Applied Mathematics E-Notes, 22(2022), 160-168 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Multiple Solutions For A Class Of $p(x)$-Kirchhoff-Type Equations*

Shapour Heidarkhani[†], Ahmad Ghobadi[‡], Mustafa Avci[§]

Received 21 December 2020

Abstract

In this paper, using variational methods and critical point theory, we establish the existence of multiple solutions for a class of $p(x)$-Kirchhoff-type problem with Dirichlet boundary data. Some recent results are extended and improved. To illustrate the application of the main results, three-dimensional equation models are presented.

1 Introduction

In the present paper, we study the following problem

$$\begin{cases} -M(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p(x) \in C(\overline{\Omega})$ with $1 < p^- := \min_{\overline{\Omega}} p(x) \le p^+ := \max_{\overline{\Omega}} p(x) < N$, $M(t) : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function and $f(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheèodory condition.

The operator $-\Delta_{p(x)}u = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian which becomes p-Laplacian when $p(x) \equiv p$ (a constant).

Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

where ρ, ρ_0, h, E, L are all constants which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the string during the vibrations. Eq. (2) received a lot of attention only after Lions [20] proposed an abstract framework for this problem. Some important and interesting results can be found in, for example, [1, 14]. The equation

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\nabla u = f(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{3}$$

is related to the stationary analogue of Eq. (2). Nonlocal problem (3) can be used for modeling, for example, physical and biological systems. Problems of Kirchhoff-type have been widely investigated. We refer the reader to [5, 10, 17, 18, 21, 22] and the references therein.

The differential equations and variational problems with variable exponent has attracted increasing attention for the last few decades. These type of differential equations are governed by the $p(x)$-Laplacian operator in general. The $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous.

*Mathematics Subject Classifications: 35J55, 35B65.

[†]Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
[‡]Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
[§]Department of Mathematics, Trent University, Peterborough, ON, Canada

The differential equations with variable exponent have been widely used for modeling many phenomena especially arising from the nonlinear elasticity theory, and the theory of electrorheological fluids, see [26]. Some other applications are image processing [7], magnetostatics [6], and capillarity phenomena [3]. We refer to [2, 4, 8, 9, 12, 19, 27] for the study of the Kirchhoff equations $p(x)$-Laplacian operators, and the corresponding variational problems.

In the present paper, we use the variational methods to obtain existence results for the problem (1) under suitable conditions imposed on f and M (see, the conditions (f_0), (f_1), (f_2), (m_0) and (m_1) of Theorem 5). In Theorem 5 we establish the existence of at least two weak solutions for the problem (1), while in Theorem 6 we discuss the existence of infinitely many solutions for the problem (1). The present paper is organized as follows. In Section 2, we recall some basic definitions and our main tools. In Section 3, we state and prove the main results of the paper. Then, we give two examples to illustrate our results.

2 Preliminaries and Basic Notation

First, we introduce some fundamental properties of the variable exponent Lebesgue $L^{p(x)}(\Omega)$ and Sobolev $W^{1,p(x)}(\Omega)$ spaces (for details, see e.g., [13, 11, 16]).

Set

$$C_+(\overline{\Omega}) = \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } \ x \in \overline{\Omega}\},$$

$$h^+ = \max_{\overline{\Omega}} h(x), \quad h^- = \min_{\overline{\Omega}} h(x) \ \text{ for any } h \in C(\overline{\Omega}).$$

We define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ \text{ measurable}; \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}$$

equipped with norm

$$|u|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

where $p(x) \in C(\overline{\Omega})$ satisfies condition

$$1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x).$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \{u : u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega)\},$$

and endowed with norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} := |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, that is to say, space $W_0^{1,p(x)}(\Omega)$ is defined as $\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,p(x)}(\Omega)}} = W_0^{1,p(x)}(\Omega)$. Thus, $u \in W_0^{1,p(x)}(\Omega)$ iff there exists a sequence $\{u_n\}$ of $C_0^\infty(\Omega)$ such that $\|u_n - u\|_{W^{1,p(x)}(\Omega)} \to 0$.

