Applied Mathematics E-Notes, 22(2022), 169-177 © ISSN 1607-2510
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A Note On The Influence Of Different Additional Regularity On The Critical Exponent*

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Received 17 December 2020

Abstract

In this paper, we consider the Cauchy problem for the semi-linear damped σ-evolution equations, where the initial data are supposed to belong to the energy space with different additional regularity, which means that,

$$(u_0,u_1)\in (H^\sigma(\mathbb{R}^n)\cap L^{m_1}(\mathbb{R}^n))\times \left(L^2(\mathbb{R}^n)\cap L^{m_2}(\mathbb{R}^n)\right),\quad m_1,m_2\in[1,2),\ \sigma\geq 1.$$

Our main goal is to study the influence of m_1 and m_2 on the critical exponent by proving the global (in time) existence of small data energy solutions where their decay estimates coincide with those to the corresponding linear equation.

1 Introduction

The semi-linear damped σ-evolution equations we want to study in this paper are:

$$\partial_t^2 u+(-\Delta)^\sigma u+\partial_t u+(-\Delta)^\sigma\partial_t u=|u|^p,\quad u(0,x)=u_0(x),\ \partial_t u(0,x)=u_1(x), \tag{1}$$

with $\sigma\in[1,\infty)$, $p\in(1,\infty)$, $t\in[0,\infty)$ and $x\in\mathbb{R}^n$, $n\geq 1$. Here, the notation $(-\Delta)^\sigma$ denotes the fractional Laplacian operator with symbol $|\xi|^{2\sigma}$, i.e.,

$$\mathcal{F}\left((-\Delta)^\sigma f\right)=|\xi|^{2\sigma}\mathcal{F}\left(f\right)(\xi),\quad \xi\in\mathbb{R}^n,\quad |\xi|=(\xi_1^2+\cdots+\xi_n^2)^{1/2},$$

where $\mathcal{F}$ is the Fourier transform. The terms $\partial_t(\cdot)$ and $(-\Delta)^\sigma\partial_t(\cdot)$ respectively denote frictional and visco-elastic damping mechanism. In this paper we will choose the initial data (u_0,u_1) that belong to the energy space $H^\sigma(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ with different additional regularity, that is,

$$u_0\in(H^\sigma(\mathbb{R}^n)\cap L^{m_1}(\mathbb{R}^n)),\quad u_1\in\left(L^2(\mathbb{R}^n)\cap L^{m_2}(\mathbb{R}^n)\right),\quad m_1,m_2\in[1,2). \tag{2}$$

Our main goal is to study the influence of m_1 and m_2 not only on the critical exponent but on the decay estimates of solutions u as well.

Indeed, critical exponent p_{crit} means global (in time) existence of small data Sobolev solutions for $p>p_{crit}$, and blow-up in finite time for $1<p\leq p_{crit}$. Additionally, when $p>p_{crit}$ the decay estimates for solution of the semi-linear Cauchy problem are the same for those of the linear problem.

The pioneering paper [1] is the first to deal with the problem of finding the critical exponent p_{crit} where the initial data (v_0,v_1) are small in $\left(H^1(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\right)\times\left(L^2(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\right)$ and $m\in[1,2]$, the authors in [1] studied the existence property of solutions to the Cauchy problem for the semi-linear wave equation with frictional damping

$$\partial_t^2 v-\Delta v+\partial_t v=|u|^p,\quad v(0,x)=v_0(x),\ \partial_t v(0,x)=v_1(x),$$

*Mathematics Subject Classifications: 35A01, 35B33, 35B45, 35G25.

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and they found the number $p_{crit}(n,m)$ which divides the range of p into $p \in (1, p_{crit}(n,m))$ (the small data global nonexistence) and $p \in (p_{crit}(n,m), \infty)$ (the small data global existence), where

$$p_{crit}(n,m) = 1 + \frac{2m}{n}, \; m \in [1,2]. \tag{3}$$

It is clear that $p_{crit}(n,m)$ interpolates the critical exponents $p_{crit}(n,1)$ and $p_{crit}(n,2)$. Here, we note that the number $p_{crit}(n,1)$ is well-known as Fujita exponent which was first found by Hiroshi Fujita for the semilinear heat equation $h_t - \Delta h = h^p$, $h(0,x) = h_0(x)$, $p > 1$.

