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General Exponentially Preinvex Functions And Their Properties

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Abstract

In this paper, we introduce some new concepts of the general exponentially preinvex functions. We investigate several properties of the general exponentially preinvex functions and discuss their relations with convex functions. Several interesting results characterizing the general exponentially preinvex functions are obtained. Optimality conditions are characterized by a class of variational-like inequalities, which are called general exponentially variational-like inequalities. Results obtained in this paper can be viewed as significant improvement of previously known results.

1 Introduction

Convex functions and convex sets have played an important and fundamental part in the development of various fields of pure and applied sciences. Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. Bernstein [8] introduced and considered the concept of exponentially convex(concave) functions. Exponentially convex(concave) functions can be considered as a significant extension of the convex functions. Avriel [4, 5] introduced and studied the concept of r-convex functions, which have important applications in information theory, big data analysis, machine learning and statistic. See, for example, [1, 2, 3, 4, 5, 10, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 35] and the references. Antczak [3] introduced these exponentially convex and preinvex functions implicitly and discuss their role in mathematical programming. Pal and Wong [34] have discussed its role in information geometry and statistics. Alirazaie and Mathur [2], Dragomir and Gomm [10] and Noor and Noor [21, 22, 23, 24, 25, 26, 27, 28] have derived several results for these exponentially convex and exponentially preinvex functions.

Hanson [12] studied the concept of invex functions involving an arbitrary bifunction to consider the mathematical programming problems. Ben-Israel and Mond [7] introduced the invex sets and preinvex functions involving the bifunction, which can be viewed as an important contribution in the field of optimization. They proved that the differentiable preinvex functions imply the invex function, but the converse is not true in general. Mohen and Neogy [13] showed that the differentiable preinvex and invex functions are equivalent under suitable conditione. Noor [15] proved that the optimality conditions of the differentiable preinvex functions can be characterized by a class of variational inequalities, which is called variational-like inequality. For the applications, formulation and other aspects of variational-like inequalities and related equilibrium-like problems, see [15, 16, 17, 18, 19, 22, 23, 29].

Motivated and inspired by the ongoing research in this interesting, applicable and dynamic field, we introduce some new classes of the general exponentially preinvex functions. It has been shown that the general exponentially convex(concave) have nice properties which convex functions enjoy. Several new concepts have been introduced and investigated. We show that the local minimum of the general exponentially preinvex functions is the global minimum. The optimal conditions of the differentiable general exponentially preinvex functions can be characterized by a class of variational inequalities, which is called general exponentially variational-like inequality, which is itself an interesting outcome of our main results. The difference (sum) of the general exponentially preinvex functions and general exponentially affine preinvex functions is again

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a general exponentially convex function. The ideas and techniques of this paper may be starting point for further research in these areas.

2 Preliminaries

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ by the inner product and norm, respectively. Let $F : K \to R$ be a continuous function.

Definition 1 **([7])** *The set K_η in H is said to be an invex set, if there exists a bifunction $\eta(.,.)$, such that*

$$u + t\eta(v,u) \in K, \quad \forall u, v \in K_\eta,\ t \in [0,1].$$

Definition 2 **([7])** *A function F on the invex set K_η in H is said to be a preinvex function, if there exists a bifunction $\eta(.,.)$, such that*

$$F(u + t\eta(v,u)) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K_\eta,\ t \in [0,1].$$

We now define the concept of general exponentially preinvex functions and their variant forms.

Definition 3 *A function F is said to be a general exponentially preinvex with respect to an arbitrary bifunction $\eta(v,u)$, if*

$$s^{F(u+t\eta(v,u))} \leq (1-t)s^{F(u)} + ts^{F(v)}, \quad \forall u, v \in K_\eta,\ t \in [0,1],\ s > 1,$$

which is equivalent to the following

Definition 4 *A function F is said to be exponentially preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if*

$$F(u + t\eta(v,u)) \leq \frac{1}{\ln s} \ln\{(1-t)s^{F(u)} + ts^{F(v)}\}, \quad \forall u, v \in K_\eta,\ t \in [0,1],$$

For $t = 1$, the Definition 3 reduces to

$$s^{F(u+\eta(v,u))} \leq s^{F(v)}, \quad \forall u, v \in K_\eta, \quad t \in [0,1],\ s > 1,$$

which is known as Condition A.