Proposition 1 ([16]) *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces if $p^- > 1$ and $p^+ < +\infty$.*

Proposition 2 ([16]) *In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is there exists a positive constant C such that*

$$|u|_{L^{p(x)}(\Omega)} \leq C|\nabla u|_{L^{p(x)}(\Omega)}, \ \ \forall u \in W_0^{1,p(x)}(\Omega).$$

As a consequence, $|\nabla u|_{L^{p(x)}(\Omega)}$ and $\|u\|$ are equivalent norms on the space $W_0^{1,p(x)}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_{L^{p(x)}(\Omega)}$ for the sake of simplicity.

Proposition 3 ([15, 16]) *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p(x) < N$. if $q \in C(\overline{\Omega})$ and $1 \leq q(x) \leq p^*(x)$ $(1 \leq q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where $p^* = \frac{Np(x)}{N-p(x)}$.*

Proposition 4 ([16]) *Let $\rho(u) = \int_\Omega |u|^{p(x)}dx$ for each $u \in L^{p(x)}(\Omega)$. Then, for any $u, u_n \in L^{p(x)}(\Omega)$ we have*

(i) $|u|_{L^{p(x)}(\Omega)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(ii) $|u|_{L^{p(x)}(\Omega)} > 1 \Longrightarrow |u|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega)}$;

(iii) $|u|_{L^{p(x)}(\Omega)} < 1 \Longrightarrow |u|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega)}$;

(iv) $|u_n|_{L^{p(x)}(\Omega)} \longrightarrow 0 \Leftrightarrow \rho(u_n) \longrightarrow 0$.

In the rest of this paper, we let $X = W_0^{1,p(x)}(\Omega)$.

Definition 1 *We say that $u \in X$ is a weak solution of (1), if*

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \int_\Omega f(x,u)v dx,$$

where $v \in X$.

Throughout this paper, weak solutions of the problem (1) mean the critical points of the associated energy functional J acting on the Sobelev space $W_0^{1,p(x)}(\Omega)$. Let's define the functionals

$$\Phi(u) := \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\right)dx$$

and

$$\Psi(u) := \int_\Omega F(x,u)dx$$

where $\widehat{M}(t) = \int_0^t M(s)ds$, and $F(x,u) = \int_0^u f(x,t)dt$.

The energy functional corresponding to problem (1) is $J : X \to \mathbb{R}$, with $J = \Phi - \Psi$, is well defined. Obviously, by the assumptions on f and M, $J \in C^1(X, \mathbb{R})$ and J is weakly lower semi-continuous. Therefore, $u \in X$ is a weak solution of (1) if and only if it holds

$$J'(u)v = M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx - \int_\Omega f(x,u)v dx$$

for all $v \in X$.

Definition 2 *Let E be a real reflexive Banach space. If any sequence $\{u_k\} \subset E$ for which $\{J(u_k)\}$ is bounded and $J'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence. Then it is said that J satisfies Palais-Smale condition.*

Theorem 1 ([24, Theorem 4.4]) *Let X be a Banach space and $J : X \to \mathbb{R}$ be a function bounded from below and differentiable on X. If J satisfies the $(PS)_c$-condition with $c = \inf_X J$, then J has a minimum on X.*

Notice that it is clear from Theorem 1 that the (PS)-condition implies the $(PS)_c$-condition for each $c \in \mathbb{R}$.

Theorem 2 ([24, Theorem 4.10]) *Let $J \in C^1(X, \mathbb{R})$, and J satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and*

$$\inf_{u \in \partial\Omega} J(u) > \max\{J(u_0), J(u_1)\},$$

then there exists a critical point u of J, i.e. $J'(u) = 0$ with $J(u) > \max\{J(u_0), J(u_1)\}$.

Theorem 3 ([25, Theorem 9.12]) *Let E be an infinite dimensional real Banach space. Let $J \in C^1(E, \mathbb{R})$ be an even functional which satisfies the (PS)-condition, and $J(0) = 0$. Suppose that $E = V \bigoplus X$, where V is finite dimensional, and J satisfies that*

- (i_1) *There exist $\alpha > 0$ and $\rho > 0$ such that $J(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$;*
- (i_2) *For any finite dimensional subspace $W \subset E$ there is $R = R(W)$ such that $J(u) \leq 0$ on $W \backslash B_R$.*

Then J possesses an unbounded sequence of critical values.

