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Positive Solutions With Exponential Decay For The Singular Fisher-Like Equation Posed On The Real-Line*

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Abstract

In this article, we are concerned with the existence of positive solutions to the boundary value problem,

$$\begin{cases} -u'' + cu' + \lambda u = F(t, u(t)), \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} e^{k|t|}u(t) = \lim_{t\to+\infty} e^{l|t|}u(t) = 0, \end{cases}$$

where λ, c are positive constants, $k, l \in \mathbb{R}$ and $F : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^+$ is a continuous function. The main existence result is proved by means of Guo-Krasnoselskii's version of expansion and compression of a cone principle in a Banach space.

1 Introduction and Main Results

This article deals with the existence of positive solutions to the boundary value problem (bvp for short),

$$\begin{cases} -u'' + cu' + \lambda u = F(t, u), \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} e^{k|t|}u(t) = \lim_{t\to+\infty} e^{l|t|}u(t) = 0, \end{cases} \tag{1}$$

where λ, c are positive constants, $k, l \in \mathbb{R}$ and $F : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^+$ is a continuous function.

By positive solution to the bvp (1), we mean a function $u \in C^2(\mathbb{R})$ such that $u(t) > 0$ for all $t \in \mathbb{R}$ and $\lim_{t\to-\infty} e^{k|t|}u(t) = \lim_{t\to+\infty} e^{l|t|}u(t) = 0$, satisfying the ordinary differential equation in (1). Imposing $k, l > 0$ in the boundary conditions in (1) means that we look for solutions having an exponential decay at $\pm\infty$.

The positivity of the solution u is required here since the bvp (1) arises in the modeling of the propagation of wave fronts in combustion theory and epidemiology, see [7, 2], where u stands to be a concentration or a density. The positive constants c and λ refer respectively to the wave speed of the front and to the removal rate. The case where the bvp (1) is autonomous, that is $F(t, u(t)) = F(u)$, with F having a prescribed form corresponds to the generalized Fisher's equation.

There are many papers in the literature considering the case of the bvp (1.1) posed on the half-line, see [1, 4, 5, 6, 8, 9, 10] and references therein. However, to the author's knowledge, there are no paper in the literature considering the singular case posed on the whole real-line and so, the purpose of this paper is to fill in the gap in this area.

Our approach in this work is based on a fixed point formulation and since the nonlinearity F is supposed to be nonnegative, we will use the Guo-Krasnoselskii's version of expansion and compression of a cone principle to prove our main existence result.

In all this paper, we assume that there exist two continuous functions $\phi : \mathbb{R} \to \mathbb{R}^+$ and $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^+$ such that

$$F(t, u) = \phi(t)f(t, u), \tag{2}$$

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$$\left\{\begin{array}{l}\text{for all } \rho>0 \text{ there exists a nonincreasing function}\\ \Psi_\rho:(0,+\infty)\rightarrow(0,+\infty) \text{ such that}\\ f(t,\frac{w}{p(t)})\leq\Psi_\rho\left(w\right) \text{ for all } t\in\mathbb{R} \text{ and all } w\in(0,\rho]\,,\\ \lim\limits_{t\rightarrow-\infty}q_-(t)\phi\left(t\right)\Psi_\rho\left(r\gamma(t)\right)=\lim\limits_{t\rightarrow+\infty}q_+(t)\phi\left(t\right)\Psi_\rho\left(r\gamma(t)\right)=0 \text{ and}\\ \int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)\Psi_\rho\left(r\gamma(s)\right)ds<\infty \text{ for all } r\in(0,\rho]\,,\end{array}\right. \tag{3}$$

where

$$p(t)=e^{-r_2|t|},$$
$$q_-(t)=\max\left(p(t),e^{k|t|}\right),$$
$$q_+(t)=\max\left(p(t),e^{l|t|}\right),$$
$$\gamma(t)=\min(e^{2r_2t},e^{(r_1-r_2)t}),$$
$$\widetilde{\gamma}\left(t\right)=\frac{\gamma(t)}{p(t)}=\min\left(e^{r_1t},e^{r_2t}\right),$$
$$\delta\left(t\right)=\min\left(e^{-r_1t},e^{-r_2t}\right)=\left(\max\left(e^{r_1t},e^{r_2t}\right)\right)^{-1},$$

r_1 and r_2 are the solutions of the characteristic equation $-X^2+cX+\lambda=0$ with $r_1<0<r_2$.

Remark 1 *Notice that Hypothesis (3) implies that* $\int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)ds<\infty$. *Indeed, for* $\rho=1$ *we have*

$$\begin{aligned}\infty &> \int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)\Psi_1\left(r\gamma(s)\right)ds\geq\Psi_1\left(\sup_{s\in\mathbb{R}}\gamma(s)\right)\int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)ds\\ &= \Psi_1\left(1\right)\int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)ds.\end{aligned}$$

Remark 2 *Notice that in the case where* $\min(k,l)\geq0$, *we have* $q_-(t)=e^{k|t|}$ *and* $q_+(t)=e^{l|t|}$. *Therefore,* $\lim\limits_{t\rightarrow-\infty}e^{k|t|}\phi\left(t\right)\Psi_\rho\left(r\gamma(t)\right)=\lim\limits_{t\rightarrow+\infty}e^{l|t|}\phi\left(t\right)\Psi_\rho\left(r\gamma(t)\right)=0$ *implies that* $\int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)\Psi_\rho\left(r\gamma(s)\right))ds<\infty$ *and Hypothesis (3) can be relaxed to*

$$\left\{\begin{array}{l}\textit{for all } \rho>0 \textit{ there exists a nonincreasing function}\\ \Psi_\rho:(0,+\infty)\rightarrow(0,+\infty) \textit{ such that}\\ f(t,\frac{w}{p(t)})\leq\Psi_\rho\left(w\right) \textit{ for all } t\in\mathbb{R} \textit{ and all } w\in(0,\rho]\,,\\ \lim\limits_{t\rightarrow-\infty}e^{k|t|}\phi\left(t\right)\Psi_\rho\left(r\gamma(t)\right)=\lim\limits_{t\rightarrow+\infty}e^{l|t|}\phi\left(t\right)\Psi_\rho\left(r\gamma(t)\right)=0 \textit{ for all } r\in(0,\rho]\,.\end{array}\right.$$

