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Local Convergence Of A Modified Chebyshev's Iterative Method For Nonlinear Ill-Posed Equations In Banach Space*

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Abstract

In this paper, we have modified Chebyshev's iterative method given in [10] for nonlinear ill-posed equations in Banach spaces involving m-accretive mappings. We have provided a local convergence for the method with some basic assumptions. The semilocal convergence analysis of this method was studied by [14]. This work provides computable convergence ball and computable error bounds.

1 Introduction

In this study we are interested in the problem of approximately solving the nonlinear ill-posed operator equation

$$F(u) = f, \tag{1}$$

where $F : D(F) \subseteq E \to E$ is an m-accretive, Fréchet differentiable and single valued nonlinear mapping from a real reflexive Banach space E into itself. The norm on E, is denoted by $\|.\|$ and the dual of E is denoted by E^*. Throughout this paper we write $\langle j, u\rangle$ instead of $j(u)$, for each $j \in E^*$ and $u \in E$. We assume that (1) has a solution, say $\hat{u}$, i.e,

$$F(\hat{u}) = f. \tag{2}$$

Recall that [1, 2, 6] F is m-accretive if it satisfies the following

1. $\langle F(x) - F(y), J(x - y)\rangle \geq 0$, where J is the dual mapping on E.
2. $R(F + \lambda I) = E$ for each $\lambda \geq 0$ where $R(F)$ and I denote the range of F and the identity mapping on E respectively.

Since F is m-accretive, for $\alpha > 0$ and for fixed $f \in E$,

$$F(u) + \alpha(u - u_0) = f \tag{3}$$

has a unique solution [4, 5, 7] denoted by u_α where u_0 is the initial guess of the exact solution $\hat{u}$. It is known [2, 4, 5, 8, 12, 13] that u_α is an approximation for $\hat{u}$ (i.e., $u_\alpha \to \hat{u}$ and $\alpha \to 0$). In practice, the available data is f^δ with

$$\|f - f^\delta\| \leq \delta. \tag{4}$$

So one has to deal with the equation

$$F(u) + \alpha(u - u_0) = f^\delta \tag{5}$$

instead of (3). The above equation has a unique solution u_α^δ. It is known that u_α^δ is a good approximation for $\hat{u}$ provided α is choosen appropriately [2, 4, 5, 8, 15, 16, 17]. Therefore, our approach in this paper is to obtain u_α^δ. In fact we have the following result (see [5, 18, 8])

$$\|u_\alpha^\delta - u_\alpha\| \leq \frac{\delta}{\alpha} \tag{6}$$

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and

$$\|u_\alpha - \hat{u}\| \leq \|\hat{u} - u_0\|. \tag{7}$$

Obtaining a closed form solution u_α^δ of (5) is difficult in general. So most of the solution methods considered for solving (5) are iterative.

Motivated by [10] we study a new local convergence analysis for modified Chebyshev's method discussed in [14] to approximate u_α^δ. Here in convergence analysis we impose conditions on u_α^δ and obtain the estimates of computed radii of the convergence of balls.

The proposed method is defined for each $k = 0, 1, \cdots,$ by

$$\begin{aligned} v_k &= u_k - [R'_\alpha(u_k)]^{-1} R_\alpha(u_k) \\ w_k &= v_k - \frac{1}{2} L_k [R'_\alpha(u_k)]^{-1} R_\alpha(u_k) \\ u_{k+1} &= w_k - H_k [R'_\alpha(u_k)]^{-1} R_\alpha(w_k) \end{aligned} \tag{8}$$

where

$$R_\alpha(u) := F(u) + \alpha(u - u_0) - f^\delta,$$

$$L_k = R'_\alpha(u_k)^{-1} F''(u_k) R'_\alpha(u_k)^{-1} R_\alpha(u_k)$$

and

$$H_k = I + L_k + \frac{3}{2} L_k^2 - \frac{1}{2} R'_\alpha(u_k)^{-1} F'''(u_k) R'_\alpha(u_k)^{-1} R_\alpha(u_k)^2.$$

Further extensive discussion of convergence rate can be seen in Ortega and Rheinbold [11] and Kelly [9].