Theorem 4 ([28, Theorem 38]) *For the functional $F : M \subseteq X \longrightarrow [-\infty, +\infty]$ with $M \neq \emptyset$, $\min_{u \in M} F(u) = \alpha$ has a solution in case the following conditions hold:*

- (i_3) *X is a real reflexive Banach space,*
- (i_4) *M is bounded and weak sequentially closed,*
- (i_5) *F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $F(u) \leq \lim_{n\to\infty} \inf F(u_n)$ holds.*

We want to remark that in the papers [5, 29], Theorems 2 and 3 have been successfully applied to show the multiple solutions of Nonlinear impulsive differential equations with Dirichlet boundary conditions and the existence of solutions for a class of degenerate nonlocal problems involving sub-linear nonlinearities, respectively. Moreover, in the paper [30], Theorem 3 has been successfully applied to obtain the existence of infinitely many solutions for a boundary value problem.

3 Main Results

We assume the following:

- (m_0) $\exists\ m_0 > 0$ such that $M(t) \geq m_0$.
- (m_1) $\exists\ 0 < k < 1$ such that $\widehat{M}(t) \geq kM(t)t$.
- (f_0) there exists a constant $\nu > \frac{p^+}{k}$ such that $0 < \nu F(x,t) \leq tf(x,t)$, $|t| > T$.
- (f_1) $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathèodory condition and

$$|f(x,t)| \leq c(1 + |t|^{\alpha(x)-1}) \quad for \quad |t| \leq T,$$

  where $\alpha \in C_+(\overline{\Omega})$ and $p^+ < \alpha(x) < p^*(x)$ for $x \in \overline{\Omega}$.
- (f_2) $f(x,t) = o(|t|^{p^+-1})$, $t \longrightarrow 0$, for $x \in \overline{\Omega}$ uniformly.

The main results of the present paper are the following.

Theorem 5 *Assume that the assumptions* (f_0), (f_1), (f_2), (m_0) *and* (m_1) *hold. Then, if* $f(x,t) \geq 0$ *for all* $(x,t) \in \overline{\Omega} \times \mathbb{R}$, *the problem* (1) *has at least two weak solutions.*

Theorem 6 *Assume that the assumptions* (f_0), (m_0) *and* (m_1) *hold. Then, if* $f(x,t)$ *is odd in* t, *the problem* (1) *has infinitely many weak solutions.*

First, we start with the following lemma.

Lemma 1 *Assume that* (f_0), (m_0) *and* (m_1) *hold. Then* $J(u)$ *satisfies the* (PS)*-condition.*

Proof. Assume that $\{u_n\} \subset X$ such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|J(u_n)| \leq c_0$. Therefore, letting $\|u_n\| > 1$, by the assumptions (f_0), (m_0), (m_1), and Proposition 4 we have

$$\begin{aligned} c_0 + \|u_n\| &\geq J(u_n) - \frac{1}{\nu} J'(u_n)(u_n) \\ &\geq \left(\frac{k}{p^+} - \frac{1}{\nu}\right) M\left(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx\right) \int_\Omega |\nabla u_n|^{p(x)} dx \\ &+ \int_\Omega \left(\frac{1}{\nu} f(x,u_n)u_n - F(x,u_n)\right) dx \\ &\geq m_0 \left(\frac{k}{p^+} - \frac{1}{\nu}\right) \|u_n\|^{p^-} \end{aligned}$$

Due to assumption $\nu > \frac{p^+}{k}$, we infer that $\{u_n\}$ is bounded. By using the same argument given in [10, Lemma 2.4], it can easily be proven that $\{u_n\}$ converges strongly to u in X. Overall, this implies J satisfies the (PS)-condition. ∎

3.1 The Proof of Theorem 5

Proof. By the definition of J, it is clear that $J(0) = 0$. Moreover, from Lemma 1 we know that J satisfies the (PS)-condition. The rest of the proof is split into two steps:

Step 1. We will show that there exists $M > 0$ such that the functional J has a local minimum $u_0 \in B_M = \{u \in X; \|u\| < M\}$. To do this, we will apply Mazur's lemma (see, e.g., [23]) which states that any weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit. Let $\{u_n\} \subseteq \overline{B}_M$ and $u_n \rightharpoonup u$ as $n \to \infty$, then there exists a sequence of convex combinations

$$v_n = \sum_{j=1}^{n} a_{n_j} u_j, \quad \sum_{j=1}^{n} a_{n_j} = 1, \qquad a_{n_j} \geq 0, \ j \in N$$

such that $v_n \to u$ in X. Since $\overline{B}_M$ is a closed convex set, we have $\{v_n\} \subseteq \overline{B}_M$ and $u \in \overline{B}_M$. Noting that J is weak sequentially lower semi-continuous on $\overline{B}_M$, and that X is a reflexive Banach space, we can infer by Theorem 4 that J has a local minimum $u_0 \in \overline{B}_M$.

Now, we assume that $J(u_0) = \min_{u \in \overline{B}_M} J(u)$, and show that

$$J(u_0) < \inf_{u \in \partial B_M} J(u).$$

Since $p^+ < \alpha^- \leq \alpha(x) < p^*(x)$, we have the embedding $X \hookrightarrow L^{p^+}(\Omega)$ which means that there exists $c_0 > 0$ such that $|u|_{p^+} \leq c_0 \|u\|$, $\forall u \in X$. Let $\varepsilon > 0$ be small enough such that $\varepsilon c_0^{p^+} < \frac{k m_0}{2p^+}$. By the assumptions (f_1) and (f_2), we have

$$F(x,t) \leq \varepsilon |t|^{p^+} + c(\varepsilon) |t|^{\alpha(x)} \text{ for } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Then, from (**??**), (m_0) and (m_1), it reads

$$\begin{aligned}J(u) &\geq km_0\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) - \varepsilon\int_\Omega |u|^{p^+}dx - c(\varepsilon)\int_\Omega |u|^{\alpha(x)}dx\\ &\geq \frac{km_0}{p^+}\|u\|^{p^+} - \varepsilon c_0^{p^+}\|u\|^{p^+} - c(\varepsilon)\|u\|^{\alpha^-}\\ &\geq \frac{km_0}{2p^+}\|u\|^{p^+} - c(\varepsilon)\|u\|^{\alpha^-},\ \text{ when } \|u\|<1.\end{aligned}$$

Therefore, there exist $r, \delta > 0$ such that $J(u) \geq \delta > 0$ for every $\|u\| = r < 1$. If we let $M = r$, then $J(u) > 0 = J(0) \geq J(u_0)$ for $u \in \partial B_M$. Hence $u_0 \in B_M$ and $J'(u_0) = 0$.

Step 2. Since $J(u_0) = \min_{u\in X} J(u)$, we can let $M > 0$ be sufficiently large such that $J(u_0) \leq 0 < \inf_{u\in\partial B_M} J(u)$, where $B_M = \{u \in X; \|u\| < M\}$.

Now we will show that there exists $u_1 \in X$ with $\|u_1\| > M$ such that $J(u_1) < \inf_{\partial B_M} J(u)$. For this, let $e_1(x) \in X$ and $u_1 = \gamma e_1, \gamma > 0$ and $\|e_1\| = 1$. By (f_0), there exist constants $a_1, a_2 > 0$ such that $F(x,t) \geq a_1|t|^\nu - a_2$ for all $x \in \overline{\Omega}$, $|t| \geq T$. When $t > t_0 > 0$, from (m_1) we can easily see that $\widehat{M}(t) \leq ct^{\frac{1}{k}}$. Indeed, by integrating over the interval (t_0, t) and using elementary calculus lead to $\widehat{M}(t) \leq \widehat{M}(t_0)(\frac{t}{t_0})^{\frac{1}{k}} = ct^{\frac{1}{k}}$. Thus