Remark 3 *Hypothesis (3) covers the case of the bvp (1) where the nonlinearity* F *satisfies the polynomial growth condition*

$$F(t,u)\leq a(t)+b(t)u^\sigma,$$

where $\sigma\geq0$ *and* $a,b\in C\left(\mathbb{R}\right)$ *are such that*

$$\left\{\begin{array}{l}\lim\limits_{t\rightarrow\nu\infty}q_\nu(t)a(t)=\lim\limits_{t\rightarrow\nu\infty}q_\nu(t)b(t)\left(p(t)\right)^{-\sigma}=0 \textit{ for } \nu=+ \textit{ or } -\\ \textit{and } \delta a,\delta bp^{-\sigma}\in L^1\left(\mathbb{R}\right).\end{array}\right.$$

To see that, take $\phi\left(t\right)=\max\left(a(t),b(t)\left(p(t)\right)^{-\sigma}\right)$ *and for* $\rho>0$, $\Psi_\rho(r)=1+\rho^\sigma$.

Let $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be the function defined by

$$G(t,s) = \frac{1}{r_2 - r_1} \begin{cases} \exp(r_1(t-s)), & \text{if } s \le t, \\ \exp(r_2(t-s)), & \text{if } t \le s. \end{cases} \tag{4}$$

Simple computations yield

$$0 < G(t,s) \le \frac{1}{r_2 - r_1} \text{ for all } t, s \in \mathbb{R}$$

and

$$G(t,s) \le \frac{\delta(s)}{\delta(t)} \text{ for all } s, t \in \mathbb{R}. \tag{5}$$

Because of (5) and Hypothesis (3) (see Remark 1), for all $\theta > 0$ we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} \left(p(t) \int_{-\theta}^{\theta} G(t,s)\phi(s)\gamma(s)ds \right) &\le \sup_{t\in\mathbb{R}} \left(p(t) \int_{-\infty}^{+\infty} G(t,s)\phi(s)ds \right) \\ &\le \sup_{t\in\mathbb{R}} \left(\frac{p(t)}{(r_2 - r_1)\,\delta(t)} \int_{-\infty}^{+\infty} \delta(s)\phi(s)ds \right) \\ &\le \frac{1}{(r_2 - r_1)} \int_{-\infty}^{+\infty} \delta(s)\phi(s)ds < \infty. \end{aligned}$$

Hence we set

$$\Gamma = \sup_{t\in\mathbb{R}} \left(p(t) \int_{-\infty}^{+\infty} G(t,s)\phi(s)ds \right),$$

$$\Theta(\theta) = \sup_{t\in\mathbb{R}} \left(p(t) \int_{-\theta}^{\theta} G(t,s)\phi(s)\gamma(s)ds \right)$$

for $\theta > 0$. The following theorem is the main result of this work. Its statement needs the introduction of the following notations. Let

$$f^0 = \lim_{\substack{t\to-\infty \\ w\to 0}} \sup \left(\sup_{t\in\mathbb{R}} \frac{f(t, \frac{w}{p(t)})}{w} \right), \qquad f^\infty = \lim_{\substack{t\to-\infty \\ w\to +\infty}} \sup \left(\sup_{t\in\mathbb{R}} \frac{f(t, \frac{w}{p(t)})}{w} \right),$$

$$f_0(\theta) = \lim_{\substack{t\to-\infty \\ w\to 0}} \inf \left(\min_{t\in I_\theta} \frac{f(t, \frac{w}{p(t)})}{w} \right), \qquad f_\infty(\theta) = \lim_{\substack{t\to-\infty \\ w\to +\infty}} \inf \left(\min_{t\in I_\theta} \frac{f(t, \frac{w}{p(t)})}{w} \right)$$

where for $\theta > 0$, $I_\theta = [-\theta, \theta]$.

Theorem 1 *Assume that Hypotheses (2) and (3) hold, $k < -r_1$, $l < r_2$ and there exists $\theta > 0$ such that one of the following situations (6) and (7) holds.*

$$f^0\Gamma < 1 < f_\infty(\theta)\,\Theta(\theta), \tag{6}$$

$$f^\infty\Gamma < 1 < f_0(\theta)\,\Theta(\theta). \tag{7}$$

Then the bvp (1) admits at least one positive solution.

We deduce from Theorem 1 the following existence result for positive solutions for the typical case of the bvp (1) where $F(t,u) = a(t)\,u^\mu$ with $\mu \in \mathbb{R}\backslash\{1\}$ and $a \in C(\mathbb{R})$.

Corollary 1 *Assume that* $k < -r_1$, $l < r_2$ *and*

$$\left\{ \begin{array}{l} F(t,u) = a\,(t)\,u^{\mu} \text{ with } \mu \neq 1,\ a \in C\,(\mathbb{R})\,, \\ \lim\limits_{t\to\nu\infty} q_{\nu}(t)a(t)p^{-\mu}(t)\max\left(1,\gamma^{\mu}\left(t\right)\right) = 0 \text{ for } \nu = + \text{ or } - \\ \text{and } \int_{-\infty}^{+\infty}\delta(s)a(s)p^{-\mu}(t)\max\left(1,\gamma^{\mu}\left(t\right)\right)ds < \infty. \end{array} \right.$$

Then the bvp (1) admits a positive solution.