A function is called the general exponentially preincave function F, if and only if, $-F$ is general exponentially preinvex function.

We remark that if $\eta(v,u) = v - u$, then the invex set $K_\eta = K$, becomes the convex set and Definition (3) reduces to:

Definition 5 *A function F is said to be general exponentially convex function, if*

$$s^{F(u+t(v-u))} \leq (1-t)s^{F(u)} + ts^{F(v)}, \quad \forall u, v \in K,\ t \in [0,1],\ s > 1,$$

which was introduced and studied by Noor and Noor [28, 29].

If $s = exp$, then Definition (3) reduces to the following concepts, which are due to Antczak [3].

Definition 6 *A function F is said to be a general exponentially preinvex with respect to an arbitrary bifunction $\eta(v,u)$, if*

$$e^{F(u+t\eta(v,u))} \leq (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K,\ t \in [0,1],\ s > 1.$$

This is equivalent to the following

Definition 7 ([3]) *A function F is said to be general exponentially preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if*

$$F(u+t\eta(v,u)) \leq \ln\{(1-t)e^{F(u)} + te^{F(v)}\}, \quad \forall u,v \in K_\eta,\ t \in [0,1].$$

If $s = exp$ and $\eta(v,u) = v-u$, then Definition (3) reduces to the following concepts, which are due to Antczak [3].

Definition 8 *A function F is said to be a exponentially convex, if*

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u,v \in K,\ t \in [0,1].$$

This is equivalent to the following

Definition 9 ([4]) *A function F is said to be exponentially convex, if*

$$F(u+t(v-u)) \leq \ln\{(1-t)e^{F(u)} + te^{F(v)}\}, \quad \forall u,v \in K,\ t \in [0,1],$$

For the applications of the exponentially convex(concave) functions in the mathematical programming and information theory, see Antczak [3] and Alirezaei and Mathar[2].

Example 1 ([2]) *The error function*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

becomes an exponentially concave function in the form $erf(\sqrt{x})$, $\quad x \geq 0$, which describes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This shows that the exponentially concave functions can play important part in communication theory and information theory.

For the properties of differentiable exponential preinvex and convex functions, see Noor and Noor [21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

Definition 10 *The function F on the invex set K_η is said to be general exponentially quasi preinvex with respect to an arbitrary bifunction $\eta(v,u)$, if*

$$s^{F(u+t\eta(v,u))} \leq \max\{s^{F(u)}, s^{F(v)}\}, \quad \forall u,v \in K_\eta,\ t \in [0,1],\ s > 1.$$

Definition 11 *A function F is said to be general* log*-preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if*

$$s^{F(u+t\eta(v,u))} \leq (s^{F(v)})^{1-t}(s^{F(v)})^t, \quad \forall u,v \in K_\eta,\ t \in [0,1],\ s > 1.$$

From the above definitions, we have

$$s^{F(u+t\eta(v,u))} \leq (s^{(F(u)})^{1-t}(s^{F(v)})^t \leq (1-t)s^{F(u)} + ts^{F(v)}) \leq \max\{s^{F(u)}, s^{F(v)}\}.$$

This shows that every general exponentially log-preinvex function is a general exponentially convex function and every general exponentially convex function is a general exponentially quasi-preinvex function. However, the converse is not true.

Let $K_\eta = I_\eta = [a, a+\eta(b,a)]$ be the interval. We now define the general exponentially preinvex function on I_η.

Definition 12 *Let $I_\eta = [a, a+\eta(b,a)]$. Then F is a general exponentially preinvex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a+\eta(b,a) \\ s^{F(a)} & s^{F(x)} & s^{F(b)} \end{vmatrix} \geq 0; \quad a \leq x \leq a+\eta(b,a).$$

One can easily show that the following are equivalent:

1. F is a general exponentially preinvex function.
2. $s^{F(x)} \leq s^{F(a)} + \frac{s^{F(b)} - s^{F(a)}}{\eta(b,a)}(a-x)$.
3. $\frac{s^{F(x)} - s^{F(a)}}{x-a} \leq \frac{s^{F(b)} - s^{F(a)}}{\eta(b,a)}$.
4. $(x-a)s^{F(b)} + \eta(b,a)s^{F(x)} + (a-x)s^{F(b)} \geq 0$.
5. $\frac{s^{F(a)}}{(\eta(b,a))(a-x)} + \frac{s^{F(x)}}{(x-a)-\eta(b,a))(a-x)} + \frac{s^{F(b)}}{\eta(b,a)(x-a-\eta(b,a))} \geq 0$,

where $x = a + t\eta(b,a) \in [0,1]$.