$$\begin{aligned}J(u_1) &= (\Phi - \Psi)(\gamma e_1)\\ &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\gamma\nabla e_1|^{p(x)}dx\right) - \int_\Omega F(x,\gamma e_1)dx\\ &\leq c\left(\int_\Omega \frac{1}{p(x)}|\gamma\nabla e_1|^{p(x)}dx\right)^{\frac{1}{k}} - a_1\gamma^\nu\int_\Omega |e_1|^\nu dx + a_2\\ &\leq \frac{c}{(p^+)^{\frac{1}{k}}}\gamma^{\frac{p^+}{k}}\left(\int_\Omega |\nabla e_1|^{p(x)}\right)^{\frac{1}{k}} - a_1\gamma^\nu\int_\Omega |e_1|^\nu dx + a_2.\end{aligned}$$

Since $\nu > \frac{p^+}{k}$, there exists sufficiently large γ such that $\gamma > M > 0$ which means $J(\gamma e_1) < 0$. Hence, $\inf_{\partial B_M} J(u) > \max\{J(u_0), J(u_1)\}$. Then, Theorem 2 assures the existence of the second critical point u^*. Therefore, u_0, u^* are two critical points of φ, which are two nontrivial solutions of the problem (1). ■

The following example illustrates Theorem 5.

Example 1 *Consider $N = 3$, and $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 \leq 8\} \subset \mathbb{R}^3$, $p(x) = \frac{3}{2} + \frac{1}{2}cos(x_1^2 + x_2^2 + x_3^2)\pi$, therefor $p \in C(\overline{\Omega})$, $p^- = 1$ and $p^+ = 2$, $M(t) = 1 + t^4$, for $t \in \mathbb{R}^+$, thus M satisfies the conditions m_0 with $m_0 = 1$ and by choosing $k = \frac{1}{5}$, $\widehat{M}(t) \geq \frac{1}{5}M(t)t$ satisfies the condition m_1. Let*

$$f(x,t) = \begin{cases} t^{10}, & |t| > 1,\\ t^6, & |t| \leq 1.\end{cases}$$

By the expression of f, we have

$$F(x,t) = \begin{cases} \frac{t^{11}}{11}, & |t| > 1,\\ \frac{t^7}{7}, & |t| \leq 1.\end{cases}$$

Moreover, $f(x,t) = o(|t|), t \to 0$ and by choosing $\alpha(x) = 4 + sin(x_1 + x_2 + x_3)\pi$ $\alpha \in C_+(\overline{\Omega})$ and $p^+ < \alpha(x) \leq p^*(x)$, such that $|f(x,t)| < c(1 + |t|^{\alpha(x)-1})$ for $|t| < 1$. And by choosing $\nu = 11$, that $\nu > \frac{p^+}{k}$ we have $11F(x,t) \leq tf(x,t)$, so we see that all conditions (f_0), (f_1), and (f_2) are satisfied therefore, the problem

$$\begin{cases} -M(\int_\Omega \frac{1}{\frac{3}{2}+\frac{1}{2}cos(x_1^2+x_2^2+x_3^2)\pi}|\nabla u|^{\frac{3}{2}+\frac{1}{2}cos(x_1^2+x_2^2+x_3^2)\pi}dx)\mathrm{div}(|\nabla u|^{\frac{-1}{2}+\frac{1}{2}cos(x_1^2+x_2^2+x_3^2)\pi}\nabla u) = f(x,u), & in\ \Omega\\ u = 0, & on\ \partial\Omega\end{cases} \tag{4}$$

has at least two nontrivial weak solutions.

3.2 The Proof of Theorem 6

Proof. From the definitions of the functionals Φ and Ψ, it is clear that J is even and $J(0) = 0$. The rest of the proof is split into two steps:

Step 1. Since its proof is straightforward, we only depict briefly how J satisfies condition $(\mathrm{i})_3$ in Theorem 3. Since, J is coercive and also satisfies (PS)-condition, by the minimization theorem [24, Theorem 4.4], the functional J has a minimum critical point $u \in X$ with $J(u) \geq \alpha > 0$ and $\|u\| = \rho$ for $\rho > 0$ small enough.

