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A Differential Inequality And The Blow-Up Of Its Solutions*

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Abstract

We analyze the blow-up of the solutions of an ordinary differential inequality, useful to prove non-existence of global solutions of evolution equations. We extend and improve previous results.

1 Introduction

The purpose of this note is to study the blow-up in finite time of any solution $\psi(t)$, of the following second order differential inequality

$$\psi(t)\frac{d^2}{dt^2}\psi(t) + \delta\psi(t)\frac{d}{dt}\psi(t) - (1+\alpha)\left(\frac{d}{dt}\psi(t)\right)^2 - \beta\psi^2(t) + \gamma\psi(t) \geq 0, \qquad t \geq 0, \tag{1}$$

where α, β, γ are strictly positive constants, and $\delta \geq 0$ is a damping coefficient.

An important topic in the analysis of evolution equations is to find sufficient conditions for the existence of global and non-global solutions due to blow-up in finite time. See the book [1] for a description of the main methods for studying blow-up in nonlinear equations of the mathematical physics. The concavity method, introduced by Levine [2, 3], has been widely used to investigate this type of behavior through differential inequalities of the type (1). See the papers [4, 5, 6] and references therein, where extensions of such method have been proposed. The blow-up theorems in these articles are based on the analysis of particular cases of (1) and are shown under sufficient conditions more restrictive than ours. The objective of this work is to expand and improve these results. See also [7, 8], for other problems.

2 The Differential Inequality

In the applications of inequalities of the type (1), $\psi(t)$ is usually a positive function of the norm in some space $\mathcal{H}$ of the solution $u(x,t)$ of a partial differential equation of the evolution type. That is, $\psi(t) = \mathsf{f}(\|u(\cdot,t)\|_{\mathcal{H}}) \geq 0$, see for instance [1] and references therein. In order to study the blow-up of $\psi(t)$ in finite time we define the following functions, for $\psi(t) > 0$.

$$\phi(t) \equiv \left(\frac{\frac{d}{dt}\psi(t)}{\psi^{\frac{1}{2}}(t)} - \frac{\delta}{\alpha}\psi^{\frac{1}{2}}(t)\right)^2 + \frac{\beta}{\alpha}\psi(t),\ \sigma_\nu(t) \equiv \frac{1+2\alpha}{2}\left(\phi(t) - \frac{\beta\nu}{\alpha}\psi(t)\right),$$

$$\mu_\lambda(t) \equiv \frac{1+2\alpha}{2}\left(\phi(t) - \frac{\beta}{\alpha(1+2\alpha)}\psi(t)\left(\lambda\frac{\beta\psi(t)}{\alpha\phi(t)}\right)^{2\alpha}\right),$$

for $t \geq 0,\ \nu > 0$, and $\lambda \in (0,1)$, and

$$\psi_0 \equiv \psi(0),\ \ \phi_0 \equiv \phi(0) = \left(\frac{\psi_0^{'}}{\psi_0^{\frac{1}{2}}} - \frac{\delta}{\alpha}\psi_0^{\frac{1}{2}}\right)^2 + \frac{\beta}{\alpha}\psi_0,\ \ \psi_0^{'} \equiv \frac{d}{dt}\psi(0).$$

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Theorem 1 *Consider any solution $\psi(t)$ of the differential inequality (1), such that*

$$\psi_0^{'} > \frac{\delta}{\alpha}\psi_0 > 0, \tag{2}$$

consequently $\phi_0 > \frac{\beta}{\alpha}\psi_0 > 0$. Then, there exists a nonempty interval

$$\mathcal{I} \equiv (\mathbf{a}, \mathbf{b}) \subset \left(0, \frac{1+2\alpha}{2}\phi_0\right),$$

with the following consequences:

(i) *If $\gamma \in \mathcal{I}$, then $\psi(t)$ blows-up at the finite time $t^* \geq \left(\alpha\frac{\psi_0'}{\psi_0} - \delta\right)^{-1}$.*

