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Analytical Solution Of Fractional Laplace Equations In Plane Polar And Spherical Coordinates*

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Abstract

The Fractional Laplace equation in plane-polar coordinates or spherical coordinates is solved. To find the solution the Mellin's transform is applied. We use Caputos definition of the fractional derivative for this approach.

1 Introduction

The Laplace equation appears in many physical phenomena and engineering applications such as electrostatic, heat conduction, and water wave propagation. The Fractional Laplace equation is based on implementing the fractional differentiation of the integer equation ([1]). It does generalize the integer Laplace equation. It is compatible with the integer Laplace equation at non-fractional order, besides that, it is available in Curvilinear and Cartesian coordinates. A variety of methods have been employed to find solutions to the Laplace fractional equation. For example, Samuel and Thomas ([1]) derived solutions of the fractional Laplace equation in terms of Mittag-Lefler function and Fox's H-function. They ([1]) used the Laplace and Fourier transform. Saxena et al. ([2]) presented analytical solutions of the fractional-order Laplace, Poisson, and Helmholtz equations. They employed the Fourier-Laplace transform method. The solutions are presented in terms of Mittag-Leffler functions, Fox H-function, and an integral operator containing a Mittag-Leffler function in the kernel. Despite the existence of solutions to the fractional Laplace equation, most of the approaches are performed in Cartesian coordinates, using separation of variables and Laplace and Fourier Transforms.

Recently, Nairat et al. ([3]) derived analytical solutions to the cylindrically symmetric fractional Helmholtz equation in an isotropic medium using the method of separation of variables. The general solution is given in terms of fractional Bessel functions attached to particular azimuthal and longitudinal exponents, it is represented in orthogonal and completeness basis.

The objective of this note is to derive analytical solutions of fractional order Laplace equation in plane polar or spherical coordinates. To obtain the solutions instead of using the method of separation of variables, we will apply the method of Mellin's transform. The Mellin's transform of a function $f(r)$ is

$$\tilde{f}(s) = \int_0^\infty r^{s-1}\, f(r)\, dr\,, \tag{1}$$

and the inverse Mellin's transform is

$$f(r) = \frac{1}{2\,\pi\, i} \int_{c-i\,\infty}^{c+i\,\infty} r^{-s}\, \tilde{f}(s)\, ds. \tag{2}$$

The Mellin's transform has been applied in problems of fractional ordinary and partial differential equations see for instance, ([4]), ([5]), ([6]), and in partial differential equations see for example, ([7]), ([8]), and the review paper by Luchko and Kiryakova ([9]).

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The work is organized as follows. In Section 2, we study the fractional form of the Laplace equation in plane polar coordinates. In particular, we consider the problem of determining a function that satisfies the fractional Laplace equation in the interior of an infinite 2-dimensional wedge. Next, in section 3, we present an analytical solution for the fractional Laplace equation in spherical coordinates, the solution is given in terms of Legendre's special functions, and finally, we draw our conclusions.

2 Solution of the Fractional Laplace Equation in Plane Polar Coordinates

Let us consider the problem of determining a function $u = u(r,\theta)$ which satisfies the fractional Laplace's equation in the interior of an infinite 2-dimensional wedge of angle 2ϕ, $\phi \leq \frac{\pi}{2}$, with conditions given on the boundary.

Consider the wedge referred to plane polar coordinates (r,θ) with the polar axis bisecting the wedge angle and the pole at the apex of the wedge. Assume the boundary conditions to be

$$u(r,\pm\phi) = 1 \text{ if } 0 < r < a \quad \text{and} \quad u(r,\pm\phi) = 0 \text{ if } r > a,$$

together with $u(r,\pm\phi) \to 0$ and $D_\alpha u(r,\pm\phi) \to 0$ as $r \to \infty$ with $|\theta| \leq \phi$. In plane polar coordinates, Laplace's equation is

$$\frac{1}{r^\alpha} D_\alpha(r^\alpha D_\alpha u) + \frac{\Gamma^2(\alpha+1)}{r^{2\alpha}} \frac{\partial^2 u}{\partial \theta^2} = 0, \tag{3}$$

where D_α represents fractional derivative operator,

$$D_\alpha u = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{1}{(x-\tau)^{\alpha+1-m}} \frac{d^m u}{d\tau^m} d\tau.$$