3 Main Results

In this section, we consider some basic properties of exponentially general preinvex functions.

Theorem 1 *Let F be a strictly general exponentially preinvex function. Then any local minimum of F is a global minimum.*

Proof. Let the exponentially preinvex function F have a local minimum at $u \in K_\eta$. Assume the contrary, that is, $F(v) < F(u)$ for some $v \in K_\eta$. Since F is a general exponentially preinvex function, we see that

$$s^{F(u+t\eta(v,u))} \leq (1-t)s^{F(u)} + ts^{F(v)}, \quad \text{for } 0 < t < 1.$$

Thus

$$s^{F(u+t\eta(v,u))} - s^{F(u)} < -t[s^{F(v)} - s^{F(u)}] < 0,$$

from which it follows that

$$s^{F(u+t\eta(v,u))} < s^{F(u)},$$

for arbitrary small $t > 0$, contradicting the local minimum. ∎

Theorem 2 *If the function F on the invex set K_η is exponentially preinvex, then the level set $L_\alpha = \{u \in K_\eta : s^{F(u)} \leq \alpha, \alpha \in R\}$ is an invex set.*

Proof. Let $u, v \in L_\alpha$. Then $s^{F(u)} \leq \alpha$ and $s^{F(v)} \leq \alpha$. Now, $\forall t \in (0,1)$, $v = u + t\eta(v,u) \in K_\eta$, since K_η is an invex set. Thus, by the general exponentially preinvexity of F, we have

$$s^{F(u+t\eta(v,u))} \leq (1-t)s^{F(u)} + ts^{F(v)} \leq (1-t)\alpha + t\alpha = \alpha,$$

from which it follows that $u + t\eta(v,u) \in L_\alpha$ Hence L_α is an invex set. ∎

Theorem 3 *The function F is a exponentially preinvex, if and only if,*

$$epi(F) = \{(u,a) : u \in K_\eta : s^{F(u)} \leq \alpha, \; \alpha \in R\}$$

is an invex set.

Proof. Assume that$F$ is general exponentially preinvex function. Let $(u,\alpha),(v,\beta)\in epi(F)$. Then it follows that $s^{F(u)}\leq\alpha$ and $s^{F(v)}\leq\beta$. Thus, $t\in[0,1]$, $\forall u,v\in K_\eta$, we have

$$s^{F(u+t\eta(v,u))}\leq(1-t)s^{F(u)}+ts^{F(v)}\leq(1-t)\alpha+t\beta,$$

which implies that $(u+t\eta(v,u),(1-t)\alpha+t\beta)\in epi(F)$. Thus $epi(F)$ is an invex set. Conversely, let $epi(F)$ be an invex set. Let $u,v\in K_\eta$. Then $(u,s^{F(u)})\in epi(F)$ and $(v,s^{F(v)})\in epi(F)$. Since $epi(F)$ is an invex set, we must have

$$(u+t\eta(v,u),(1-t)s^{F(u)}+ts^{F(v)}\in epi(F),$$

which implies that

$$s^{F(u+t\eta(v,u)}\leq(1-t)s^{F(u)}+ts^{F(v)}.$$

This shows that F is a general exponentially preinvex function. ■

Theorem 4 *The function F is general exponentially quasi preinvex function, if and only if, the level set $L_\alpha=\{u\in K_\eta,\alpha\in R:s^{F(u)}\leq\alpha\}$ is an invex set.*

Proof. Let $u,v\in L_\alpha$. Then $u,v\in K_\eta$ and $\max\{s^{F(u)},s^{F(v)}\}\leq\alpha$. Now for $t\in(0,1)$, $w=u+t\eta(v,u)\in K_\eta$, by the invexity of K_η. We have to prove that $u+t\eta(v,u)\in L_\alpha$. By the general exponentially preinvexity of F, we have

$$s^{F(u+t\eta(v,u))}\leq\max\{(s^{F(u)},s^{F(v)}\}\leq\alpha,$$

which implies that $u+t\eta(v,u)\in L_\alpha$, showing that the level set L_α is indeed an invex set.