(ii) *$\mathbf{a} = \sigma_\nu(0)$ and $\mathbf{b} = \mu_\lambda(0)$, moreover*

$$\mathbf{a} = \frac{\beta\psi_0}{((1+2\alpha)\nu^*)^{\frac{1}{2\alpha}}} < \frac{\beta\psi_0}{(1+2\alpha)^{\frac{1}{\alpha}}},$$

$$\mathbf{b} = \frac{\alpha\phi_0}{\lambda^*} > \frac{1+2\alpha}{2}\phi_0 - \left(\frac{1+2\alpha}{2\alpha} - \zeta(\lambda^*)\right)\beta\psi_0 > \frac{1+2\alpha}{2}\phi_0 - \frac{\beta}{2\alpha}\psi_0,$$

for some $\frac{2\alpha}{1+2\alpha} < \lambda^* < 1$ and $\nu^* > 1+2\alpha$, where $1 < \zeta(\lambda^*) < \frac{1+2\alpha}{2\alpha}$ is a function of λ^*, defined in the proof.

(iii) *For fixed ψ_0*

$$\psi'_0 \mapsto |\mathcal{I}| = \mathbf{b} - \mathbf{a},$$

is strictly increasing and such that

$$0 < \frac{1+2\alpha}{2}\phi_0 - |\mathcal{I}| < \left(\frac{1+2\alpha}{2\alpha} - \zeta(\lambda^*) + \frac{1}{((1+2\alpha)\nu^*)^{\frac{1}{2\alpha}}}\right)\beta\psi_0.$$

Furthermore, we have the limit values as $\psi'_0 \to \infty$

$$\mathbf{a} \to 0,\ \left|\mathbf{b} - \frac{1+2\alpha}{2}\phi_0\right| \to 0,\ \nu^* \to \infty,\ \lambda^* \to \frac{2\alpha}{1+2\alpha},\ \zeta(\lambda^*) \to \frac{1+2\alpha}{2\alpha},\ \ t^* \to 0.$$

Corollary 1 *Assume that (2) is met. For every number $\gamma > 0$, we can choose $\psi_0^{'}$ large enough, so that $\gamma \in \mathcal{I}$, and then the corresponding solution $\psi(t)$ of (1) blows-up in finite time.*

Remark 1 *The analysis of blow up for some quasilinear equations of parabolic and hyperbolic type is done in [4], applying inequality (1) with $\gamma = 0$, if a condition similar to (2) is fulfilled. The blow up property for nonlinear Kirchooff type and wave equations is proved in [5], by means of the inequality (1) with $\beta = 0$, if (2) and an additional condition on ϕ_0 are satisfied. Finally, the blow up behavior in nonlinear Klein-Gordon and the double dispersive equations is showed in [6], using (1) with $\delta = 0$, if (2) and an implicit condition on γ similar to (9) hold. We point out that if (9) is true, we will show the existence of the blow up interval $\mathcal{I}$ whose measure is estimated in terms of $\beta > 0$. We also observe that $\delta > 0$ allows us to apply our result to evolution equations with linear dissipation, see [1, 4, 5]. Furthermore, $\gamma > 0$ is proportional to the size of the initial energy in applications to hyperbolic equations, see [1, 5, 6]. Since the blow-up problem for any positive value of the initial energy is a current research topic, it is important to consider any $\gamma > 0$, see [5, 6] and references therein. Hence, our result improves and extends the previous ones in the literature.*

Proof of Theorem 1. We assume that the solution is global, then $\psi(t)$ is well defined for any $t \geq 0$. From (2), the following holds for $t \in [0, t_1)$ and some $t_1 > 0$,

$$\frac{d}{dt}\psi(t) > \frac{\delta}{\alpha}\psi(t) > 0. \tag{3}$$

Hence, the following differential inequality for $\psi(t)$ holds

$$\frac{d^2}{dt^2}\psi(t) + \delta\frac{d}{dt}\psi(t) - (1+\alpha)\frac{\left(\frac{d}{dt}\psi(t)\right)^2}{\psi(t)} - \beta\psi(t) + \gamma \geq 0 \tag{4}$$

If we define $\mathcal{F}(t) \equiv \psi^{-\alpha}(t)$, the differential inequality (4) becomes

$$\frac{d^2}{dt^2}\mathcal{F}(t) + \delta\frac{d}{dt}\mathcal{F}(t) + \alpha\beta\mathcal{F}(t) - \alpha\gamma\mathcal{F}^{\frac{1+\alpha}{\alpha}}(t) \leq 0. \tag{5}$$

