}^{n-2}C_1 + \dots + x_{n-1} {}^{n-2}C_{n-2} &= 0, \\ \text{i.e } -x_n \{ {}^{n-2}C_0 + {}^{n-2}C_1 + \dots + {}^{n-2}C_{n-2} \} &= 0 \\ \implies x_n = 0 \implies x_i = 0, & \forall i, 1 \leq i \leq n. \end{aligned}$$

Now, consider the partition $\{V_1, V_2\}$ of $V(G)$ where V_1 is the vertex set of T'_{n-1} and $V_2 = V(G) \setminus V_1$. For an edge e_i , $i > n$ with end vertices u_s and u_t , since e_i is of size two, we have $x_s + x_t = 0$. If $u_s \in V_1$ and $u_t \in V_2$, then $s \leq n$ and $t > n$ and $x_s = 0$, which implies $x_t = 0$. Suppose that u_s and u_t both are in V_2 . Then, since G is connected, there exists a path between u_s and u_t which consists only of edges of size two and which passes through at least one vertex in V_1 . This implies that $x_s = 0$ and $x_t = 0$. Hence, $x_i = 0 \quad \forall i, 1 \leq i \leq p$. Thus, null space of $\mathcal{B}(G)$ is of dimension zero and rank of $\mathcal{B}(G)$ is equal to $|V(G)|$.

Continuing in similar lines, one can prove that rank of $\mathcal{B}(G) = |V(G)|$ when G is a connected semigraph which has only one semiedge and has Z_n^{n-1} as a sub semigraph. ■

Applications In communication networks, not all nodes are equally important. Some of them may just receive and transmit the data whereas, some others may be used to process the data. So, in the graph representing the communication networks, all the vertices need not have the same importance, such situations could be better represented by a graph model called semigraph. In particular, E_n^c is an edge with n vertices on the same edge with exactly two end vertices and the remaining are mid vertices, where any two vertices are mutually adjacent to each other. Hence, in communication networks, E_n^c represents the situation where the communication takes place between every pair of vertices and only two of them which are placed at the end are of more importance.

In theoretical computer science, communication complexity studies the amount of communication required to solve a problem when the input to the problem is distributed among two or more parties. In the field of communication complexity, the rank of the communication matrix of a function gives bounds on the amount of communication needed for two or more parties to compute the function.

In this article, we have studied the rank of the binomial incidence matrix of the semigraph which represents the semigraph uniquely, which in turn gives a better picture of communication complexity when E_n^c is part of the communication network.

Conclusions We have studied the structure of a semigraph in terms of its substructures and also change in the structure of semigraph by adding graphs and vertices to a specific semigraph. Also the effect on the rank of the binomial incidence matrix of a semigraph have been discussed. Majority of the semigraphs resulted from addition operation in this article are of full rank.

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