Step 2. Now, we will show that J satisfies condition $(\mathrm{i})_4$ in Theorem 3. Let $W \subset X$ be a finite dimensional subspace. Any non-zero vector $u \in W$ has a unique representation $u = \theta e_1$, where $\theta = \|u\|$ and $\|e_1\| = 1$. Then, similar to Step 2 in the proof of Theorem 5, it follows

$$\begin{aligned}
J(\theta e_1) &= (\Phi - \Psi)(\theta e_1) \\
&= \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\theta \nabla e_1|^{p(x)} dx\right) - \int_\Omega F(x, \theta e_1) dx \\
&\leq c \left(\int_\Omega \frac{1}{p(x)} |\theta \nabla e_1|^{p(x)} dx\right)^{\frac{1}{k}} - a_1 \theta^\nu \int_\Omega |e_1|^\nu dx + a_2 \\
&\leq \frac{c}{(p^+)^{\frac{1}{k}}} \theta^{\frac{p^+}{k}} \left(\int_\Omega |\nabla e_1|^{p(x)}\right)^{\frac{1}{k}} - a_1 \theta^\nu \int_\Omega |e_1|^\nu dx + a_2.
\end{aligned}$$

The above inequality implies that there exists θ_0 such that $\|\theta e_1\| > \rho$ and $J(\theta e_1) < 0$ for every $\theta \geq \theta_0 > 0$. Since W is a finite dimensional subspace, there exists $R = R(W) > 0$ such that for all $u \in W \setminus B_R$, that is, when $\|u\| \geq R$, we have $J(u) \leq 0$. According to Theorem 3, the functional $J(u)$ possesses infinitely many critical points, i.e., the problem (1) has infinitely many weak solutions. ∎

The following example illustrates Theorem 6.

Example 2 *Consider $N = 3$ and $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 \leq 8\} \subset \mathbb{R}^3$, $p(x) = 2 + \frac{1}{2} sin(x_1^2 + x_2^2 + x_3^2)\pi$, therefor $p \in C(\overline{\Omega})$, $p^- = \frac{3}{2}$ and $p^+ = \frac{5}{2}$, $M(t) = 1 + t^2$, for $t \in \mathbb{R}^+$, thus M satisfies the conditions m_0 with $m_0 = 1$ and by choosing $k = \frac{1}{3}$, $\widehat{M}(t) \geq \frac{1}{3} M(t) t$ satisfies the condition m_1. Let*

$$f(x,t) = \begin{cases} t^7, & |t| > 1, \\ t^5 + sin(\pi t), & |t| \leq 1. \end{cases}$$

By the expression of f, we have

$$F(x,t) = \begin{cases} \frac{1}{8} t^8, & |t| > 1, \\ \frac{1}{6} t^6 - \frac{1}{\pi} cos(\pi t), & |t| \leq 1. \end{cases}$$

Moreover, by choosing $\nu = 8$, that $\nu > \frac{p^+}{k}$ we have $8F(x,t) \leq t f(x,t)$, so we see that all condition (f_0), is satisfied. Therefore, the problem

$$\begin{cases} -M(\int_\Omega \frac{1}{2 + \frac{1}{2} sin(x_1^2 + x_2^2 + x_3^2)\pi} |\nabla u|^{2 + \frac{1}{2} sin(x_1^2 + x_2^2 + x_3^2)\pi} dx) \operatorname{div}(|\nabla u|^{\frac{1}{2} sin(x_1^2 + x_2^2 + x_3^2)\pi} \nabla u) = f(x,u), & in \ \Omega \\ u = 0, & on \ \partial\Omega \end{cases}$$

has infinitely many weak solutions.

References

[1] A. Arosio and S. Pannizi, On the well-posedness of Kirchhoff string, Trans. Amer. Math. Soc., 348(1996), 305–330.

[2] M. Avci, B. Cekic and R.A. Mashiyev, Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory, Math. Methods Appl. Sci., 34(2011), 1751–1759.

[3] M. Avci, Ni-Serrin type equations arising from capillarity phenomena with non-standard growth, Bound. Value Probl., 55(2013), 1–13.

[4] F. Cammaroto and L. Vilasi, Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator, Nonlinear Anal. TMA, 74(2011), 1841–1852.