Next, if we define $\mathcal{G}(t) \equiv e^{\delta t}\mathcal{F}(t)$, (5) implies that

$$\frac{d^2}{dt^2}\mathcal{G}(t) - \delta\frac{d}{dt}\mathcal{G}(t) + \alpha\beta\mathcal{G}(t) - \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t)e^{-\frac{\delta t}{\alpha}} \leq 0, \tag{6}$$

and (3) is characterized by $\frac{d}{dt}\mathcal{F}(t) < -\delta\mathcal{F}(t) < 0$, $-\delta\frac{d}{dt}\mathcal{G}(t) > 0$. Hence, (6) becomes

$$\frac{d^2}{dt^2}\mathcal{G}(t) + \alpha\beta\mathcal{G}(t) \leq \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t)e^{-\frac{\delta t}{\alpha}} \leq \alpha\gamma\mathcal{G}^{\frac{1+\alpha}{\alpha}}(t),$$

and the following integral of this differential inequality is obtained

$$\left(\frac{d}{dt}\mathcal{G}(t)\right)^2 \geq \mathcal{J}\left(\mathcal{G}(t)\right), \tag{7}$$

where, for $s > 0$,

$$\mathcal{J}(s) \equiv \frac{2\alpha^2}{1+2\alpha}\gamma s^{\frac{1+2\alpha}{\alpha}} - \alpha\beta s^2 + C_0, \quad C_0 \equiv \alpha^2\psi_0^{-(1+2\alpha)}\left(\phi_0 - \frac{2\gamma}{1+2\alpha}\right).$$

From (7), if there exists a constant $\kappa_0 > 0$ such that

$$\left(\frac{d}{dt}\mathcal{G}(t)\right)^2 \geq \mathcal{J}\left(\mathcal{G}(t)\right) \geq \kappa_0^2, \tag{8}$$

then $\frac{d}{dt}\mathcal{G}(t) \leq -\kappa_0 < 0$, and t_1 is never reached. Hence, $0 < \mathcal{G}(t) \leq \psi_0^{-\alpha} - t\kappa_0$, which is impossible for $t \geq t^* \equiv (\kappa_0\psi_0^{\alpha})^{-1}$. Then, the solution blows-up at the finite time t^*. We notice that $\mathcal{J}(s)$ attains an absolute minimum at $s_0 \equiv \left(\frac{\beta}{\gamma}\right)^{\alpha}$, that is

$$\mathcal{J}\left(\mathcal{G}(t)\right) \geq \mathcal{J}(s_0) = -\frac{\alpha\beta}{1+2\alpha}\left(\frac{\beta}{\gamma}\right)^{2\alpha} + \mathcal{C}_0 = \alpha^2\psi_0^{-(1+2\alpha)}\left(\phi_0 - \mathcal{K}(\gamma)\right),$$

where

$$\mathcal{K}(\gamma) \equiv \frac{2\gamma}{1+2\alpha} + \frac{\beta}{\alpha(1+2\alpha)}\left(\frac{\beta}{\gamma}\right)^{2\alpha}\psi_0^{1+2\alpha}.$$

In order to satisfy (8), we set $\kappa_0^2 \equiv \mathcal{J}(s_0)$, then, $\kappa_0^2 > 0$ if and only if

$$\mathcal{K}(\gamma) < \phi_0. \tag{9}$$

We notice that $\mathcal{K}(\gamma) \to \infty$ as, either $\gamma \to 0$ or $\gamma \to \infty$. Consequently $\mathcal{K}$ attains an absolute minimum at $\gamma_0 \equiv \beta\psi_0$, that is $\mathcal{K}(\gamma) \geq \mathcal{K}(\gamma_0) = \frac{\beta}{\alpha}\psi_0$. Hence, (9) holds due to (2), and there exist two different roots of $\mathcal{K}(\gamma) = \phi_0$, denoted by $\mathbf{a}$ and $\mathbf{b}$, such that $0 < \mathbf{a} < \gamma_0 < \mathbf{b} < \frac{1+2\alpha}{2}\phi_0$, and $\frac{\beta}{\alpha}\psi_0 < \mathcal{K}(\gamma) < \phi_0$, $\gamma \in \mathcal{I} \equiv (\mathbf{a}, \mathbf{b}), \gamma \neq \gamma_0$. And since $\mathcal{K}$ is strictly monotone for $\gamma < \gamma_0$ and $\gamma > \gamma_0$, it follows that, for fixed ψ_0, the interval $\mathcal{I}$ grows as ψ_0' grows. That is,

$$\lim_{\psi_0' \to \infty} \left| \frac{1+2\alpha}{2}\phi_0 - \mathbf{b} \right| = 0 = \lim_{\psi_0' \to \infty} \mathbf{a}.$$