Proof. We have that $F(t,u) = \phi\,(t)\,f(t,u)$ with $\phi\,(t) = a(s)\,(p(s))^{-\mu}$ and $f(t,u) = (p(t)u)^{\mu}$. We have to show that all hypotheses of Theorem 1 are fulfilled.

For all $\rho > 0$ and $w \in (0,\rho]$, we have

$$f\left(t, \frac{w}{p(t)}\right) = w^{\mu} \leq \Psi_{\rho}(w) = \left\{ \begin{array}{l} \rho^{\mu},\ \text{if } \mu \geq 0, \\ w^{\mu},\ \text{if } \mu < 0 \end{array} \right.$$

and

$$\Psi_{\rho}(r\gamma\,(t)) = \left\{ \begin{array}{l} \rho^{\mu},\ \text{if } \mu \geq 0, \\ r^{\mu}\gamma^{\mu}\,(t)\,,\ \text{if } \mu < 0 \end{array} \right. = \max\left(\rho^{\mu}, r^{\mu}\right)\max\left(1,\gamma^{\mu}\left(t\right)\right).$$

Thus, we obtain from the above calculation that for $\nu = +$ or $-$

$$\lim_{t\to\nu\infty} q_{\nu}(t)\phi\,(t)\,\Psi_{\rho}(r\gamma\,(t)) = \max\left(\rho^{\mu}, r^{\mu}\right)\lim_{t\to\nu\infty} q_{\nu}(t)a(t)p^{-\mu}(t)\max\left(1,\gamma^{\mu}\left(t\right)\right) = 0$$

and

$$\begin{aligned} &\int_{-\infty}^{+\infty}\delta(s)\phi\,(s)\,\Psi_{\rho}(r\gamma\,(t)))ds \\ = &\int_{-\infty}^{+\infty}\delta(s)a(s)\left(p^{-\mu}(s)\right)\max\left(\rho^{\mu}, r^{\mu}\right)\max\left(1,\left(\gamma\,(s)\right)^{\mu}\right)ds \\ \leq &\ \max\left(\rho^{\mu}, r^{\mu}\right)\int_{-\infty}^{+\infty}\delta(s)a(s)p^{-\mu}(s)\max\left(1,\gamma^{\mu}\left(s\right)\right)ds < \infty. \end{aligned}$$

Moeover, we have

$$\left\{ \begin{array}{llll} f^{0} = 0 & \text{and} & f_{\infty}\,(\theta) = +\infty \text{ for all } \theta > 0, & \text{if } \mu > 0, \\ f^{\infty} = 0 & \text{and} & f_{0}\,(\theta) = +\infty \text{ for all } \theta > 0, & \text{if } \mu \leq 0. \end{array} \right.$$

Therefore, Theorem 1 guarantees existence of a positive solution to such a case of bvp (1). ■

Example 1 *Consider the bvp*

$$\left\{ \begin{array}{l} -u'' + u' + 2u = F(t,u),\ t \in \mathbb{R}, \\ \lim\limits_{t\to-\infty} e^{-|t|}u(t) = \lim\limits_{t\to+\infty} e^{|t|}u(t) = 0, \end{array} \right. \tag{8}$$

where

$$F(t,u) = e^{-8|t|}\left(\frac{ae^{4|t|}u}{e^{3|t|} + u} + \frac{be^{2|t|}u^{2}}{e^{2|t|} + u}\right)$$

and a, b *are positive constants.*

We have then $r_1 = -1$, $r_2 = 2$, $k = -1$, $l = 1$, $p(t) = e^{-2|t|}$, $q_{-}(t) = e^{-|t|}$, $q_{+}(t) = e^{|t|}$, $\gamma(t) = \min(e^{4t}, e^{-3t})$, $\delta\,(t) = \min\left(e^{t}, e^{-2t}\right)$, $\Gamma = \frac{7}{30}$ *and* $\lim_{\theta\to+\infty}\Theta\,(\theta) = \frac{2}{21}$.

Taking

$$\phi\,(t) = e^{-4|t|} \quad \text{and } f(t,u) = \frac{au}{e^{|t|} + u} + \frac{bu^{2}}{1+u},$$

we obtain $\Psi_\rho(x) = a\rho + b\rho^2$ *and*

$$\lim_{t\to-\infty} q_-(t)\phi(t)\,\Psi_\rho(r\gamma(t)) = \lim_{t\to-\infty} e^{-5|t|} = 0,$$

$$\lim_{t\to+\infty} q_+(t)\phi(t)\,\Psi_\rho(r\gamma(t)) = \lim_{t\to+\infty} e^{-3|t|} = 0,$$

Since $f^0 = a$ *and* $f_\infty(\theta) = b$ *for all* $\theta > 0$, *we conclude from Theorem 1 that if* $a < \frac{30}{7}$ *and* $b > \frac{21}{2}$, *then the bvp (8) admits a positive solution.*

Example 2 *Consider the bvp*

$$\begin{cases} -u'' + u' + 6u = e^{-8|t|}u^{-2}, \ t \in \mathbb{R}, \\ \lim\limits_{t\to-\infty} e^{|t|}u(t) = \lim\limits_{t\to+\infty} e^{2|t|}u(t) = 0. \end{cases} \tag{9}$$

We have then $r_1 = -2$, $r_2 = 3$, $k = 1$, $l = 2$, $p(t) = e^{-3|t|}$, $q_-(t) = e^{|t|}$, $q_+(t) = e^{2|t|}$, $\gamma(t) = \min(e^{6t}, e^{-5t})$ *and* $\delta(t) = \min\left(e^{2t}, e^{-3t}\right)$.

