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# Nineteen Limit Cycles In Discontinuous Quartic Differential System With Two Zones*

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**Abstract**

In this paper we study the number of limit cycles for discontinuous differential systems, which can bifurcate from the periodic orbits of the quartic isochronous centers $\dot{x} = -y + xy(x^2 + y^2), \dot{y} = x + y^2(x^2 + y^2)$ when they are perturbed inside the class of all discontinuous quartic polynomial systems with the straight line of discontinuity $y = 0$. Using the averaging theory of first order for discontinuous differential systems, we show that this system has at least 19 limit cycles. Comparing the obtained results for the discontinuous with the results for the continuous quartic polynomial differential systems, this work shows that the discontinuous systems have at least 11 more limit cycles surrounding the origin than the continuous ones.

## 1 Introduction

In recent years, there has been intensive study of the bifurcations of limit cycles in planar polynomial vector fields. The search for the maximum number of limit cycles that polynomial differential systems of a given degree can have is part of $16^{th}$ Hilbert's Problem and many contributions have been made in this direction, see for instance [8]. Recently the theory of limit cycles has also been studied in discontinuous piecewise differential systems, see [5, 6, 11, 12, 16].

Itikawa and Llibre considered in [9] the uniform isochronous center of the quartic polynomial differential system

$$\dot{x} = -y + xy(x^2 + y^2), \quad \dot{y} = x + y^2(x^2 + y^2), \tag{1}$$

when perturbed inside the whole class of quartic polynomial differential systems, they proved using the averaging theory of first order that the number of limit cycles which bifurcate from the periodic solutions of uniform isochronous center located at the origin of system (1) is at least 8.

For recent studies on isochronous and uniform isochronous centers see for instance [1, 4, 7, 15].

Our objective is to study the number of limit cycles of the discontinuous quartic differential systems with two zones separated by a straight line. Related studies about the number of limit cycles bifurcating from center and isochronous centers some of them perturbed in discontinuous quadratic and cubic systems can be found in [3, 10, 14].

In the present work, using the averaging theory of first order we study the limit cycles which bifurcate from the periodic solutions of the uniform isochronous center located at the origin of system (1) when it is perturbed inside the whole class of quartic polynomial differential systems. More precisely we consider the following differential system

$$\dot{X} = Z(x, y) = \begin{cases} Y_1 & \text{if } y > 0 \ , \\ Y_2 & \text{if } y < 0 \ , \end{cases} \tag{2}$$

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where

$$Y_1 = \begin{pmatrix} -y + xy(x^2+y^2) + \varepsilon p_1(x,y) \\ x + y^2(x^2+y^2) + \varepsilon q_1(x,y) \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} -y + xy(x^2+y^2) + \varepsilon p_2(x,y) \\ x + y^2(x^2+y^2) + \varepsilon q_2(x,y) \end{pmatrix},$$

and $\varepsilon$ is a small parameter, $p_i(x,y)$, $q_i(x,y)$, $i = 1,2$ are respectively the quartic polynomials in the variables $x$ and $y$, given by

$$\begin{aligned} p_1(x,y) = {} & a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3 \\ & + a_{10} x^4 + a_{11} x^3 y + a_{12} x^2 y^2 + a_{13} xy^3 + a_{14} y^4, \end{aligned}$$

$$\begin{aligned} q_1(x,y) = {} & b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3 \\ & + b_{10} x^4 + b_{11} x^3 y + b_{12} x^2 y^2 + b_{13} xy^3 + b_{14} y^4, \end{aligned}$$

$$\begin{aligned} p_2(x,y) = {} & c_1 x + c_2 y + c_3 xy + c_4 x^2 + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3 \\ & + c_{10} x^4 + c_{11} x^3 y + c_{12} x^2 y^2 + c_{13} xy^3 + c_{14} y^4, \end{aligned}$$

$$\begin{aligned} q_2(x,y) = {} & d_1 x + d_2 y + d_3 xy + d_4 x^2 + d_5 y^2 + d_6 x^3 + d_7 x^2 y + d_8 xy^2 + d_9 y^3 \\ & + d_{10} x^4 + d_{11} x^3 y + d_{12} x^2 y^2 + d_{13} xy^3 + d_{14} y^4. \end{aligned}$$

In other words, we extend the work done by Itikawa and Llibre [9] for the continuous quartic polynomial differential systems to the discontinuous ones with the straight line of discontinuity $y = 0$.