Conversely, assume that L_α is an invex set. Then, for any $u,v\in L_\alpha$, $t\in[0,1]$, $u+t\eta(v,u)\in L_\alpha$. Let $u,v\in L_\alpha$ for $\alpha=\max s^{F(u)},s^{F(v)}$ and $s^{F(v)}\leq s^{F(u)}$. Then from the definition of the level set L_α, it follows that

$$s^{F(u+t\eta(v,u))}\leq\max(s^{F(u)},s^{F(v)})\leq\alpha.$$

Thus F is a general exponentially quasi preinvex function. This completes the proof. ■

Theorem 5 *Let F be a general exponentially preinvex function. Let $\mu=\inf_{u\in K_\eta}F(u)$. Then the set $E=\{u\in K_\eta:s^{F(u)}=\mu\}$ is an invex set K_η. If F is strictly exponentially preinvex function, then E is a singleton.*

Proof. Let $u,v\in E$. For $0<t<1$, let $w=u+t\eta(v,u)$. Since F is a general exponentially preinvex function, we see that

$$F(w)=s^{F(u+t\eta(v,u))}\leq(1-t)s^{F(u)}+ts^{F(v)}=t\mu+(1-t)\mu=\mu,$$

which implies that $w\in E$. and hence E is an invex set. For the second part, assume to the contrary that $F(u)=F(v)=\mu$. Since K_η is an invex set, we see that for $0<t<1$, $u+t\eta(v,u)\in K_\eta$. Further, since F is strictly exponentially preinvex function,

$$s^{F(u+t\eta(v,u))}<(1-t)s^{F(u)}+ts^{F(v)}=(1-t)\mu+t\mu=\mu.$$

This contradicts the fact that $\mu=\inf_{u\in K_\eta}F(u)$ and hence the result follows. ■

Theorem 6 *If F is a general exponentially preinvex function such that*

$$s^{F(v)}<s^{F(u)},\quad\forall u,v\in K_\eta,$$

then F is a strictly general exponentially quasi preinvex function.

Proof. By the general exponentially preinvexity of F, we have

$$s^{F(u+t\eta(v,u))} \leq (1-t)s^{F(u)} + ts^{F(v)} < s^{F(u)}, \quad \forall u,v \in K_\eta, \ t \in [0,1],$$

since $s^{F(v)} < s^{F(u)}$, which shows that the function F is a strictly general exponentially quasi preinvex function. ■

We now discuss the properties of the differentiable general preinvex functions. For this, we need the following.

Condition C ([13]) Let $\eta(.,.) : K_\eta \times K_\eta \to H$ satisfy the following assumption

$$\eta(u, u + t\eta(v,u)) = -t\eta(v,u)),$$

$$\eta(v, u + t\eta(v,u))) = (1-t)\eta(v,u), \quad \forall u,v \in K_\eta, \ t \in [0,1].$$

Remark 1 *It is worth mentioning that for $t = 0$, we have $\eta(u,u) = 0$. In particular, it follow that $\eta(v,u) = 0$, if and only if, $v = u$. Also one can show that $\eta(v,u) = -\eta(u,v)$, $\forall u,v \in K_\eta$. That is the bifunction $\eta(.,.)$ is skew symmetric.*

Theorem 7 *Let F be a differentiable function and Condition C hold. Then the function F is a general exponentially preinvex function, if and only if,*

$$s^{F(v)} - s^{F(u)} \geq \langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle, \quad \forall v,u \in K_\eta. \tag{1}$$

Proof. Let F be a general exponentially preinvex function. Then

$$s^{F(u+t\eta(v,u))} \leq (1-t)s^{F(u)} + ts^{F(v)}, \quad \forall u,v \in K_\eta,$$

which can be written as

$$s^{F(v)} - s^{F(u)} \geq \{\frac{s^{F(u+t\eta(v,u))} - s^{F(u)}}{t}\}.$$

Taking the limit in the above inequality as $t \to 0$, we have

$$s^{F(v)} - s^{F(u)} \geq \langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle,$$

which is (1), the required result.