Taking $a(t) = e^{-8|t|}$ *and* $\mu = -2$ *we have*

$$\lim_{t\to-\infty} q_-(t)a(t)p^{-\mu}(t)\max\left(1, \gamma^\mu(t)\right) = \lim_{t\to-\infty} e^t = 0,$$

$$\lim_{t\to+\infty} q_+(t)a(t)p^{-\mu}(t)\max\left(1, \gamma^\mu(t)\right) = \lim_{t\to+\infty} e^{-2t} = 0$$

and

$$\int_{-\infty}^{+\infty} \delta(s)a(s)p^{-\mu}(s)\max\left(1, \gamma^\mu(s)\right)ds = \int_{-\infty}^{0} e^{4s}ds + \int_{0}^{+\infty} e^{-7s}ds < \infty.$$

Hence, all the conditions in Corollary 1 are satisfied and the bvp (9) admits a positive solution.

2 Abstract Background

It has been mentioned in the above section that Theorem 1 will be obtained by means of Guo-Krasnoselskii's fixed point theorem. Let us recall this powerfull theorem and the necessary theorical background to its statement.

Let $(E, ||.||)$ be a real Banach space. A nonempty closed convex subset C of E is said to be a cone in E if $C \cap (-C) = \{0_E\}$ and $tC \subset C$ for all $t \geq 0$.

Let Ω be a nonempty subset in E. A mapping $A : \Omega \to E$ is said to be compact if it is continuous and $A(\Omega)$ is relatively compact in E.

The Guo-Krasnoselskii's version of expansion and compression of a cone principle in a Banach space is the following theorem.

Theorem 2 *Let* P *be a cone in* E *and let* Ω_1, Ω_2 *be bounded open subsets of* E *with* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *If* $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ *is a compact mapping such that either:*

1. $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*

2. $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$.

Then T *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Fixed Point Formulation

We start this section by the following important lemma. It proposes a cone in a specific functional favorable to the use of Theorem 2.

Lemma 1 *For all* $t, \tau, s \in \mathbb{R}$,

$$p(t)G(t,s) \geq \gamma(t)\, p(\tau)G(\tau,s).$$

Proof. Set $Q(t,\tau,s) = \dfrac{p(t)G(t,s)}{p(\tau)G(\tau,s)}$. Then we distinguish between four cases.

a) $\tau, t \geq 0$, in this case we have

$$Q(t,\tau,s) = \begin{cases} \exp\left(-(r_2-r_1)\,t + (r_2-r_1)\,\tau\right) \geq e^{-(r_2-r_1)t}, \text{ if } s \leq \tau \leq t, \\ \exp\left(-(r_2-r_1)\,t + (r_2-r_1)\,s\right) \geq e^{-(r_2-r_1)t}, \text{ if } \tau \leq s \leq t, \\ 1, \text{ if } \tau \leq t \leq s, \\ \exp\left(-(r_2-r_1)\,t + (r_2-r_1)\,\tau\right) \geq e^{-(r_2-r_1)t}, \text{ if } s \leq t \leq \tau, \\ \exp\left((r_2-r_1)\,\tau - (r_2-r_1)\,s\right) \geq 1, \text{ if } t \leq s \leq \tau, \\ 1, \text{ if } t \leq \tau \leq s \end{cases} \geq \gamma(t).$$

b) $\tau, t \leq 0$, in this case we have

$$Q(t,\tau,s) = \begin{cases} \exp\left((r_2+r_1)\,t - (r_2+r_1)\,\tau\right) \geq e^{(r_2+r_1)t}, \text{ if } s \leq \tau \leq t, \\ \exp\left(-(r_2-r_1)\,t - 2r_2\tau + (r_2-r_1)\,s\right) \geq e^{-(r_2-r_1)t}, \text{ if } \tau \leq s \leq t, \\ \exp\left(2r_2 t - 2r_2\tau\right) \geq e^{2r_2 t}, \text{ if } \tau \leq t \leq s, \\ \exp\left((r_2+r_1)\,t - (r_2+r_1)\,\tau\right) \geq e^{(r_2+r_1)t}, \text{ if } s \leq t \leq \tau, \\ \exp\left(2r_2 t - (r_2+r_1)\,\tau - (r_2-r_1)\,s\right) \geq e^{2r_2 t}, \text{ if } t \leq s \leq \tau, \\ \exp\left(2r_2 t - 2r_2\tau\right) \geq e^{2r_2 t}, \text{ if } t \leq \tau \leq s \end{cases} \geq \gamma(t).$$

c) $\tau \leq 0, t \geq 0$, in this case we have

$$Q(t,\tau,s) = \begin{cases} \exp\left(-(r_2-r_1)\,t - (r_2+r_1)\,\tau\right) \geq e^{-(r_2-r_1)t}, \text{ if } s \leq \tau \leq t, \\ \exp\left(-(r_2-r_1)\,t - 2r_2\tau + (r_2-r_1)\,s\right) \geq e^{-(r_2-r_1)t}, \text{ if } \tau \leq s \leq t, \\ \exp\left(-2r_2\tau\right) \geq 1, \text{ if } \tau \leq t \leq s \end{cases} \geq \gamma(t).$$

d) $\tau \geq 0,\ t \leq 0$, in this case we have

$$Q(t,\tau,s) = \begin{cases} \exp\left((r_2+r_1)\,t + (r_2-r_1)\,\tau\right) \geq e^{(r_2+r_1)t}, \text{ if } s \leq t \leq \tau, \\ \exp\left(2r_2 t + (r_2-r_1)\,\tau - (r_2-r_1)\,s\right) \geq e^{2r_2 t}, \text{ if } t \leq s \leq \tau, \\ \exp\left(2r_2 t\right), \text{ if } t \leq \tau \leq s \end{cases} \geq \gamma(t).$$