System (2) can be written using the sign function in the form

$$\dot{X} = Z(x,y) = G_1(x,y) + sing(y) G_2(x,y),$$

where $G_1(x,y) = \frac{1}{2}\left(Y_1(x,y) + Y_2(x,y)\right)$ and $G_2(x,y) = \frac{1}{2}\left(Y_1(x,y) - Y_2(x,y)\right)$.

Our main result is the following.

**Theorem 1** *For $|\varepsilon| \neq 0$ sufficiently small there are discontinuous quartic polynomial differential systems (2) having at least 19 limit cycles bifurcating from the periodic orbits of the isochronous center (1).*

All our calculations were performed with the assistance of the software *Mathematica* and *Maple*. This paper is structured as follows. In section 2 we present some preliminary results on uniform isochronous centers and on the averaging theory for discontinuous differential systems that we shall use in this paper. The proof of Theorem 1 is given in section 3.

# 2 Main Results

In this section we summarise the main results that we will use to study the discontinuous quartic differential systems (1). As it was stated in [14] the next theorem is the first-order averaging theory developed for discontinuous differential systems in [11].

**Theorem 2 ([11])** . *We consider the following discontinuous differential system*

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{3}$$

*with*

$$F(t,x) = F_1(t,x) + sign(h(t,x)) F_2(t,x),$$

$$R(t,x,\varepsilon) = R_1(t,x,\varepsilon) + sign(h(t,x))R_2(t,x,\varepsilon),$$

*where* $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ *and* $h : \mathbb{R} \times D \to \mathbb{R}$ *are continuous functions,* $T-$*periodic in the variable* $t$ *and* $D$ *is an open subset of* $\mathbb{R}^n$. *We also suppose that* $h$ *is a* $C^1$ *function having* $0$ *as a regular value. Denote by* $\mathcal{M} = h^{-1}(0)$, *by* $\sum = \{0\} \times D \nsubseteq \mathcal{M}$, *by* $\sum_0 = \sum \setminus \mathcal{M} \neq \varnothing$, *and its elements by* $z \equiv (0,z) \notin \mathcal{M}$.

*Define the averaged function* $f : D \to \mathbb{R}^n$ *as*

$$f(x) = \int_0^T F(t,x)dt. \tag{4}$$

*We assume the following three conditions.*

*(i)* $F_1, F_2, R_1, R_2$ *and* $h$ *are locally L-Lipschitz with respect to* $x$;

*(ii) for* $a \in \sum_0$ *with* $f(a) = 0$, *there exist a neighborhood* $V$ *of* $a$ *such that* $f(z) \neq 0$ *for all* $z \in \bar{V} \setminus \{a\}$ *and* $d_B(f,V,a) \neq 0$, *(i.e. the Brouwer degree of* $f$ *at* $a$ *is not zero).*

*(iii) If* $\partial h \setminus \partial t(t_0,z_0) = 0$ *for some* $(t_0,z_0) \in M$, *then* $\left(\langle \nabla_x h, F_1\rangle^2 - \langle \nabla_x h, F_2\rangle^2\right)(t_0,z_0) > 0$.

*Then, for* $|\varepsilon| > 0$ *sufficiently small, there exists a* $T-$*periodic solution* $x(.,\varepsilon)$ *of system* (3) *such that* $x(t,\varepsilon) \to a$ *as* $\varepsilon \to 0$.

As it was stated in [2], The next theorem provides a method to write a perturbed differential system under the form (5).

**Theorem 3** *Consider the unperturbed system* $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, *where* $P, Q : \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions, and assume that this system has a continuous family of period solutions* $\{\Gamma_h\} \subset \{(x,y) : \mathcal{H}(x,y) = h, h_1 < h < h_2\}$, *where* $\mathcal{H}$ *is a first integral of the system. For a given first integral* $H$ *assume that* $xQ(x,y) - yP(x,y) \neq 0$ *for all* $(x,y)$ *in the period annulus formed by the ovals* $\{\Gamma_h\}$. *Let* $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0,2\pi) \to [0,\infty)$ *be a continuous function such that*

$$H(\rho(R,\theta)\cos\theta, \rho(R,\theta)\sin\theta) = R^2$$

*for all* $R \in (\sqrt{h_1}, \sqrt{h_2})$ *and all* $\theta \in [0,2\pi)$. *Then the differential equation which describes the dependence between the square root of the energy* $R = \sqrt{h}$ *and the angle* $\theta$ *for the perturbed system*

$$\begin{cases} \dot{x} = P(x,y) + \varepsilon p(x,y), \\ \dot{y} = Q(x,y) + \varepsilon q(x,y), \end{cases} \tag{5}$$