The rest of the paper is organized as follows. Basic assumptions and preliminaries are discussed in Section 2. Local convergence analysis of our method is given in Section 3. As an illustration to our work we have provided a numerical example in Section 4. Finally, the paper ends with a conclusion in Section 5.

2 Basic Assumptions and Preliminaries

The results in this paper are based on the following assumptions $(\mathcal{A})$:

- $(\mathcal{A}_1)$ There exists a constant $k_0 \geq 0$ such that for every $u, v \in B(u_\alpha^\delta, r_0)$ and $w \in E$ there exists an element $\Phi(u, v, w) \in E$ such that $[F'(u) - F'(v)]w = F'(v)\Phi(u, v, w)$, $\|\Phi(u, v, w)\| \leq k_0 \|w\| \|v - u\|$.
- $(\mathcal{A}_2)$ There exists $v \in E$ such that $u_0 - \hat{u} = F'(u_0)^\nu v \quad 0 < \nu \leq 1$.

Theorem 1 ([13, Theorem 3.3]) *Let condition $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. If $3L_0 r < 1$, then*

$$\|u_\alpha - \hat{u}\| \leq c_1 \alpha^\nu$$

for some constant $c_1 > 0$, where ν is as in $(\mathcal{A}_2)$.

Let $k_0 > 0$ be a given parameter. Let us define functions g_1, g_2, g_3 and h on the interval $[0, 1/2k_0)$,

$$g_1(r) = \frac{3k_0 r}{1 - 2k_0 r},$$

$$g_2(r) = \frac{r}{1 - 2k_0 r}\left(3k_0 + \frac{M_1^2 M_2}{(1 - 2k_0 r)^2}\right),$$

$$h(r) = 1 + \frac{M_1 M_2 r}{1 - 2k_0 r} + \frac{3}{2} \frac{M_1^2 M_2^2 r^2}{(1 - 2k_0 r)^2} + \frac{1}{2} \frac{M_3 M_1 r^2}{(1 - 2k_0 r)^3}$$

and

$$g_3(r) = r g_2(r)\left(1 + \frac{h(r) M_1}{1 - 2k_0 r}\right).$$

Moreover, define polynomial g on the interval $[0, \frac{1}{2k_0})$ by

$$g(r) = g_3(r) - 1.$$

We have that $g(0) < 0$ and $g(r) \to +\infty$ as $r \to \frac{1}{2k_0}^-$. Then, it follows from the intermediate value theorem that polynomial g has roots in the domain. Denote r_0 the smallest root of polynomial g on the domain. From the definitions of the functions g_1, g_2, g_3, h, polynomial g and point r_0 that for each $r \in (0, r_0)$

$$0 < g_1(r) < 1,$$

$$0 < g_2(r) < 1,$$

$$1 < h(r),$$

and

$$0 < g_3(r) < 1.$$

Now using the above results and notations, we can show the local convergence result of the method (8).

3 Local Convergence

Using basic assumptions and observations discussed in above section we arrive in following theorem which describes the local convergence of (8).

Theorem 2 *Let $F : D \subseteq E \to E$ be twice Fréchet differentiable operator with $B(u_\alpha^\delta, r_0) \subseteq D$ and k_0, M_1 and $M_2 > 0$. Suppose that for each $u \in D$*

$$\|R'_\alpha(u_\alpha^\delta)^{-1}F'(u)\| \leq M_1 \tag{9}$$

$$\|R'_\alpha(u_\alpha^\delta)^{-1}F''(u)\| \leq M_2 \tag{10}$$

holds. Then sequence $\{u_k\}$ defined in (8) is well defined and remains in $B(u_\alpha^\delta, r_0)$ $\forall k = 0, 1, \cdots$ and converges to u_α^δ. Moreover the following estimates holds for each $k = 0, 1, \cdots$,

$$\|v_k - u_\alpha^\delta\| \leq g_1(\|u_k - u_\alpha^\delta\|)\|u_k - u_\alpha^\delta\| \tag{11}$$

$$\|w_k - u_\alpha^\delta\| \leq g_2(\|u_k - u_\alpha^\delta\|)\|u_k - u_\alpha^\delta\| \tag{12}$$

and

$$\|u_{k+1} - u_\alpha^\delta\| \leq g_3(\|u_k - u_\alpha^\delta\|)\|u_k - u_\alpha^\delta\| \tag{13}$$