Now, concerning the linear wave equations with frictional and visco-elastic damping terms, the authors in [2] studied the asymptotic profiles of solutions and showed that the effect of the frictional damping u_t is more dominant than that of the visco-elastic one $(-\Delta u_t)$ as $t \to \infty$, this interesting result tell us that the critical value for the corresponding semi-linear Cauchy problem is exactly $p_{crit}(n,m)$ as defined above in (3). For more results, one can see the works [3, 4] or [5] for initial data in energy space with additional L^1 or L^m regularity.

More recently, the authors in [6] used unified $(L^m \cap L^2) - L^2$ linear estimates to prove the global (in time) existence of small data solutions for the problem (1) and they found the following critical exponent:

$$p_{crit}(n,m,\sigma) = 1 + \frac{2m\sigma}{n}, \quad m \in [1,2), \tag{4}$$

where they choose the initial data (u_0, u_1) as in (2) with $m_1 = m_2 = m$.
Since $p_{crit}(n,m,\sigma)$ always depends on the parameter m, this fact leads us to ask the following question:

If we choose the initial data as in (2), what happens to the critical exponent (4)?

To answer this question, we will prove in this paper the global (in time) existence of small data solutions to (1). Our method is standard and is based on Banach fixed point theorem, Gagliardo-Nirenberg inequality as well as the application of mixed $L^m - L^2$ linear estimates.

For the best reading of this paper, we use the following notation:

- We write $f \lesssim g$ when there exists a constant $c > 0$ such that $f \leq cg$.
- $H^\sigma(\mathbb{R}^n)$ stands for Sobolev space as defined below (see [7, p 445])
$$H^\sigma(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{H^\sigma(\mathbb{R}^n)} = \|(1 + |\cdot|^2)^{\frac{\sigma}{2}} \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$
- $L^m(\mathbb{R}^n)$ is the usual Lebesgue space with $m \in [1,2)$.

The paper is structured as follows: In Section 2 we recall some estimates for solutions to the corresponding linear equation (5) and two inequalities which play an essential role to prove our results in Section 3.

2 Main Tools

The so-called $L^m - L^2$ linear estimates, Gagliardo-Nirenberg inequality and integral inequality are very important tools to demonstrate Theorem 1. We introduce them in the following section.

Lemma 1 (Proposition 2.1 [6]) *Let $m \in [1,2)$. Then, the Sobolev solutions u^{lin} to the following linear equation:*

$$\partial_t^2 u^{lin} + (-\Delta)^\sigma u^{lin} + \partial_t u^{lin} + (-\Delta)^\sigma \partial_t u^{lin} = 0, \quad u^{lin}(0,x) = u_0(x), \quad \partial_t u^{lin}(0,x) = u_1(x), \tag{5}$$

satisfy the $(L^m \cap L^2) - L^2$ estimates:

$$\begin{aligned} \|\partial_t^j (-\Delta)^{a/2} u^{lin}(t,\cdot)\|_{L^2} \lesssim {}& (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|u_0\|_{(L^m \cap H^a)} \\ &+ (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|u_1\|_{(L^m \cap H^{[a+2(j-1)\sigma]^+})}, \end{aligned} \tag{6}$$

and the $L^2 - L^2$ estimates:

$$\|\partial_t^j (-\Delta)^{a/2} u^{lin}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{a}{2\sigma}-j} \|(u_0,u_1)\|_{H^a \times H^{[a+2(j-1)\sigma]^+}}, \tag{7}$$

for any $a \geq 0$, $j = 0,1$ and for all space dimensions $n \geq 1$, where $[\cdot]^+ = \max\{0,\cdot\}$.

The above linear estimates are proved in [6].
Recalling the data spaces (2), we may obtain the following corollary.

Corollary 1 *Let $m_1, m_2 \in [1,2)$. Then, the Sobolev solutions u^{lin} to the linear equation (5) satisfy the $(L^m \cap L^2) - L^2$ estimates:*

$$\begin{aligned} & \|\partial_t^j (-\Delta)^{a/2} u^{lin}(t,\cdot)\|_{L^2} \\ \lesssim\ & (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|u_0\|_{(L^{m_1}\cap H^a)} \\ & +(1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|u_1\|_{(L^{m_2}\cap H^{[a+2(j-1)\sigma]^+})} \\ \lesssim\ & \begin{cases} (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|(u_0,u_1)\|_{(L^{m_1}\cap H^a)\times(L^{m_2}\cap H^{[a+2(j-1)\sigma]^+})} & \text{if } m_2 \leq m_1, \\ (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|(u_0,u_1)\|_{(L^{m_1}\cap H^a)\times(L^{m_2}\cap H^{[a+2(j-1)\sigma]^+})} & \text{if } m_1 \leq m_2, \end{cases} \end{aligned} \tag{8}$$

as well as (7).