The proof is complete. ■

The functional framework in which we will solve the bvp (1) consists in the following Banach space E and the cone P given below and suggested by Lemma 1. In this paper, we let E be the linear space defined by

$$E = \left\{ u \in C(\mathbb{R},\mathbb{R}) : \lim_{|t|\to\infty} p(t)u(t) = 0 \right\}.$$

Equipped with the norm $\|\cdot\|$, where for $u \in E$, $\|u\| = \sup_{t\in\mathbb{R}} (p(t) |u(t)|)$, E becomes a Banach space. The subset P of E given by

$$P = \{u \in E : u(t) \geq \widetilde{\gamma}(t) \|u\| \text{ for all } t \in \mathbb{R}\}$$

is a cone of E.

The following lemma is an adapted version to the case of the space E of Corduneanu's compactness criterion ([3], p. 62). It will be used in this work to prove that the operator in the fixed point formulation coresponding to the bvp (1), maps bounded sets of $P\backslash B(0,\epsilon)$ (for arbitrary $\epsilon > 0$), into relatively compact sets.

Lemma 2 *A nonempty subset M of E is relatively compact if the following conditions hold:*

(a) M is bounded in E,

(b) the set $\{u : u(t) = p(t)x(t),\ x \in M\}$ is locally equicontinuous on $\mathbb{R}$, and

(c) the set $\{u : u(t) = p(t)x(t),\ x \in M\}$ is equiconvergent at $\pm\infty$.

Lemma 3 *Assume that Hypotheses (2) and (3) hold $l < r_2$ and $k < -r_1$. Then there exists a continuous operator $T : P\backslash\{0\} \to P$ such that for all r, R with $0 < r < R$, $T(P \cap (B(0,R)\backslash B(0,r)))$ is relatively compact and fixed points of T are positive solutions to the bvp (1).*

Proof. The proof is divided into four steps.

Step 1. In this step we prove the existence of the operator T. To this aim let $u \in P\backslash\{0\}$. By means of Hypothesis (3) with $R = \|u\|$, for all $t \in \mathbb{R}$ we have from (5) and Hypothesis (3),

$$\begin{aligned}\int_{-\infty}^{+\infty} G(t,s)\phi(s) f(s,u(s))ds &\leq \int_{-\infty}^{+\infty} G(t,s)\phi(s)\Psi_R(R\gamma(s))\,ds \\ &\leq \frac{1}{(r_2-r_1)\,\delta(t)} \int_{-\infty}^{+\infty} \delta(s)\phi(s)\Psi_R(R\gamma(s))\,ds < \infty.\end{aligned}$$

Thus, let v be the function defined by

$$v(t) = \int_{-\infty}^{+\infty} G(t,s)\phi(s) f(s,u(s))ds.$$

Clealy, v is continuous on $\mathbb{R}$ and $v(t) > 0$ for all $t \in \mathbb{R}$. Moreover, we have

$$p(t)v(t) \leq \frac{1}{(r_2-r_1)} (J_1(t) + J_2(t)),$$

where

$$J_1(t) = \frac{\int_{-\infty}^{t} e^{-r_1 s}\phi(s)\Psi_R(R\gamma(s))\,ds}{\exp(r_2|t| - r_1 t)} \text{ and } J_2(t) = \frac{\int_{t}^{+\infty} \phi(s)\Psi_R(R\gamma(s))\,ds}{\exp(r_2|t| - r_2 t)}.$$

Since for $t \leq 0$,

$$J_1(t) \leq \int_{-\infty}^{t} \delta(s)\phi(s)\Psi_R(R\gamma(s))\,ds$$

and for $t \geq 0$,

$$J_2(t) = \int_{t}^{+\infty} \delta(s)\phi(s)\Psi_R(R\gamma(s))\,ds,$$

we obtain from Hypothesis (3) that $\lim\limits_{t\to-\infty} J_1(t) = \lim\limits_{t\to+\infty} J_2(t) = 0$.

Now applying L'Hopital's rule, we obtain from Hypothesis (3) that

$$\lim_{t\to+\infty} J_1(t) = \lim_{t\to+\infty} \frac{e^{-r_1 t}\phi\left(t\right)\Psi_R\left(R\gamma(t)\right)}{\left(r_2-r_1\right)\exp\left(\left(r_2-r_1\right)t\right)} = \frac{1}{\left(r_2-r_1\right)}\lim_{t\to+\infty} p(t)\phi\left(t\right)\Psi_R\left(R\gamma(t)\right) = 0$$

and

$$\lim_{t\to-\infty} J_2(t) = \lim_{t\to-\infty} \frac{e^{-r_2 t}\phi\left(t\right)\Psi_R\left(R\gamma(t)\right)}{2r_2\exp\left(-2r_2 t\right)} = \frac{1}{2r_2}\lim_{t\to-\infty} p(t)\phi\left(t\right)\Psi_R\left(R\gamma(t)\right) = 0.$$

Hence, we conclude that $\lim_{|t|\to+\infty} p(t)v(t) = 0$ and $v \in E$.

Finally, Lemma 1 leads to

$$p(t)v(t) = \int_{-\infty}^{+\infty} p(t)G(t,s)\phi\left(s\right)f(s,u(s))ds \geq \gamma\left(t\right)\int_{-\infty}^{+\infty} p(\tau)G(\tau,s)\phi\left(s\right)f(s,u(s))ds$$

for all $t,\tau \in \mathbb{R}$.