To apply the Mellin's transform we multiply the equation by $r^{s-1+2\alpha}$ and integrate with respect to r from zero to infinity

$$\int_0^\infty r^{s-1+\alpha} D_\alpha(r^\alpha D_\alpha u)\, dr + \Gamma^2(\alpha+1) \int_0^\infty r^{s-1} \frac{\partial^2 u}{\partial \theta^2}\, dr = 0, \tag{4}$$

using the definition of the Mellin's transform

$$\tilde{u}(s,\theta) = \int_0^\infty r^{s-1}\, u\, dr\,, \tag{5}$$

and

$$\int_0^\infty r^{s-1+\alpha} D_\alpha u\, dr = \frac{\Gamma(1-s)}{\Gamma(1-s-\alpha)} \tilde{u}(s,\theta), \tag{6}$$

(see ([10]) we get

$$\int_0^\infty r^{s-1+\alpha} D_\alpha(r^\alpha D_\alpha u)\, dr = \frac{\Gamma^2(1-s)}{\Gamma^2(1-s-\alpha)} \tilde{u}(s,\theta) \tag{7}$$

reduces the equation (3) to

$$\frac{d^2\tilde{u}}{d\theta^2} + \Big(\frac{\Gamma^2(1-s)}{\Gamma^2(\alpha+1)\,\Gamma^2(1-s-\alpha)}\Big)\tilde{u} = 0, \tag{8}$$

with solution

$$\tilde{u}(s,\theta) = A(s)\cos\left(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\theta\right) + B(s)\sin\left(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\theta\right), \tag{9}$$

using the boundary conditions we see that $B(s) = 0$, thus

$$\tilde{u}(s,\theta) = A(s)\cos\Big(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\theta\Big). \tag{10}$$

The transformed boundary conditions are

$$\tilde{u}(s, \pm\phi) = \int_0^a r^{s-1}\, dr \;=\; \frac{a^s}{s}, \tag{11}$$

thus

$$A(s) \;=\; \frac{a^s}{s\,\cos\big(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\phi\big)}, \tag{12}$$

and

$$\tilde{u}(s,\theta) \;=\; \frac{a^s}{s\,\cos\Big(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\phi\Big)}\,\cos\left(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\theta\right). \tag{13}$$

Taking the inverse Mellin's transform of (13), we get

$$u(r,\theta) = \frac{1}{2\,\pi\,i}\int_{c\,-\,i\,\infty}^{c\,+\,i\,\infty} (\frac{a}{r})^s\, \frac{1}{s\,\cos\big(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\phi\big)}\,\cos\big(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\theta\big)\, ds, \tag{14}$$

where because of the nature of the integrand, it follows that the line integral must be taken along a line $0 < Re(s) = c < \frac{\pi}{2\,\phi}$. The mtegrand has poles at $s = 0$, and

$$\left(\frac{\Gamma(1-s)}{\Gamma(\alpha+1)\,\Gamma(1-s-\alpha)}\phi\right) = \pm\frac{(2k-1)}{2}\pi,$$

$$\frac{\Gamma(1-s)}{\Gamma(1-s-\alpha)} = \pm\frac{(2k-1)}{2\,\phi}\,\pi\,\Gamma(\alpha+1)\,, \tag{15}$$

putting $\alpha = 1$, we obtain the condition derived by Lomen [7]

$$s \;=\; \pm\frac{(2k-1)}{2\,\phi}\pi, \tag{16}$$

and using the Cauchy's integral formula we obtain

$$u(r,\theta) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\,(\frac{a}{r})^{\frac{(2k-1)\,\pi}{2\,\phi}}\,\cos\big(\frac{(2k-1)\,\pi}{2\,\phi}\,\theta\big), \quad r > a, \tag{17}$$

$$u(r,\theta) \;=\; 1 \;-\; \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\,(\frac{a}{r})^{\frac{-(2k-1)\,\pi}{2\,\phi}}\,\cos\big(\frac{(2k-1)\,\pi}{2\,\phi}\,\theta\big), \quad r < a. \tag{18}$$