Proof. The general solutions u^{lin} to (5) can be written in the following form:

$$\begin{aligned} \mathcal{F}(u^{lin})(t,\xi) &= \frac{e^{-|\xi|^{2\sigma}t} - |\xi|^{2\sigma}e^{-t}}{1-|\xi|^{2\sigma}}\mathcal{F}(u_0)(\xi) + \frac{e^{-|\xi|^{2\sigma}t} - e^{-t}}{1-|\xi|^{2\sigma}}\mathcal{F}(u_1)(\xi) \\ &= \mathcal{F}(K_\sigma(t,x))(t,\xi)\mathcal{F}(u_0)(\xi) + \mathcal{F}(G_\sigma(t,x))(t,\xi)\mathcal{F}(u_1)(\xi). \end{aligned} \tag{9}$$

Using now $L^{m_1} - L^2$ estimates for the first kernel and $L^{m_2} - L^2$ estimates for the second kernel as well as the Hausdorff-Young inequality or Young convolution inequality and

$$\|fg\|_{L^2} \leq \|f\|_{\frac{2m}{2-m}} \|g\|_{L^{m'}}, \quad \frac{1}{m} + \frac{1}{m'} = 1, \ \forall m \in [1,2],$$

this immediately leads to the required estimates. One can directly verify these estimates from Lemma (1). ■

We recall the fractional Gagliardo-Nirenberg and integral inequalities in the following lemmas.

Lemma 2 ([8, 7]) *Let $1 < q < \infty$, $\sigma > 0$ and $s \in [0,\sigma)$. Then, the following fractional Gagliardo-Nirenberg inequality holds for all $y \in H^\sigma(\mathbb{R}^n)$*

$$\|(-\Delta)^{s/2} y\|_{L^q(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\sigma/2} y\|_{L^2(\mathbb{R}^n)}^{\theta_q} \|y\|_{L^2(\mathbb{R}^n)}^{1-\theta_q},$$

where

$$\theta_q = \frac{n}{\sigma}\left(\frac{1}{2} - \frac{1}{q} + \frac{s}{n}\right) \in \left[\frac{s}{\sigma}, 1\right].$$

Lemma 3 ([7]) *Let $a, b \in \mathbb{R}$ such that $\max\{a,b\} > 1$. Then, it holds*

$$\int_0^t (1+t-s)^{-a}(1+s)^{-b} ds \lesssim (1+t)^{-\min\{a,b\}}.$$

3 Main Results

Our main results are divided into two cases,

$$m_1 \le m_2 \quad \text{and} \quad m_2 \le m_1.$$

In the following theorem we will see the nice influence of m_1 and m_2 on the critical exponent (4) when $m_2 \le m_1$.

Theorem 1 *Let us consider the Cauchy problem (1) with $\sigma \ge 1$ and $p > 1$. Let $m_1, m_2 \in [1, 2)$ such that*

$$m_2 \le m_1.$$

We assume the following conditions for p and the dimension n:

$$\begin{cases} \frac{2}{m_2} \le p \le \frac{n}{n-2\sigma} & \text{if } \ 2\sigma < n \le \frac{4\sigma}{2-m_2}, \\ \frac{2}{m_2} \le p & \text{if } \ 1 \le n \le 2\sigma. \end{cases} \tag{10}$$

Moreover, we suppose

$$p > \frac{m_1}{m_2} + \frac{2m_1\sigma}{n}. \tag{11}$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any data

$$(u_0, u_1) \in \mathcal{R}^{m_1,m_2,\sigma}(\mathbb{R}^n) := (H^\sigma(\mathbb{R}^n) \cap L^{m_1}(\mathbb{R}^n)) \times \left(L^2(\mathbb{R}^n) \cap L^{m_2}(\mathbb{R}^n)\right),$$

with $\|(u_0, u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}} < \varepsilon_0$, we have a uniquely determined globally (in time) solution

$$u \in \mathcal{C}\left([0,\infty), H^\sigma(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,\infty), L^2(\mathbb{R}^n)\right)$$

to (1). Furthermore, the solution satisfies the estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \|(u_0,u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)},$$

$$\|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1} \|(u_0,u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)},$$

$$\left\|(-\Delta)^{\sigma/2} u(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}} \|(u_0,u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)}.$$

In the following theorem, when $m_1 \le m_2$, then the critical exponent (4) can only be influenced by the parameter m_2 of the additional regularity of the second initial data.