Taking the supremum on τ yields

$$v(t) \geq \widetilde{\gamma}\left(t\right)\|v\|,$$

proving that $v \in P$ and the operator $T : P\backslash\{0\} \to P$, where for $u \in P\backslash\{0\}$

$$Tu(t) = \int_{-\infty}^{+\infty} G(t,s)\phi\left(s\right)f(s,u(s))ds,$$

is well defined.

Step 2. In this step we prove that the operator T is continuous. Let (u_n) be a sequence in $P\backslash\{0\}$ such that $\lim_{n\to\infty} u_n = u$ in E with u in $P\backslash\{0\}$ and let $R > r > 0$ be such that $(u_n) \subset B(0,R)\backslash B(0,r)$. If Ψ_R is the function given by Hypothesis (3), then for all $n \geq 1$ we have

$$\begin{aligned}
\|Tu_n - Tu\| &= \sup_{t\in\mathbb{R}}\left(p(t)\left|Tu_n\left(t\right) - Tu\left(t\right)\right|\right) \\
&\leq \sup_{t\in\mathbb{R}}\left(\frac{p(t)}{\left(r_2-r_1\right)\delta\left(t\right)}\int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)\left|f(s,u_n(s)) - f((s,u(s))\right|ds\right) \\
&\leq \frac{1}{\left(r_2-r_1\right)}\int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)\left|f(s,u_n(s)) - f((s,u(s))\right|ds.
\end{aligned}$$

Because of

$$\left|f(s,u_n(s)) - f(s,u(s))\right| \to 0, \text{ as } n \to +\infty$$

for all $s > 0$ and

$$\delta\left(s\right)\phi\left(s\right)\left|f(s,u_n(s)) - f((s,u(s))\right| \leq \delta\left(s\right)\phi\left(s\right)\Psi_R\left(r\gamma(s)\right)$$

with $\int_{-\infty}^{+\infty}\delta\left(s\right)\phi\left(s\right)\Psi_R\left(r\gamma(s)\right)ds < \infty$, the Lebesgue dominated convergence theorem guarantees that $\lim_{n\to\infty}\|Tu_n - Tu\| = 0$. Hence, we have proved that T is continuous.

Step 3. In this step, we prove that for $R > r > 0$, $T\left(P \cap \left(B(0,R)\backslash B(0,r)\right)\right)$ is relatively compact. Set $\Omega = P \cap \left(B(0,R)\backslash B(0,r)\right)$ and let Φ be defined by

$$\Phi(s) = \phi\left(s\right)\Psi_R\left(r\gamma(s)\right),$$

where Ψ_R is the function given by Hypothesis (3). For all $u \in \Omega$, we have

$$\|Tu\| \leq \sup_{t\geq 0}\left(\frac{p(t)}{\left(r_2-r_1\right)\delta\left(t\right)}\int_{-\infty}^{+\infty}\delta\left(s\right)\Phi(s)ds\right) \leq \frac{1}{r_2-r_1}\int_{-\infty}^{+\infty}\delta\left(s\right)\Phi(s)ds < \infty,$$

proving that $T\Omega$ is bounded in E.

Let $t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}$, for all $u \in \Omega$ we have

$$\begin{aligned} |p(t_2)Tu(t_2) - p(t_1)Tu(t_1)| &\leq |p_1(t_2) - p_1(t_1)| \int_{-\infty}^{\zeta} e^{-r_1 s}\Phi(s)ds \\ &\quad + |p_2(t_2) - p_2(t_1)| \int_{\eta}^{+\infty} e^{-r_2 s}\Phi(s)ds + C_{\eta,\zeta}\int_{t_1}^{t_2}\Phi(s)ds, \end{aligned}$$

where for $i = 1, 2,\ p_i(t) = e^{-r_2|t| + r_i t}$ and $C_{\eta,\zeta} = 2\sup_{t,s\in[\eta,\zeta]} p(t)G(t,s)$.

Because that p_1, p_2 and $t \to \int_0^t \Phi_{r,R}(s)ds$ are uniformly continuous on compact intervals, the above estimates prove that $T\Omega$ is equicontinuous on compact intervals.

For all $u \in \Omega$ and $t > 0$, we have

$$p(t)Tu(t) \leq p(t)\int_{-\infty}^{+\infty} G(t,s)\Phi(s)ds = H(t).$$

By means of L'Hopital's rule, we obain from Hypothesis (3) that

$$\lim_{|t|\to\infty} H(t) = \lim_{|t|\to\infty} p(t)\Phi(t) = 0,$$

proving the equiconvergence of $T\Omega$.

In view of Lemma 2, $T\Omega$ is relatively compact in E.

Step 4. We claim that fixed points of T are positive solutions to the bvp (1). Let $u \in P\backslash\{0\}$ be a fixed point of T with $\|u\| = R$. For all $t \in \mathbb{R}$ we have

$$u(t) = \frac{1}{r_2 - r_1}\left(e^{r_1 t}\int_{-\infty}^{t} e^{-r_1 s} f(s,u(s))ds + e^{r_2 t}\int_{t}^{+\infty} e^{-r_2 s} f(s,u(s))ds\right),$$

$$u'(t) = \frac{r_1 e^{r_1 t}}{r_2 - r_1}\int_{-\infty}^{t} e^{-r_1 s} f(s,u(s))ds + \frac{r_2 e^{r_2 t}}{r_2 - r_1}\int_{t}^{+\infty} e^{-r_2 s} f(s,u(s))ds$$

and

$$u''(t) = \frac{(r_1)^2 e^{r_1 t}}{r_2 - r_1}\int_{-\infty}^{t} e^{-r_1 s}\phi(s) f(s,u(s))ds + \frac{(r_2)^2 e^{r_2 t}}{r_2 - r_1}\int_{t}^{+\infty} e^{-r_2 s}\phi(s) f(s,u(s))ds - \phi(s) f(t,u(t)).$$