Proof. For easiness we use the notation

$$e_k = \|u_k - u_\alpha^\delta\| \text{ for each } k = 0, 1, 2. \cdots,$$

$$\hat{e}_k = \|v_k - u_\alpha^\delta\| \text{ for each } k = 0, 1, 2. \cdots,$$

$$\bar{e}_k = \|w_k - u_\alpha^\delta\| \text{ for each } k = 0, 1, 2. \cdots,$$

We have, for some $v \in D$ with $\|v\| = 1$,

$$\begin{aligned}
\|(I - R'_\alpha(u_\alpha^\delta)^{-1}R'_\alpha(u_0))(v)\| &\leq \|R'_\alpha(u_\alpha^\delta)^{-1}(F'(u_\alpha) - F'(u_0))(v)\| \\
&\leq \|R'_\alpha(u_\alpha^\delta)^{-1}(F'(u_\alpha^\delta))\phi(u_\alpha^\delta, u_0, v)\| \\
&\leq 2k_0\|u_\alpha^\delta - u_0\| \\
&\leq 2k_0\|e_0\| \\
&< 1.
\end{aligned}$$

Therefore, by Banach lemma on invertible operators [3], we have

$$\|R'_\alpha(u_0)^{-1}R'_\alpha(u_\alpha^\delta)\| \leq \frac{1}{1-2k_0\|e_0\|}. \tag{14}$$

Consider

$$\begin{aligned}
& R'_\alpha(u_\alpha^\delta)^{-1}(R_\alpha(u_0)-R_\alpha(u_\alpha^\delta)) - R'_\alpha(u_0)e_0 \\
= \; & R'_\alpha(u_\alpha^\delta)^{-1}(F(u_0)-F(u_\alpha^\delta)-F'(u_0)e_0) \\
= \; & \int_0^1 R'_\alpha(u_\alpha^\delta)^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta+te_0,u_\alpha^\delta,e_0)dt + (-R'_\alpha(u_\alpha^\delta)^{-1}F'(u_\alpha^\delta)\phi(u_\alpha^\delta+te_0,u_\alpha^\delta,e_0)).
\end{aligned}$$

By taking norm on both sides we have

$$\begin{aligned}
\|R'_\alpha(u_\alpha^\delta)^{-1}(R_\alpha(u_0)-R_\alpha(u_\alpha^\delta)) - R'_\alpha(u_0)e_0\| &\leq k_0\|e_0\|^2 + 2k_0\|e_0\|^2 \\
&\leq 3k_0\|e_0\|^2|
\end{aligned}$$

From definition (8),

$$\begin{aligned}
\hat{e_0} &= e_0 - (R'_\alpha(u_0))^{-1}R_\alpha(u_0) \\
&= e_0 - (R'_\alpha(u_0))^{-1}R'_\alpha(u_\alpha^\delta)R'_\alpha(u_\alpha^\delta)^{-1}R_\alpha(u_0) \\
&= -R'_\alpha(u_0)^{-1}R'_\alpha(u_\alpha^\delta)[R'_\alpha(u_\alpha^\delta)^{-1}R_\alpha(u_0) - R'_\alpha(u_\alpha^\delta)^{-1}R'_\alpha(u_0)e_0] \\
&= -R'_\alpha(u_0)^{-1}R'_\alpha(u_\alpha^\delta)[R'_\alpha(u_\alpha^\delta)^{-1}R_\alpha(u_0) - R_\alpha(u_\alpha^\delta) - R'_\alpha(u_0)e_0].
\end{aligned}$$

Therefore,

$$\|\hat{e_0}\| \leq \frac{1}{1-2k_0\|e_0\|}[3k_0\|e_0\|^2] \leq g_1(\|e_0\|)\|e_0\| \leq r_0.$$

So, $v_0 \in B(u_\alpha^\delta, r_0)$ and

$$L_0 = R'_\alpha(u_0)^{-1}F''(u_0)R'_\alpha(u_0)^{-1}R_\alpha(u_0) = R'_\alpha(u_0)^{-1}R'_\alpha(u_\alpha^\delta).$$