Theorem 2 *Let us consider the Cauchy problem (1) with $\sigma \ge 1$ and $p > 1$. Let $m_1, m_2 \in [1, 2)$ such that*

$$m_1 \le m_2.$$

We assume the same conditions for p and the dimension n as in (10). Moreover, we suppose

$$p > 1 + \frac{2m_2\sigma}{n}. \tag{12}$$

Then we have the same conclusion as in Theorem (1). Furthermore, the solution u satisfies the estimates:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \|(u_0,u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)},$$

$$\|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-1} \|(u_0,u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)},$$

$$\left\|(-\Delta)^{\sigma/2} u(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}} \|(u_0,u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}(\mathbb{R}^n)}.$$

Remark 1 *The conditions (11), (12) are assumed to get the same decay estimates of the semi-linear model with those of the corresponding linear model (5). The bounds (10) on p and n appear due to the application of Gagliardo-Nirenberg inequality from Lemma (2).*

Remark 2 *It is clear that when $m_1 = m_2$ our results in Theorems 1, 2 coincide with those in the cited papers [5], [6]. Theorem (1) showed the influence of the additional L^{m_1} regularity of u_0 not only on the critical exponent but also on the decay estimates. While, the decay estimates in the second theorem are related to u_1 (see again Corollary 1).*

Remark 3 *On one hand, Theorem (1) says it is better to choose a uniform additional regularity of the initial data, since the following exponent:*

$$p_{glob}(n, \sigma, m_1, m_2) = \frac{m_1}{m_2} + \frac{2m_1\sigma}{n}$$

is not more sharper than that in (4) for any $m_1, m_2 \in [1, 2)$, but if the initial data (u_0, u_1) are chosen only from the energy space $H^\sigma(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then we can prove p_{glob} is better than the critical exponent $p_{crit}(n, 2, \sigma)$ if and only if

$$\frac{m_1}{m_2} + \frac{2m_1\sigma}{n} < 1 + \frac{4\sigma}{n}, \ i, e. \ n < \left(\frac{(4 - 2m_1)\sigma}{m_1 - m_2}\right) m_2.$$

On the other hand, Theorem (2) says it is not necessary to choose the same additional regularity for the initial data (u_0, u_1) because the lower bound in (12) does not depend on m_1.

Example 1 *Let us consider the following wave equation with frictional and visco-elastic damping:*

$$\partial_t^2 u - \Delta u + \partial_t u - \Delta \partial_t u = |u|^p, \quad (u, \partial_t u)(0, x) = (u_0, u_1)(x), \quad (\sigma = 1).$$

- *From Theorem 1, if we fix $n = 3$, $u_1 \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and $u_0 \in H^1(\mathbb{R}^3) \cap L^{m_1}(\mathbb{R}^3)$ then we have the global (in time) existence of small data solutions to the above equation for any p and m_1 satisfy:*

$$p \in [2, 3], \quad m_1 \in \left[1, \frac{6}{5}\right].$$

- *From Theorem 1, if we fix $n = 2$, $u_1 \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $u_0 \in H^1(\mathbb{R}^2) \cap L^{m_1}(\mathbb{R}^2)$ then we have the global (in time) existence of small data solutions to the above equation for any p and m_1 satisfy:*

$$p \in [2m_1, \infty), \qquad m_1 \in [1, 2).$$

Example 2 *Let us consider now the following plate equation with frictional and visco-elastic damping:*

$$\partial_t^2 u + \Delta^2 u + \partial_t u + \Delta^2 \partial_t u = |u|^p, \quad (u, \partial_t u)(0, x) = (u_0, u_1)(x), \quad (\sigma = 2).$$

- *From Theorem 1, if we fix $n = 5$, $u_1 \in L^2(\mathbb{R}^5) \cap L^1(\mathbb{R}^5)$ and $u_0 \in H^2(\mathbb{R}^5) \cap L^{m_1}(\mathbb{R}^5)$ then we have the global (in time) existence of small data solutions to the above equation for any p and m_1 satisfy:*

$$p \in [2, 5], \quad m_1 \in \left[1, \frac{10}{9}\right].$$

- *From Theorem 1, if we fix $n = 3$, $u_1 \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and $u_0 \in H^2(\mathbb{R}^3) \cap L^{m_1}(\mathbb{R}^3)$ then we have the global (in time) existence of small data solutions to the above equation for any p and m_1 satisfy:*

$$p \in \left[\frac{7m_1}{3}, \infty\right), \qquad m_1 \in [1, 2).$$

Now, we prove Theorem 1 using the Banach's fixed point theorem.