Thus, we obtain

$$\begin{aligned} -u''(t) + cu'(t) + \lambda u(t) &= \frac{-r_1^2 + cr_1 + \lambda}{r_2 - r_1}\int_{-\infty}^{t} G(t,s)\phi(s) f(s,u(s))ds \\ &\quad + \frac{-r_2^2 + cr_2 + \lambda}{r_2 - r_1}\int_{t}^{+\infty} G(t,s)\phi(s) f(s,u(s))ds + \phi(t) f(t,u(t)) \\ &= \phi(t) f(t,u(t)). \end{aligned}$$

Now, we need to prove that u satisfies the boundary conditions, $\lim_{t\to-\infty} e^{l|t|}u(t) = \lim_{t\to+\infty} e^{k|t|}u(t) = 0$. We have

$$e^{l|t|}u(t) \leq \frac{1}{r_2 - r_1}(L_1(t) + L_2(t))$$

and

$$e^{k|t|}u(t) \leq \frac{1}{r_2 - r_1}(K_1(t) + K_2(t)),$$

where

$$L_1(t) = \frac{\int_{-\infty}^{t} e^{-r_1 s}\phi(s)\Psi_R(R\gamma(s))\,ds}{\exp(-l|t| - r_1 t)},\quad L_2(t) = \frac{\int_{t}^{+\infty} e^{-r_2 s}\phi(s)\Psi_R(R\gamma(s))\,ds}{\exp(-l|t| - r_2 t)},$$

$$K_1(t) = \frac{\int_{-\infty}^{t} e^{-r_1 s}\phi(s)\,\Psi_R(R\gamma(s))\,ds}{\exp(-k\,|t| - r_1 t)} \text{ and } K_2(t) = \frac{\int_{t}^{+\infty} e^{-r_2 s}\phi(s)\,\Psi_R(R\gamma(s))\,ds}{\exp(-k\,|t| - r_2 t)}.$$

Since for $t \leq 0$,

$$L_1(t) \leq \begin{cases} \int_{-\infty}^{t} \delta(s)\phi(s)\,\Psi_R(R\gamma(s))\,ds, & \text{if } l \leq r_1, \\ \frac{\int_{-\infty}^{t} \delta(s)\phi(s)\Psi_R(R\gamma(s))ds}{\exp((l-r_1)t)}, & \text{if } l > r_1 \end{cases}$$

and for $t \geq 0$,

$$K_2(t) \leq \begin{cases} \int_{t}^{+\infty} \delta(s)\phi(s)\,\Psi_R(R\gamma(s))\,ds, & \text{if } k \leq -r_2, \\ \frac{\int_{t}^{+\infty} \delta(s)\phi(s)\Psi_R(R\gamma(s))ds}{\exp(-(k+r_2)t)}, & \text{if } k > -r_2, \end{cases}$$

Hypothesis (3) and L'Hopital's rule lead to $\lim_{t\to-\infty} L_1(t) = \lim_{t\to+\infty} K_2(t) = 0$.

Taking in account the conditions $k < -r_1$ and $l < r_2$ and Hypothesis (3), the L'Hopital's rule leads to

$$\lim_{t\to-\infty} L_2(t) = \lim_{t\to-\infty} \frac{-e^{-r_2 t}\phi(t)\,\Psi_R(R\gamma(t))}{(l-r_2)\exp((l-r_2)\,t)} = \frac{1}{r_2 - l}\lim_{t\to-\infty} e^{l|t|}\phi(t)\,\Psi_R(R\gamma(t)) = 0$$

and

$$\lim_{t\to+\infty} K_1(t) = \lim_{t\to+\infty} \frac{e^{-r_1 t}\phi(t)\,\Psi_R(R\gamma(t))}{-(k+r_1)\exp(-(k+r_1)\,t)} = \frac{-1}{(k+r_1)}\lim_{t\to+\infty} e^{k|t|}\phi(t)\,\Psi_R(R\gamma(t)) = 0.$$

Hence, we have proved that $\lim_{t\to-\infty} e^{l|t|}u(t) = \lim_{t\to+\infty} e^{k|t|}u(t) = 0$, completing the proof of the lemma. ■

4 Proof of Theorem 1

Step 1. Existence in the case where (6) holds

Let $\epsilon > 0$ be such that $(f^0 + \epsilon)\Gamma < 1$. For such a ϵ, there exists $R_1 > 0$ such that $f(t, \frac{w}{p(t)}) \leq (f^0 + \epsilon)w$ for all $w \in (0, R_1)$. Let $\Omega_1 = \{u \in E, \|u\| < R_1\}$.

Therefore, for all $u \in P \cap \partial\Omega_1$ and all $t \in \mathbb{R}$, we have

$$\begin{aligned} p(t)Tu(t) &= p(t)\int_{-\infty}^{+\infty} G(t,s)\phi(s)f(s, \frac{1}{p(s)}(p(s)u(s)))ds \\ &\leq \left(f^0 + \epsilon\right)p(t)\int_{-\infty}^{+\infty} G(t,s)\phi(s)(p(s)u(s))ds \\ &\leq \|u\|\left(f^0 + \epsilon\right)p(t)\int_{-\infty}^{+\infty} G(t,s)\phi(s)ds \\ &\leq \Gamma\left(f^0 + \epsilon\right)\|u\| \leq \|u\|, \end{aligned}$$

leading to $\|Tu\| \leq \|u\|$.