*where* $p, q : \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions is*

$$\frac{dR}{d\theta} = \varepsilon \frac{\mu(x^2+y^2)(Qp - Pq)}{2R(Qx - Py)} + \mathcal{O}(\varepsilon^2) \tag{6}$$

*where* $\mu = \mu(x,y)$ *is the integrating factor corresponding to the first integral* $H$ *of the unperturbed system and* $x = \rho(R,\theta)\cos\theta$, $y = \rho(R,\theta)\sin\theta$.

For more details see [2]. We also need the next result, which can be found in [13, Prop. 1].

**Proposition 1** *Let* $f_0, \ldots, f_n$ *be analytic functions defined on an open interval* $I \subset \mathbb{R}$. *If* $f_0, \ldots, f_n$ *are linearly independent then there exists* $s_1, \ldots, s_n \in I$ *and* $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ *such that for every* $j \in \{1, \ldots, n\}$ *we have* $\sum_{i=0}^n \lambda_i f_i(s_j) = 0$.

## 3 Proof of Theorem 1

In order to apply the averaging method for studying limit cycles of (5) for $\varepsilon$ sufficiently small, we need to write system (5) in the standard form (3) for applying the averaging theory. The following result of Theorem 3 provides a way for transforming (5) in this standard form.

We recall that the period annulus of a center is the topological annulus formed by all the periodic orbits surrounding the center contained in the connected component whose inner boundary is the center.

A first integral $H$ and an integrating factor $\mu$ in the period annulus of the center of the quartic differential system (1) have the expressions

$$H(x,y)=\frac{1}{3(x^2+y^2)^{3/2}}-\frac{x}{(x^2+y^2)^{1/2}},\quad \mu(x,y)=\frac{1}{(x^2+y^2)^{5/2}},$$

respectively. For this system we note that $h_1=0, h_2=1$, and that the function $\rho$ that satisfies the hypotheses of Theorem 3 is given by

$$\rho(R,\theta)=\frac{1}{(R^2+3\cos\theta)^{1/3}},$$

then $H(\rho\cos\theta,\rho\sin\theta)=R^2/3$ for all $R>\sqrt{3}$ and $\theta\in[0,2\pi)$. Then using Theorem 3 where

$$Qp-Pq=\begin{cases}
a_1x^2+(a_2+b_1)xy+b_2y^2+a_4x^3+(a_5+b_3)xy^2+(a_3+b_4)x^2y\\
+b_5y^3+a_6x^4+(a_7+b_6)x^3y+(a_8+b_7)x^2y^2+(a_9+b_8)xy^3+b_9y^4\\
+a_{10}x^5+(a_{11}-b_1+b_{10})x^4y+(a_1+a_{12}-b_2+b_{11})x^3y^2\\
+(a_2+a_{13}-b_1+b_{12})x^2y^3+(a_1+a_{14}-b_2+b_{13})xy^4\\
+(a_2+b_{14})y^5-b_4x^5y+(a_4-b_3)x^4y^2+(a_3-b_4-b_5)x^3y^3\\
+(a_4+a_5-b_3)x^2y^4+(a_3-b_5)xy^5+a_5y^6-b_6x^6y+(a_6-b_7)x^5y^2\\
+(a_7-b_6-b_8)x^4y^3+(a_6+a_8-b_7-b_9)x^3y^4+(a_7+a_9-b_8)x^2y^5\\
+(a_8-b_9)xy^6+a_9y^7-b_{10}x^7y+(a_{10}-b_{11})x^6y^2+(a_{11}-b_{10}-b_{12})x^5y^3\\
+(a_{10}+a_{12}-b_{11}-b_{13})x^4y^4+(a_{11}+a_{13}-b_{12}-b_{14})x^3y^5\\
+(a_{12}+a_{14}-b_{13})x^2y^6+(a_{13}-b_{14})xy^7+a_{14}y^8 \qquad \text{for } y>0,\\
\\
c_1x^2+(c_2+d_1)xy+d_2y^2+c_4x^3+(c_5+d_3)xy^2+(c_3+d_4)x^2y\\
+d_5y^3+c_6x^4+(c_7+d_6)x^3y+(c_8+d_7)x^2y^2+(c_9+d_8)xy^3+d_9y^4\\
+c_{10}x^5+(c_{11}-d_1+d_{10})x^4y+(c_1+c_{12}-d_2+d_{11})x^3y^2\\
+(c_2+c_{13}-d_1+d_{12})x^2y^3+(c_1+c_{14}-d_2+d_{13})xy^4\\
+(c_2+d_{14})y^5-d_4x^5y+(c_4-d_3)x^4y^2+(c_3-d_4-d_5)x^3y^3\\
+(c_4+c_5-d_3)x^2y^4+(c_3-d_5)xy^5+c_5y^6-d_6x^6y+(c_6-d_7)x^5y^2\\
+(c_7-d_6-d_8)x^4y^3+(c_6+c_8-d_7-d_9)x^3y^4+(c_7+c_9-d_8)x^2y^5\\
+(c_8-d_9)xy^6+c_9y^7-d_{10}x^7y+(c_{10}-d_{11})x^6y^2+(c_{11}-d_{10}-d_{12})x^5y^3\\
+(c_{10}+c_{12}-d_{11}-d_{13})x^4y^4+(c_{11}+c_{13}-d_{12}-d_{14})x^3y^5\\
+(c_{12}+c_{14}-d_{13})x^2y^6+(c_{13}-d_{14})xy^7+c_{14}y^8 \qquad \text{for } y<0.
\end{cases}$$