Proof of Theorem 1. Since we are dealing with semi-linear Cauchy problems, we use the Banach's fixed point theorem inspired from the book [7, Page 303] or the paper [9]. This powerful method needs to define a family of evolution spaces $B(T)$ for any $T>0$ with *suitable norm* $\|\cdot\|_{B(T)}$ as well as an operator

$$\mathcal{S}: u \in B(T) \longmapsto \mathcal{S}u \in B(T).$$

If this operator satisfies the contraction property:

$$\|\mathcal{S}u\|_{B(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}} + \|u\|^p_{B(T)}, \quad \forall u \in B(T), \tag{13}$$

$$\|\mathcal{S}u-\mathcal{S}\bar{u}\|_{B(T)} \lesssim \|u-\bar{u}\|_{B(T)}\Big(\|u\|^{p-1}_{B(T)}+\|\bar{u}\|^{p-1}_{B(T)}\Big), \quad \forall u,\bar{u} \in B(T), \tag{14}$$

then, one can deduce the existence and uniqueness of a global (in time) solution of (1) for small norm of initial data. Here, it is clear that the smallness of the initial data together with the condition (13) imply that $\mathcal{S}$ maps balls of $B(T)$ into balls of $B(T)$. Moreover, the existence of a unique solution is guaranteed by proving that the recurrence sequence

$$u_{-1}=0,\ u_k=\mathcal{S}u_{k-1}, \quad k=0,1,\cdots$$

is a Cauchy sequence in the Banach space $B(T)$ converging to the unique solution u of the equation $u=\mathcal{S}u$. Let us now go back and try to define the operator $\mathcal{S}$. To do this, applying Duhamel's principle to the Cauchy problem (1) we find the solutions u written as follows:

$$\begin{aligned} u(t,x) &= K_\sigma(t,x)*u_0(x)+G_\sigma(t,x)*u_1(x)+\int_0^t G_\sigma(t-s,x)*|u(s,x)|^p ds \\ &= u^{lin}(t,x)+u^{nol}(t,x), \end{aligned} \tag{15}$$

where $*$ denotes the convolution product with respect to x, and the kernels K_σ, G_σ are given by the representation (9). We can define the operator $\mathcal{S}$ by the same formula above:

$$\mathcal{S}: B(T) \longrightarrow B(T): u \longmapsto \mathcal{S}u = u^{lin}+u^{nol},$$

where the Banach space $B(T)$ is defined for all $T>0$ as follows:

$$B(T) := \mathcal{C}\left([0,T],H^\sigma\right) \cap \mathcal{C}^1\left([0,T],L^2\right).$$

The main step now is to choose a suitable norm for the above space. Fortunately, this choice is based on the linear estimates. From Corollary 1 we choose

$$\begin{aligned} \|u\|_{B(T)} = \sup_{0\le t\le T}\Big(&(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac12\right)}\|u(t,\cdot)\|_{L^2}+(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac12\right)+\frac12}\|(-\Delta)^{\sigma/2}u(t,\cdot)\|_{L^2} \\ &+(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac12\right)+1}\|\partial_t u(t,\cdot)\|_{L^2}\Big). \end{aligned} \tag{16}$$

The proof is divided into three steps.

Step 1*:* Using linear estimates when $m_2 \le m_1$ from (8) we have

$$\begin{aligned} \|u^{lin}\|_{B(T)} = \sup_{0\le t\le T}\Big(&(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac12\right)}\|u^{lin}(t,\cdot)\|_{L^2}+(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac12\right)+\frac12}\|(-\Delta)^{\sigma/2}u^{lin}(t,\cdot)\|_{L^2} \\ &+(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac12\right)+1}\|\partial_t u^{lin}(t,\cdot)\|_{L^2}\Big) \lesssim \|(u_0,u_1)\|_{\mathcal{R}^{m_1,m_2,\sigma}}. \end{aligned} \tag{17}$$