Now, suppose that $f_\infty(\theta)\,\Theta(\theta) > 1$ for some $\theta > 0$ and let $\varepsilon > 0$ be such that

$$(f_\infty(\theta) - \varepsilon)\Theta(\theta) > 1.$$

There exists $R_2 > R_1$ such that $f(t, \frac{w}{p(t)}) > (f_\infty(\theta) - \varepsilon)w$ for all $t \in I_\theta$ and all $w \geq R_2$. Let $\gamma_\theta =$

$\min\{\widetilde{\gamma}(s) : s \in I_\theta\}$, $\widetilde{R}_2 = R_2/\gamma_\theta$ and $\Omega_2 = \{u \in E : \|u\| < \widetilde{R}_2\}$. For all $u \in P \cap \partial\Omega_2$ and all $t \in \mathbb{R}$, we have

$$\begin{aligned}
\|Tu\| &\geq \sup_{t\in\mathbb{R}}\left(p(t)\int_{-\theta}^{\theta} G(t,s)\phi(s)f(s,\frac{1}{p(s)}(p(s)u(s)))ds\right)\\
&\geq (f_\infty(\theta)-\varepsilon)\sup_{t\in\mathbb{R}}\left(p(t)\int_{-\theta}^{\theta} G(t,s)\phi(s)\left(p(s)u(s)\right)ds\right)\\
&\geq (f_\infty(\theta)-\varepsilon)\sup_{t\in\mathbb{R}}\left(p(t)\int_{-\theta}^{\theta} G(t,s)\phi(s)\left(\gamma(s)\|u\|\right)ds\right)\\
&\geq \|u\|(f_\infty(\theta)-\varepsilon)\Theta(\theta)\geq\|u\|
\end{aligned}$$

We deduce from Assertion 1 of Theorem 2, that T admits a fixed point $u \in P$ with $R_1 \leq \|u\| \leq \widetilde{R}_2$ which is, by Lemma 3, a positive solution to the bvp (1).

Step 2. Existence in the case where (7) holds

Let $\varepsilon > 0$ be such that $(f_0(\theta)-\varepsilon)\Theta(\theta) > 1$, there exists R_1 such that $f(t, \frac{w}{p(t)}) > (f_0(\theta)-\varepsilon)w$ for all $t \in I_\theta$ and all $w \in (0, R_1)$. Let $\Omega_1 = \{u \in E : \|u\| < R_1\}$, for all $u \in P \cap \partial\Omega_1$ and all $t \in \mathbb{R}$, we have

$$\begin{aligned}
\|Tu\| &\geq \sup_{t\in\mathbb{R}}\left(p(t)\int_{-\theta}^{\theta} G(t,s)\phi(s)f(s,\frac{1}{p(s)}(p(s)u(s)))ds\right)\\
&\geq (f_0(\theta)-\varepsilon)\sup_{t\in\mathbb{R}}\left(p(t)\int_{-\theta}^{\theta} G(t,s)\phi(s)\left(p(s)u(s)\right)ds\right)\\
&\geq (f_0(\theta)-\varepsilon)\sup_{t\in\mathbb{R}}\left(p(t)\int_{-\theta}^{\theta} G(t,s)\phi(s)\left(\gamma(s)\|u\|\right)ds\right)\\
&\geq \|u\|(f_0(\theta)-\varepsilon)\Theta(\theta)\geq\|u\|.
\end{aligned}$$

Let $\epsilon > 0$ be such that $(f^\infty+\epsilon)\Gamma < 1$, there exists $R_\epsilon > 0$ such that

$$f(t,\frac{w}{p(t)}) \leq (f^\infty+\epsilon)w + \Psi_{R_\epsilon}(w), \text{ for all } t\in\mathbb{R} \text{ and } w>0,$$

where Ψ_{R_ϵ} is the functions given by Hypothesis (3) for $R = R_\epsilon$. Let

$$\Phi_\epsilon(t) = \phi(s)\Psi_{R_\epsilon}(R_\epsilon\gamma(s)) \quad\text{and}\quad \widetilde{R}_2 = \frac{\overline{\Phi}_\epsilon\Gamma}{1-(f^\infty+\epsilon)\Gamma} \text{ with } \overline{\Phi}_\epsilon = \sup_{t\geq 0}\left(p(t)\int_{-\infty}^{+\infty}G(t,s)\Phi_\epsilon(s)\,ds\right).$$

and notice that $\Gamma^{-1}(f^\infty+\epsilon)R + \overline{\Phi}_\epsilon \leq R$ for all $R \geq \widetilde{R}_2$.

Let $R_2 > \max(R_1, \widetilde{R}_2, R_\epsilon)$ and $\Omega_2 = \{u \in E, \|u\| < R_2\}$. For all $u \in P \cap \partial\Omega_2$ and all $t \in \mathbb{R}$, we have

$$\begin{aligned}
p(t)Tu(t) &= p(t)\int_{-\infty}^{+\infty} G(t,s)\phi(s)f(s,\frac{1}{p(s)}(p(s)u(s)))ds\\
&\leq p(t)\int_{-\infty}^{+\infty} G(t,s)\phi(s)\left((f^\infty+\epsilon)(p(s)u(s))+\Psi_\epsilon(p(s)u(s))\right)ds\\
&\leq (f^\infty+\epsilon)\|u\|\,p(t)\int_{-\infty}^{+\infty}G(t,s)\phi(s)ds+\overline{\Phi}_\epsilon\\
&\leq (f^\infty+\epsilon)\Gamma\|u\|+\overline{\Phi}_\epsilon\leq\|u\|,
\end{aligned}$$

leading to

$$\|Tu\| \leq \|u\|.$$

We deduce from Assertion 2 of Theorem 2, that T admits a fixed point $u \in P$ with $R_1 \leq \|u\| \leq R_2$ which is, by Lemma 3, a positive solution to the bvp (1).

Thus, the proof of Theorem 1 is complete.

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