Consider

$$\begin{aligned}
\|R'_\alpha {u_\alpha^\delta}^{-1}R_\alpha(u_0)\| &= \|R_\alpha(u_\alpha^\delta)^{-1}(F(u_0)-F(u_\alpha^\delta)+\alpha(u_0-u_\alpha^\delta)) \\
&= \|R'_\alpha(u_\alpha^\delta)^{-1}\int_0^1 F'(u_\alpha^\delta+te_0)e_0dt\| + \|R'_\alpha(u_\alpha^\delta)^{-1}\alpha(u_0-u_\alpha^\delta)\| \\
&\leq M_1\|e_0\|.
\end{aligned}$$

Therefore,

$$\|L_0\| \leq \|R'_\alpha(u_0)^{-1}R'_\alpha(u_\alpha^\delta)\|^2\|R'_\alpha(u_\alpha^\delta)F''(u_0)\|\|R'_\alpha(u_\alpha^\delta)^{-1}R_\alpha(u_0)\| \leq \frac{M_1M_2\|e_0\|}{(1-2k_0\|e_0\|)^2}.$$

From definition (8),

$$\tilde{e_0} = \hat{e_0} - \frac{1}{2}L_0(R'_\alpha(u_0))^{-1}R'_\alpha(u_\alpha^\delta)R'_\alpha(u_\alpha^\delta)^{-1}R_\alpha(u_0).$$

By taking norm on both sides,

$$\begin{aligned}
\|\tilde{e_0}\| &\leq \frac{3k_0\|e_0\|^2}{1-2k_0\|e_0\|} + \frac{1}{2}\frac{M_1M_2\|e_0\|}{1-2k_0\|e_0\|}(\frac{1}{1-2k_0\|e_0\|})(M_1\|e_0\|) \\
&\leq \frac{\|e_0\|^2}{1-2k_0e_0}(3k_0 + \frac{M_1^2M_2}{(1-2k_0\|e_0\|^2)}) \\
&\leq g_2(\|e_0\|)\|e_0\| \\
&< r_0.
\end{aligned}$$

Therefore, $w_0 \in B(u_\alpha^\delta, r_0)$.

Now from definition (8),

$$\|H_0\| \leq 1 + \frac{M_1 M_2 \|e_0\|}{1 - 2k_0\|e_0\|} + \frac{3}{2}\frac{M_1^2 M_2^2 \|e_0\|^2}{(1 - 2k_0\|e_0\|)^2} + \frac{1}{2}\frac{M_3 M_1 \|e_0\|^2}{(1 - 2k_0\|e_0\|)^3} \leq h(\|e_0\|),$$

$$\begin{aligned}
\|e_1\| &\leq \|\tilde{e_0}\| + \frac{\|H_0\| M_1 \|\tilde{e_0}\|}{1 - 2k_0\|e_0\|} \\
&\leq \frac{\|e_0\|^2}{1 - 2k_0\|e_0\|}(3k_0 + \frac{M_1^2 M_2}{(1 - 2k_0\|e_0\|)^2})(1 + \frac{h(\|e_0\|)M_1}{1 - 2k_0\|e_0\|}) \\
&= g_3(\|e_0\|)\|e_0\| \\
&< r_0.
\end{aligned}$$

Hence, $u_1 \in B(u_\alpha^\delta, r_0)$ and (11)–(13) holds for $k = 1$. If we simply replace u_0, v_0, w_0, u_1 by u_k, v_k, w_k, u_{k+1} in the preceding estimates, we arrive at the estimates (11)–(13) and through these estimates $u_k, v_k, w_k, u_{k+1} \in B(u_\alpha^\delta, r_0)$. ∎

4 Numerical Example

In this section we present a numerical example.

Example 1 (see [12], section 4.3) *Let $F : D(F) \subseteq C[0,1] \longrightarrow C[0,1]$ be defined by*

$$F(u) := \int_0^1 k(t,s)u^3(s)ds, \tag{15}$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1, \\ (1-s)t, & 0 \leq t \leq s \leq 1. \end{cases}$$

Then for all u, v

$$\langle F(u) - F(v), J(u-v)\rangle = \|\int_0^1 k(t,s)(u^3 - v^3)(s)ds\|^2 \geq 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)u^2(s)w(s)ds. \tag{16}$$

In our computation, we take $f(t) = \frac{6sin(\pi t) + sin^3(\pi t)}{9\pi^2}$ and $f^\delta = f + \delta$. Then the exact solution is

$$\hat{u}(t) = sin(\pi t).$$

We use

$$u_0(t) = sin(\pi t) + \frac{3[t\pi^2 - t^2\pi^2 + sin^2(\pi t)]}{4\pi^2},$$

as our initial guess.