Conversely, let (1) hold. Then

$$\forall u,v \in K_\eta, t \in [0,1], \quad v_t = u + t\eta(v,u) \in K_\eta.$$

Using Condition C, we have

$$s^{F(v)} - s^{F(v_t)} \geq \langle s^{F(v_t)}F'(v_t)\ln s, \eta(v,v_t)\rangle = (1-t)\langle s^{F(v_t)}F'(v_t)\ln s, \eta(v,u))\rangle. \tag{2}$$

In a similar way, we have

$$s^{F(u)} - s^{F(v_t)} \geq \langle s^{F(v_t)}F'(v_t)\ln s, \eta(u,v_t)\rangle = -t\langle s^{F(v_t)}F'(v_t)\ln s, \eta(v,u)\rangle. \tag{3}$$

Multiplying (2) by t and (3) by $(1-t)$ and adding the resultant, we have

$$s^{F(u+t\eta(v,u)} \leq (1-t)s^{F(u)} + ts^{F(v)},$$

showing that F is a general exponentially preinvex function. ■

Theorem 7 enables us to introduce the concept of the exponentially η-monotone operators, which appears to be new.

Definition 13 *The differential $F'(.)$ is said to be exponentially η-monotone, if*

$$\langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle + \langle s^{F(v)}F'(v)\ln s, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in H.$$

Definition 14 *The differential $F'(.)$ is said to be exponentially pseudo η-monotone, if*

$$\langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle \geq 0, \quad \Rightarrow -\langle s^{F(v)}F'(v)\ln s, \eta(v,u)\rangle \geq 0, \quad \forall u, v \in H.$$

From these definitions, it follows that exponentially η-monotonicity implies exponentially pseudo η-monotonicity, but the converse is not true.

Theorem 8 *Let F be a differentiable general exponentially preinvex function on the invex set K_η and Condition C hold. Then*

$$s^{F(v)} - s^{F(u)} \geq \langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle, \quad \forall v, u \in K_\eta. \tag{4}$$

if and only if, F' satisfies

$$\langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle + \langle s^{F(v)}F'(v)\ln s, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in K_\eta. \tag{5}$$

Proof. Let F be a general exponentially preinvex function on the invex set K_η. Then, from Theorem 7, we have

$$s^{F(v)} - s^{F(u)} \geq \langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle, \quad \forall u, v \in K_\eta. \tag{6}$$

Changing the role of u and v in (6), we have

$$s^{F(u)} - s^{F(v)} \geq \langle s^{F(v)}F'(v)\ln s, \eta(u,v)\rangle, \quad \forall u, v \in K_\eta. \tag{7}$$

Adding (6) and (7), we have

$$\langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle + \langle s^{F(v)}F'(v)\ln s, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in K_\eta,$$

which shows that F' is a exponentially η-monotone operator.

Conversely, from (5), we have

$$\langle s^{F(v)}F'(v)\ln s, \eta(v,u)\rangle \leq -\langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle. \tag{8}$$

Since K_η is an invex set, $\forall u, v \in K_\eta$, $t \in [0,1]$, $v_t = u + t\eta(v,u) \in K_\eta$. Taking $v = v_t$ in (8), we have

$$\langle s^{F(v_t)}F'(v_t)\ln s, \eta(u,v_t)\rangle \leq \langle -s^{F(u)}F'(u)\ln s, \eta(v_t,u)\rangle,$$

which implies, using the Condition C, that

$$\langle s^{F(v_t)}F'(v_t)\ln s, \eta(v,u)\rangle \geq \langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle. \tag{9}$$

Consider the auxiliary function

$$\xi(t) = s^{F(u+t\eta(v,u))},$$

from which, we have

$$\xi(1) = s^{F(u+\eta(v,u))}, \quad \xi(0) = s^{F(u)}.$$

Then, from (9), we have

$$\xi'(t) = \langle s^{F(v_t)}F'(v_t)\ln s, \eta(v,u)\rangle \geq \langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle. \tag{10}$$

Integrating (10) between 0 and 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq \langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle.$$

Thus it follows using the fact $e^{F(u+\eta(v,u))} \leq s^{F(v)}$, that

$$s^{F(v)} - s^{F(u)} \geq \langle s^{F(u)}F'(u)\ln s, \eta(v,u)\rangle,$$

which is the required (4). ■

We now give a necessary condition for general exponentially pseudo-preinvex function.