Then, (9) holds if and only if $\gamma \in \mathcal{I}$. From the definition of κ_0^2, we obtain $t^* \geq \left(\alpha\frac{\psi_0'}{\psi_0} - \delta\right)^{-1}$, and $\lim_{\psi_0' \to \infty} t^* = 0$. We shall use σ_ν and μ_λ to find the values of $\mathbf{a}$ and $\mathbf{b}$, respectively. First, we consider the equation

$$\mathcal{K}(\sigma_\nu) = \phi_0, \quad \sigma_\nu \equiv \sigma_\nu(0). \tag{10}$$

This holds if and only if,

$$p(\nu) \equiv \frac{\nu}{\alpha} + \left(\frac{2}{(1+2\alpha)^{\frac{1+2\alpha}{2\alpha}}}\right)\frac{1}{\nu^{\frac{1}{2\alpha}}} = \frac{\phi_0}{\beta\psi_0}. \tag{11}$$

We notice that $p(\nu) \to \infty$, as $\nu \to 0$ and $\nu \to \infty$. Furthermore, p has an absolute minimum, that is

$$p(\nu) \geq p(1+2\alpha) = \frac{1}{\alpha}, \quad \nu > 0.$$

Moreover, from the definition of ϕ_0, $\frac{\phi_0}{\beta\psi_0} > \frac{1}{\alpha}$. Then, equation (10) equivalently (11), has two roots and only one such that $\nu^* > 1+2\alpha$. Furthermore,

$$\mathbf{a} = \sigma_{\nu^*} = \frac{\beta\psi_0}{((1+2\alpha)\nu^*)^{\frac{1}{2\alpha}}} < \frac{\beta\psi_0}{(1+2\alpha)^{\frac{1}{\alpha}}}.$$

Also, $\lim_{\psi_0' \to \infty} \nu^* = \infty$. Next, we consider the equation

$$\mathcal{K}(\mu_\lambda) = \phi_0, \quad \mu_\lambda \equiv \mu_\lambda(0). \tag{12}$$

We observe that, for $\lambda \in [\lambda_0, 1]$, $\lambda_0 = 2\alpha(1+2\alpha)^{-1}$, this holds if and only if

$$q(\lambda) \equiv \frac{1}{1+2\alpha}\left(\lambda\frac{\beta\psi_0}{\alpha\phi_0}\right)^{1+2\alpha} = \lambda - \frac{2\alpha}{1+2\alpha} \equiv r(\lambda). \tag{13}$$

Both functions are strictly monotone increasing and, due to the definition of ϕ_0,

$$q(\lambda_0) > r(\lambda_0) = 0, \quad q(1) < r(1) = \frac{1}{1+2\alpha}.$$

Then, there exists one and only one $\lambda^* \in (2\alpha(1+2\alpha)^{-1}, 1)$ where $q(\lambda^*) = r(\lambda^*)$. That is, only one root λ^* of equation (13), equivalently (12). Moreover,

$$\mathbf{b} = \mu_{\lambda^*} = \frac{\alpha\phi_0}{\lambda^*},$$

and $\lim_{\psi_0' \to \infty} \lambda^* = \frac{2\alpha}{1+2\alpha}$. Next, we show the following lower bound for $\mathbf{b}$

$$\mathbf{b} > \frac{1+2\alpha}{2}\phi_0 - \frac{1}{2\alpha}\beta\psi_0 = \frac{1+2\alpha}{2\psi_0}\left(\psi_0' - \frac{\delta}{\alpha}\psi_0\right)^2 + \beta\psi_0,$$