For $0 < \alpha < 1$, the general solution can be written as follows

$$u(r,\theta) = \sum_{k=0}^{\infty} A_{k,\alpha}\,(\frac{a}{r})^{|s(k,\alpha)|}\,\cos\big(\frac{(2k-1)\,\pi}{2\,\phi}\,\theta\big), \quad r < a, \tag{19}$$

and

$$u(r,\theta) = \sum_{k=0}^{\infty} B_{k,\alpha}\,(\frac{a}{r})^{-|s(k,\alpha)|}\,\cos\big(\frac{(2k-1)\,\pi}{2\,\phi}\,\theta\big), \quad r > a, \tag{20}$$

where $s(k,\alpha)$ are the solutions of the transcendant Equation (15).

3 Solution of the Fractional Laplace Equation in Spherical Polar Coordinates

In this section, we will consider the problem of determining a function $u(r,\theta)$ which satisfies the fractional Laplace's equation in the interior of an infinite right circular cone of vertex angle 2ϕ, $\phi \leq \frac{\pi}{2}$, and given conditions on the boundary. Consider the cone referred to spherical coordinates (r, Φ, θ) with the vertex of the cone at the origin. Assume the solution is independent of the meridian angle and $u(r,\theta) = f(r)$ for $\theta = \phi$. Thus the boundary value problem is governed by the fractional Laplace equation.

$$\frac{1}{r^{\alpha}} D_{\alpha}(r^{\alpha} D_{\alpha} u + u) + \frac{1}{r^{2\alpha} \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta \frac{\partial u}{\partial\theta}) = 0, \quad 0 < r < \infty, \quad 0 \leq \theta < \phi, \tag{21}$$

together with the boundary condition

$$u(r,\phi) = f(r). \tag{22}$$

Application of the Mellin transform reduces the differential equation (21) to

$$\frac{1}{\sin\theta} \frac{d}{d\theta}(\sin\theta \frac{d\tilde{u}}{d\theta}) + \big(\frac{\Gamma^2(1-s)}{\Gamma^2(1-s-\alpha)} + \frac{\Gamma(1-s)}{\Gamma(1-s-\alpha)}\big)\tilde{u} = 0, \tag{23}$$

in the particular case $\alpha = 1$, we get

$$\frac{1}{\sin\theta} \frac{d}{d\theta}(\sin\theta \frac{d\tilde{u}}{d\theta}) + s\,(s+1)\tilde{u} = 0, \tag{24}$$

see Lomen [7]. Using the change $x = \cos\theta$, Equation (23) becomes

$$\frac{d}{dx}\big((1-x^2)\frac{d\tilde{u}}{dx}\big) + l\,\big(\,l+1\big)\tilde{u} = 0, \tag{25}$$

where $l = l(s,\alpha) = \frac{\Gamma(1-s)}{\Gamma(1-s-\alpha)}$. The solution of (25) can be written in terms of Legendre's functions,

$$\tilde{u}(s,\theta) = A(s,\alpha)\,P_l(\cos\theta) + B(s,\alpha)\,Q_l(\cos\theta). \tag{26}$$

In order that the solution be bounded for $\theta = 0$, let $B = 0$. Applying the boundary condition gives

$$A(s,\alpha) = \frac{\tilde{f}(s)}{P_l(\cos\phi)}, \tag{27}$$

taking the inverse Mellin's transform, we get

$$u(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-s}\frac{\tilde{f}(s)}{P_l(\cos\phi)} P_l(\cos\theta) ds. \tag{28}$$

The Legendre's functions $P_l(\cos\phi)$ has an infinite number of real simple zeros for each choice of ϕ, whose numerical values are known only approximately. Except for very special cases the integral can only be evaluated numerically.

4 Conclusion

The solution of the fractional Laplace equation in plane polar and spherical coordinates is presented. The solutions were obtained using Mellin's transform. The solution generalized the results obtained by Lomen (1962) ([7]). Moreover, the derived solution in both cases is represented interms of an orthogonal and complete set. It is worth noticing that Mellin's transform can be applied to the fractional Poisson and fractional biharmonic equations in polar coordinates.

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