[5] G. Caristi, S. Heidarkhani, A. Salari and S.A. Tersian, Multiple solutions for degenerate nonlocal problems, Appl. Math. Lett., 84(2018), 26–33.

[6] B. Cekic, A. V. Kalinin, R. A. Mashiyev and M. Avci, $L^{p(x)}(\Omega)$-estimates of vector fields and some applications to magnetostatics problems, J. Math. Anal. Appl., 389(2012), 838–851.

[7] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math., 4(2006), 1383–1406.

[8] N. T. Chung, Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities, Complex Var. Elliptic Equ., 58(2013), 1637–1646.

[9] N. T. Chung, On a class of $p(x)$-Kirchhoff type problems with robin boundary conditions and indefinite weights, Turkish World Mathematical Society Journal of Applied and Engineering Mathematics, 10(2020), 400–410.

[10] N. T. Chung and H. Q. Toan, Multiple solutions for a class of degenerate nonlocal problems involving sublinear nonlinearities, Matematiche, 69(2014), 171–182.

[11] D. V. Cruz-Uribe and A. Fiorenza, Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Springer, Basel, 2013.

[12] G. Dai and R. Hao, Existence of solutions for a $p(x)$-Kirchhoff-type equation, J. Math. Anal. Appl., 359(2009), 704–710.

[13] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Heidelberg, 2011.

[14] P. D'Ancona and S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math., 108(1992), 247–262.

[15] X. L. Fan, J. S. Shen and D. Zhao, Sobolve embedding theorem for spaces $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl., 262(2001), 749–760.

[16] X. L. Fan and D. Zhao, On the space $L^{p(x)}$ and $W^{m,p(x)}$, J. Math. Anal. Appl., 263(2001), 424–446.

[17] J. R. Graef, S. Heidarkhani and L. Kong, A variational approach to a Kirchhoff-type problem involving two parameters, Results Math. 63(2013), 877–889.

[18] X. He and W. Zou, Infinitely many positive solutions for Kirchhoff-type problems, Nonlinear Anal. TMA, 70(2009), 1407–1414.

[19] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[20] J. L. Lions, On some equations in boundary value problems of mathematical physics, in: Contemporary Developments in Continuum Mechanics and partial Differential Equations, Proc. Internat. Sympos. Inst. Mat.Univ. Fed. Rio de Janeiro, 1997, in: North-Holland Math. Stud. 30 North-Holland, Amsterdam, (1978), 284–346.

[21] D. Liu, On a p-Kirchhoff equation via fountain theorem and dual fountain theorem, Nonlinear Anal. TMA, 72(2010), 302–308.

[22] T. F. Ma, Remarks On an elliptic equation of Kirchhoff type, Nonlinear Anal. TMA, 63(2005), 1967–1977.

[23] J. Mawhin, Some boundary value problems for Hartman-type perturbations of the ordinary vector p-Laplacian, Nonlinear Anal. TMA, 40(2000), 497–503.

[24] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, Berlin, 1989.

[25] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, in: CBMS Regional Conference Series in Mathematics, vol. 65, The American Mathematical Society, Providence, RI, 1986 Published for the Conference Board of the Mathematical Sciences, Washington, DC.

[26] M. Ruzicka, Electro-rheological Fluids: Modeling and Mathematical Theory, in: Lecture Notes in Math., vol. 1784, Springer, Berlin, 2000.

[27] Z. Yucedag, M. Avci and R.A. Mashiyev, On an elliptic system of $p(x)$-Kirchhoff-type under Neumann boundary condition, Math. Model. Anal., 17(2012), 161–170.

[28] E. Zeidler, Nonlinear Functional Analysis and its Applications, vol. III. Springer, Berlin (1985).

[29] D. Zhang, Multiple solutions of nonlinear impulsive Ddifferential equations with Dirichlet boundary conditions via variational method, Results. Math., 63(2013), 611–628.

[30] D. Zhang and B. Dai, Infinitely many solutions for a class of nonlinear impulsive differential equations with periodic boundary conditions, Compu. Math. Appl., 61(2011), 3153–3160.