To this end, consider first the following inequality

$$\frac{\alpha}{\psi_0\lambda^*}\left(\left(\psi_0' - \frac{\delta}{\alpha}\psi_0\right)^2 + \frac{\beta}{\alpha}\psi_0^2\right) = \frac{\alpha\phi_0}{\lambda^*} = \mathbf{b} > \frac{1+2\alpha}{2\psi_0}\left(\psi_0' - \frac{\delta}{\alpha}\psi_0\right)^2,$$

which, for $s \equiv \left(\psi_0' - \frac{\delta}{\alpha}\psi_0\right)^2 > 0$, is equivalent to

$$l(s) \equiv \left(\lambda^*\left(\frac{1+2\alpha}{2\alpha}\right) - 1\right)\left(\psi_0' - \frac{\delta}{\alpha}\psi_0\right)^2 < \frac{\beta}{\alpha}\psi_0^2, \quad \frac{2\alpha}{1+2\alpha} < \lambda^* < 1.$$

We calculate the following limits taking in account that λ^* is a function of s,

$$\lim_{s\to\infty}\lambda^* = \frac{2\alpha}{1+2\alpha}, \quad \lim_{s\to 0}\lambda^* = 1,$$

$$\lim_{s\to\infty} l(s) = \frac{1}{2\alpha}\lim_{s\to\infty} s\left(\lambda^*\frac{\beta\psi_0}{\alpha\phi_0}\right)^{1+2\alpha} = \frac{1}{2\alpha}\lim_{s\to\infty} s\left(\lambda^*\ \frac{\frac{\beta}{\alpha}\psi_0^2}{\frac{\beta}{\alpha}\psi_0^2 + s}\right)^{1+2\alpha} = 0.$$

Also, $\lim_{s\to 0} l(s) = 0$. Consequently, there is some $s^* \in (0,\infty)$, such that $l(s^*) = \max_{s\in(0,\infty)} l(s)$. After some calculations, we find that

$$s^* = \psi_0^2\left(\frac{2\beta(1-\lambda^*)}{(1+2\lambda)\lambda^* - 2\alpha(1-\lambda^*)}\right), \; l(s^*) = \frac{\beta}{\alpha}\psi_0^2\left(\frac{((1+2\alpha)\lambda^* - 2\alpha)(1-\lambda^*)}{(1+2\alpha)\lambda^* - 2\alpha(1-\lambda^*)}\right),$$

and then $l(s) \le l(s^*) = \frac{\beta}{\alpha}\psi_0^2\eta(\lambda^*)$, $\eta(\lambda^*) \equiv \frac{((1+2\alpha)\lambda^*-2\alpha)(1-\lambda^*)}{(1+2\alpha)\lambda^*-2\alpha(1-\lambda^*)} < 1$. Notice that this is equivalent to

$$\mathbf{b} > \left(\frac{1+2\alpha}{2\psi_0}\right)\left(\psi_0' - \frac{\delta}{\alpha}\psi_0\right)^2 + \zeta(\lambda^*)\beta\psi_0 = \frac{1+2\alpha}{2}\phi_0 - \left(\frac{1+2\alpha}{2\alpha} - \zeta(\lambda^*)\right)\beta\psi_0,$$

where

$$\frac{1+2\alpha}{2\alpha} > \zeta(\lambda^*) \equiv \frac{\lambda^*(1+2\alpha)}{\lambda^*(1+2\alpha) - 2\alpha(1-\lambda^*)} > 1, \quad \text{since } \frac{2\alpha}{1+2\alpha} < \lambda^* < 1.$$

Also, $\lim_{\psi_0'\to\infty}\zeta(\lambda^*) = \frac{1+2\alpha}{2\alpha}$. Finally, from the bounds for $\mathbf{a}, \mathbf{b}$, we have that

$$0 < \frac{1+2\alpha}{2}\phi_0 - |\mathcal{I}| < \left(\frac{1+2\alpha}{2\alpha} - \zeta(\lambda^*) + \frac{1}{((1+2\alpha)\nu^*)^{\frac{1}{2\alpha}}}\right)\beta\psi_0.$$

■

Proof of Corollary 1. Since $\psi_0' \to \infty \Rightarrow \mathbf{a} \to 0$ and $\mathbf{b} \to \infty$, we see that, for every $\xi > 0$ there exists $\eta > 0$, such that $\psi_0' > \eta \Rightarrow \xi \in \mathcal{I} = (\mathbf{a}, \mathbf{b})$. Hence, any solution of (1) with $\gamma = \xi$ blows-up in finite time. ■

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