Theorem 9 *Let F' be a general exponentially pseudomonotone. Then F is a general exponentially pseudo-invex function.*

Proof. Let F' be a general exponentially pseudomonotone operator. Then, $\forall u, v \in K_\eta$,

$$\langle s^{F(u)} F'(u) \ln s, \eta(v, u)\rangle \geq 0$$

which implies that

$$-\langle s^{F(v)} F'(v) \ln s, \eta(v, u)\rangle \geq 0. \tag{11}$$

Since K is an invex set, $\forall u, v \in K_\eta$, $t \in [0,1]$, $v_t = u + t\eta(v, u) \in K_\eta$.

Taking $v = v_t$ in (11), we have

$$\langle s^{F(v_t)} F'(v_t), \eta(v, u)\rangle \geq 0. \tag{12}$$

Consider the auxiliary function

$$\xi(t) = s^{F(u+t\eta(v,u))} = s^{F(v_t)}, \quad \forall u, v \in K_\eta, \ t \in [0,1],$$

which is differentiable, Then, using (12), we have

$$\xi'^{F(v_t)} F'(v_t) \ln s, \eta(v, u)\rangle \geq 0.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t) dt \geq 0,$$

that is,

$$s^{F(v)} - s^{F(u)} \geq s^{F(u+\eta(v,u))} - s^{F(u)} \geq 0,$$

showing that F is a general exponentially pseudo-invex function. ■

Definition 15 *The function F is said to be sharply general exponentially pseudo invex if*

$$\langle s^{F(u)} F'(u) \ln s, \eta(v, u)\rangle \geq 0 \Rightarrow F(v) \geq s^{F(v+t(u-v))}, \quad \forall u, v \in K_\eta, \ t \in [0,1].$$

Theorem 10 *Let F be a sharply general exponentially pseudo invex function on K_η. Then*

$$\langle s^{F(v)} F'(v), v - u\rangle \geq 0, \quad \forall u, v \in K_\eta.$$

Proof. Let F be a sharply general exponentially pseudo invex function on K_η, Then

$$s^{F(v)} \geq s^{F(v+t\eta(u,v))}, \quad \forall u, v \in K_\eta, \ t \in [0,1].$$

from which we have

$$0 \leq \lim_{t\to 0}\{\frac{s^{F(v+t\eta(v,u)))} - s^{F(v)}}{t}\} = \langle s^{F(v)} F'(v) \ln s, \eta(v, u)\rangle,$$

which is the required result. ■

Definition 16 *A function F is said to be a general exponentially pseudo preinvex function, if there exists a strictly positive bifunction $b(.,.)$, such that if $s^{F(v)} < s^{F(u)}$, then*

$$s^{F(u+t\eta(v,u))} < s^{F(u)} + t(t-1)b(v, u), \quad \forall u, v \in K_\eta, \ t \in [0,1].$$

Theorem 11 *If the function F is a general exponentially preinvex function such that $s^{F(v)} < s^{F(u)}$, then the function F is a general exponentially pseudo preinvex.*

Proof. Since $s^{F(v)} < s^{F(u)}$ and F is the general exponentially preinvex function, we see that $\forall u, v \in K_\eta$, $t \in [0,1]$, we have

$$\begin{aligned} s^{F(u+t\eta(v,u)))} &\leq s^{F(u)} + t(s^{F(v)} - s^{F(u)}) \\ &< s^{F(u)} + t(1-t)(s^{F(v)} - s^{F(u})) \\ &= s^{F(u)} + t(t-1)(s^{F(u)} - s^{F(v)})) \\ &< s^{F(u)} + t(t-1)b(u,v), \end{aligned}$$

where $b(u,v) = s^{F(u)} - s^{F(v)} > 0$, the required result. This shows that the function F is a general exponentially pseudo preinvex function. ∎

We now discuss the optimality condition for the differentiable general exponentially preinvex functions, which is the main motivation of our next result.