Step 2: To conclude (13), we come to prove

$$\|u^{nol}\|_{B(T)} \lesssim \|u\|^p_{B(T)}. \tag{18}$$

As usual, we divide the interval $[0,t]$ into two sub-intervals $[0,t/2]$ and $[t/2,t]$. We use the $L^{m_2}-L^2$ linear estimates if $\tau\in[0,t/2]$ and L^2-L^2 estimates if $\tau\in[t/2,t]$. From Corollary 1 we may estimate:

$$\|u^{nol}(t,\cdot)\|_{L^2} \lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)}\left(\|u(\tau,\cdot)\|^p_{L^{2p}}+\|u(\tau,\cdot)\|^p_{L^{m_2p}}\right)d\tau+\int_{t/2}^t\|u(\tau,\cdot)\|^p_{L^{2p}}\,d\tau, \tag{19}$$

$$\begin{aligned}\|\partial_t u^{nol}(t,\cdot)\|_{L^2} \lesssim &\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-1}\left(\|u(\tau,\cdot)\|^p_{L^{2p}}+\|u(\tau,\cdot)\|^p_{L^{m_2p}}\right)d\tau\\ &+\int_{t/2}^t(1+t-\tau)^{-1}\|u(\tau,\cdot)\|^p_{L^{2p}}\,d\tau,\end{aligned} \tag{20}$$

$$\begin{aligned}\|(-\Delta)^{\sigma/2}u^{nol}(t,\cdot)\|_{L^2} \lesssim &\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}}\left(\|u(\tau,\cdot)\|^p_{L^{2p}}+\|u(\tau,\cdot)\|^p_{L^{m_2p}}\right)d\tau\\ &+\int_{t/2}^t(1+t-\tau)^{-\frac{1}{2}}\|u(\tau,\cdot)\|^p_{L^{2p}}\,d\tau.\end{aligned} \tag{21}$$

Now, we are in a position to use the fractional Gagliardo-Nirenberg inequality from Lemma 2 to estimate these norms:

$$\|u(\tau,\cdot)\|^p_{L^{2p}},\quad \|u(\tau,\cdot)\|^p_{L^{m_2p}}.$$

Here, we have from (16):

$$(1+\tau)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\frac{1}{2}}\|(-\Delta)^{\sigma/2}u(\tau,\cdot)\|_{L^2}\lesssim\|u\|_{B(T)},$$

$$(1+\tau)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)}\|u(\tau,\cdot)\|_{L^2}\lesssim\|u\|_{B(T)}.$$

So, we can estimate the above norms as follows:

$$\|u(\tau,\cdot)\|^p_{L^{\nu p}}\lesssim(1+\tau)^{-\frac{np}{2m_1\sigma}+\frac{n}{2\nu\sigma}}\|u\|^p_{B(T)},\ \nu=m_2,2, \tag{22}$$

provided that the conditions (10) are satisfied for p and n. Hence, we conclude

$$\|u(\tau,\cdot)\|^p_{L^{m_2p}}+\|u(\tau,\cdot)\|^p_{L^{2p}}\lesssim(1+\tau)^{-\frac{np}{2m_1\sigma}+\frac{n}{2m_2\sigma}}\|u\|^p_{B(T)}, \tag{23}$$

The first integral of u^{nol} over $[0,t/2]$ can be estimated using the following equivalences:

$$(1+t-\tau)\approx(1+t)\text{ if }\tau\in[0,t/2],\quad(1+\tau)\approx(1+t)\text{ if }\tau\in[t/2,t]$$

and Lemma 3 as follows:

$$\begin{aligned}&\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)}(1+\tau)^{-\frac{np}{2m_1\sigma}+\frac{n}{2m_2\sigma}}\|u\|^p_{B(T)}d\tau\\ \lesssim\ &(1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)}\|u\|^p_{B(T)}\int_0^{t/2}(1+\tau)^{-\frac{np}{2m_1\sigma}+\frac{n}{2m_2\sigma}}d\tau\\ \lesssim\ &(1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)}\|u\|^p_{B(T)}\\ \lesssim\ &(1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)}\|u\|^p_{B(T)},\end{aligned}$$

provided that $p > \frac{m_1}{m_2} + \frac{2m_1\sigma_2}{n}$ and $m_2 \leq m_1$. For the second integral over $[t/2, t]$ we also derive:

$$\int_{t/2}^{t} (1+\tau)^{-\frac{np}{2m_1\sigma}+\frac{n}{4\sigma}} \|u\|^p_{B(T)} d\tau \lesssim (1+t)^{1-\frac{np}{2m_1\sigma}+\frac{n}{4\sigma}} \|u\|^p_{B(T)}.$$