We choose $\alpha_0 = \mu\delta$ and $\mu = 1.01$. We use the Gauss-Legendre quadrature formula:

$$\int_0^1 f(t)dt \approx \sum_{j=1}^n w_j f(t_j),$$

Table 1: Abscissa and weights of Gauss-Legendre quadrature formula

i	t_i	w_i
1	0.0022215151047509	0.0056968992505131
2	0.0116680392702412	0.0131774933075160
3	0.0285127143855128	0.0204695783506531
4	0.0525040010608623	0.0274523479879176
5	0.0832786856195830	0.0340191669061784
6	0.1203703684813212	0.0400703501675005
7	0.1632168157632658	0.0455141309914818
8	0.2111685348793885	0.0502679745335253
9	0.2634986342771425	0.0542598122371318
10	0.3194138470953061	0.0574291295728558
11	0.3780665581395058	0.0597278817678923
12	0.4385676536946448	0.0611212214951550
13	0.5000000000000000	0.0615880268633577
14	0.5614323463053552	0.0611212214951550
15	0.6219334418604942	0.0597278817678923
16	0.6805861529046939	0.0574291295728558
17	0.7365013657228575	0.0542598122371318
18	0.7888314651206115	0.0502679745335253
19	0.8367831842367342	0.0455141309914818
20	0.8796296315186788	0.0400703501675005
21	0.9167213143804170	0.0340191669061784
22	0.9474959989391377	0.0274523479879176
23	0.9714872856144872	0.0204695783506531
24	0.9883319607297588	0.0131774933075160
25	0.9977784848952490	0.0056968992505131

where the absissas t_j and the weight w_j for $n = 25$ are given in Table 1, to discretize equation (15). The discretized form of (8) is as follows:

$$\begin{aligned} v_k(t_i) &= u_k(t_i) - [R'_\alpha(u_k(t_i))]^{-1} R_\alpha(u_k(t_i)), \\ w_k(t_i) &= v_k(t_i) - \frac{1}{2} L_k(t_i)[R'_\alpha(u_k(t_i))]^{-1} R_\alpha(u_k(t_i)), \\ u_{k+1}(t_i) &= w_k(t_i) - H_k(t_i)[R'_\alpha(u_k(t_i))]^{-1} R_\alpha(w_k(t_i)), \end{aligned}$$

where

$$F(u(t_i)) = \sum_{j=1}^{25} a_{ij} u(t_j)^3, \qquad F'(u(t_i)) = \sum_{j=1}^{25} 3a_{ij} u(t_j)^2,$$

$$F''(u(t_i)) = \sum_{j=1}^{25} 6a_{ij} u(t_j), \qquad F'''(u(t_i)) = \sum_{j=1}^{25} 6a_{ij},$$

Table 2: The relative error and residual error

δ	α	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|F(u_k)-f^\delta\|\|}{\|\|f^\delta\|\|}$
0.01	0.011156683466653	0.579437886300696	1.000000000000000
0.005	0.005578341733327	0.510365663178468	1.000000000000000
0.001	0.001115668346665	0.366910099833426	1.000000000000000

and

$$R_\alpha(u(t_i)) = F(u(t_i)) + \alpha(u(t_i) - u_0(t_0)) - (f(t_i) + \delta)$$

with

$$a_{ij} = \begin{cases} w_j t_j(1-t_i), & \text{if } j \leq i, \\ w_j t_i(1-t_j), & \text{if } i < j. \end{cases}$$

The relative error $\frac{||u_k-\hat{u}||}{||\hat{u}||}$ *and the residual error* $\frac{||F(u_k)-f^\delta||}{||f^\delta||}$ *are given in Table 2.*

5 Conclusion

In this paper, we study a modern Chebyshev's iterative method given in [10] for nonlinear ill-posed equations in Banach spaces involving m-accretive mappings. We provide a local convergence for the method with some basic assumptions.This work provides computable convergence ball and computable error bounds. We have also provided a numerical example which illustrates our work.

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