Theorem 12 *Let F be a differentiable general exponentially preinvex function. Then $u \in K_\eta$ is the minimum of the function F, if and only if, $u \in K_\eta$ satisfies the inequality*

$$\langle s^{F(u)} F'(u) \ln s, \eta(v,u) \rangle \geq 0, \quad \forall u, v \in K_\eta. \tag{13}$$

Proof. Let $u \in K_\eta$ be a minimum of the function F. Then

$$F(u) \leq F(v), \quad \forall v \in K_\eta.$$

from which, we have

$$s^{F(u)} \leq s^{F(v)}, \quad \forall v \in K_\eta. \tag{14}$$

Since K_η is an invex set, we see that $\forall u, v \in K_\eta$, $t \in [0,1]$,

$$v_t = u + t\eta(v,u) \in K_\eta.$$

Taking $v = v_t$ in (14), we have

$$0 \leq \lim_{t \to 0} \{ \frac{s^{F(u+t\eta(v,u))} - s^{F(u)}}{t} \} = \langle s^{F(u)} F'(u) \ln s, \eta(v,u) \rangle. \tag{15}$$

Since F is differentiable general exponentially preinvex function, we see that

$$s^{F(u+t\eta(v,u))} \leq s^{F(u)} + t(s^{F(v)} - s^{F(u)}), \quad u, v \in K_\eta, \ t \in [0,1],$$

from which, using (15), we have

$$s^{F(v)} - s^{F(u)} \geq \lim_{t \to 0} \{ \frac{s^{F(u+t\eta(v,u))} - s^{F(u)}}{t} \} = \langle s^{F(u)} F'(u) \ln s, \eta(v,u) \rangle \geq 0,$$

from which , we have

$$s^{F(v)} - s^{F(u)} \geq 0,$$

which implies that

$$F(u) \leq F(v), \quad \forall v \in K_\eta.$$

This shows that $u \in K_\eta$ is the minimum of the differentiable general exponentially preinvex function the required result. ∎

Remark 2 *The inequality of the type (13) is called the general exponentially variational-like inequality, which appears to be a new one. It is an interesting problem to investigate the existence of uniqueness solution of the inequality (13) and its various properties.*

We now show that the difference of general exponentially preinvex functions and general exponentially affine preinvex functions is again a general exponentially preinvex function.

Theorem 13 *Let f be a general exponentially affine preinvex function. Then F is a general exponentially preinvex function, if and only if, $H = F - f$ is a general exponentially preinvex function.*

Proof. Let f be general exponentially affine preinvex function. Then

$$s^{f(u+t\eta(v,u))} = (1-t)s^{f(u)} + ts^{f(v)}, \quad \forall u, v \in K_\eta,\ t \in [0,1]. \tag{16}$$

From the general exponentially preinvexity of F, we have

$$s^{F(u+t\eta(v,u))} \leq (1-t)s^{F(u)} + ts^{F(v)}, \quad \forall u, v \in K_\eta,\ t \in [0,1]. \tag{17}$$

From (16) and (17), we have

$$s^{F((u+t\eta(v,u))} - s^{f((u+t\eta(v,u))} \leq (1-t)(s^{F(u)} - s^{f(u)}) + t(s^{F(v)} - s^{f(v)}),$$

from which, it follows that

$$\begin{aligned} s^{H((u+t\eta(v,u))} &= s^{F((u+t\eta(v,u))} - s^{f((1-t)f(u+t\eta(v,u))} \\ &\leq (1-t)(s^{F(u)} - s^{f(u)}) + t(s^{F(v)} - s^{f(v)}), \end{aligned}$$

which show that $H = F - f$ is a general exponentially preinvex function.

The inverse implication is obvious. ■

Conclusion

In this paper, we have introduced and studied a new class of preinvex functions which is called the general exponentially preinvex function. It has been shown that general exponentially preinvex functions enjoy several properties which convex functions have. Several new properties of the general exponentially preinvex functions have been established. We have proved that the minimum of the general exponentially differentiable preinvex functions can be characterized by a new class of variational inequalities, which is called the general exponentially variational-like inequalities. It is an interesting problem to investigate various properties of general exponentially variational-like inequalities. Results in this paper can be viewed as significant and important improvements of the known results.

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