Thanks to $m_2 \leq m_1$, we reach the following desired estimate for u^{nol}

$$(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \|u^{nol}(t,\cdot)\|_{L^2} \lesssim \|u\|^p_{B(T)}.$$

We can proceed as above to prove again:

$$(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1} \|\partial_t u^{nol}(t,\cdot)\|_{L^2} \lesssim \|u\|^p_{B(T)},$$

$$(1+t)^{\frac{n}{2\sigma}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}} \|(-\Delta)^{\sigma/2} u^{nol}(t,\cdot)\|_{L^2} \lesssim \|u\|^p_{B(T)}.$$

In this way we proved inequality (18).
Step 3: To prove (14) we choose two elements u, $\bar{u}$ belong to $B(T)$, and we write

$$\mathcal{S}u - \mathcal{S}\bar{u} = \int_0^t G_\sigma(t-\tau,x) * (|u(\tau,x)|^p - |\bar{u}(\tau,x)|^p) d\tau.$$

So, we divide $[0,t]$ as above, we have again:

$$\begin{aligned} \|(u^{nol}-\bar{u}^{nol})(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \||u(\tau,\cdot)|^p - |\bar{u}(\tau,\cdot)|^p\|_{L^{m_2}\cap L^2}\, d\tau \\ &\quad + \int_{t/2}^{t} \||u(\tau,\cdot)|^p - |\bar{u}(\tau,\cdot)|^p\|_{L^2}\, d\tau, \end{aligned}$$

$$\begin{aligned} \|\partial_t(u^{nol}-\bar{u}^{nol})(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-1} \||u(\tau,\cdot)|^p - |\bar{u}(\tau,\cdot)|^p\|_{L^{m_2}\cap L^2}\, d\tau \\ &\quad + \int_{t/2}^{t} (1+t-\tau)^{-1} \||u(s,\cdot)|^p - |\bar{u}(s,\cdot)|^p\|_{L^2}\, d\tau, \end{aligned}$$

$$\begin{aligned} \|(-\Delta)^{\sigma/2}(u^{nol}-\bar{u}^{nol})(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{2\sigma}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}} \||u(\tau,\cdot)|^p - |\bar{u}(\tau,\cdot)|^p\|_{L^{m_2}\cap L^2}\, d\tau \\ &\quad + \int_{t/2}^{t} (1+t-\tau)^{-\frac{1}{2}} \||u(\tau,\cdot)|^p - |\bar{u}(\tau,\cdot)|^p\|_{L^2}\, d\tau. \end{aligned}$$

By employing the Hölder's inequality, we derive for $\nu = m_2, 2$, the following

$$\||u(\tau,\cdot)|^p - |\bar{u}(\tau,\cdot)|^p\|_{L^\nu} \leq \|u(\tau,\cdot)-\bar{u}(\tau,\cdot)\|_{L^{\nu p}} \left(\|u(\tau,\cdot)\|^{p-1}_{L^{\nu p}} + \|\bar{u}(\tau,\cdot)\|^{p-1}_{L^{\nu p}}\right). \tag{24}$$

Using again the norm of the solution space $B(T)$ and fractional Gagliardo-Nirenberg inequality we can prove the estimates for $u^{nol} - \bar{u}^{nol}$. Hence, the proof of Theorem 1 is completed. ■

Conclusion 1 *We have proven in this paper how the different additional regularity of the initial data could possibly affect the critical exponent and also the decay estimates of the solutions to the semi-linear Cauchy problem (1). In our forthcoming paper, we would like to generalize this idea and study another Cauchy problem of the form:*

$$\partial_t^2 w + (-\Delta)^\sigma w + (-\Delta)^\delta \partial_t w = |w|^p, \quad w(0,x) = w_0(x), \quad \partial_t w(0,x) = w_1(x),$$

where $\sigma \geq 1$, $\delta \in (0,\sigma)$ and $p > 1$, see [10].

Acknowledgment. The authors would like to thank the referees for their constructive comments and suggestions that helped to improve the original manuscript. We thank Directorate-General for Scientific Research and Technological Development in Algeria (DGRSDT) for providing research conditions under code: C00L03UN220120220008.

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