We transform system (2) into the form

$$\frac{dR}{d\theta} = \begin{cases} \varepsilon \dfrac{3}{2(R^2+3\cos(\theta))}\Big(A(\theta;a,b)R + B(\theta;a,b)R^2 + C(\theta;a,b)R^3 + D(\theta;a,b)R^4 \\ \qquad + E(\theta;a,b)R^5 + F(\theta;a,b)R^6 + G(\theta;a,b)R^7\Big) + O(\varepsilon^2) \qquad \text{if } y>0, \\ \\ \varepsilon \dfrac{3}{2(R^2+3\cos(\theta))}\Big(A(\theta;c,d)R + B(\theta;c,d)R^2 + C(\theta;c,d)R^3 + D(\theta;c,d)R^4 \\ \qquad + E(\theta;c,d)R^5 + F(\theta;c,d)R^6 + G(\theta;c,d)R^7\Big) + O(\varepsilon^2) \qquad \text{if } y<0, \end{cases} \tag{7}$$

where

$$A(\theta;a,b) = 9(a_1\cos(\theta)^4 + (a_2+b_1)\cos(\theta)^3\sin(\theta) + b_2\cos(\theta)^2\sin(\theta)^2),$$

$$\begin{aligned} B(\theta;a,b) &= (R^2+3\cos(\theta))^{\frac{2}{3}}(3a_4\cos(\theta)^4 + 3(a_5+b_3)\cos(\theta)^2\sin(\theta)^2 \\ &\quad + 3(a_3+b_4)\cos(\theta)^3\sin(\theta) + 3b_5\cos(\theta)\sin(\theta)^3), \end{aligned}$$

$$\begin{aligned} C(\theta;a,b) &= 6(a_1\cos(\theta)^3 + (a_2+b_1)\cos(\theta)^2\sin(\theta) + b_2\cos(\theta)\sin(\theta)^2 \\ &\quad + 3(R^2+3\cos(\theta))^{\frac{1}{3}}(a_6\cos(\theta)^5 + (a_7+b_6)\cos(\theta)^4\sin(\theta) \\ &\quad + (a_8+b_7)\cos(\theta)^3 sin(\theta)^2 + (a_9+b_8)\cos(\theta)^2\sin(\theta)^3 + b_9\cos(\theta)\sin(\theta)^4), \end{aligned}$$

$$\begin{aligned} D(\theta;a,b) &= (R^2+3\cos(\theta))^{\frac{2}{3}}(a_4\cos(\theta)^3 + (a_5+b_3)\cos(\theta)\sin(\theta)^2 \\ &\quad + (a_3+b_4)\cos(\theta)^2\sin(\theta) + b_5\sin(\theta)^3 + 3(a_{10}\cos(\theta)^6 \\ &\quad + (a_{11}-b_1+b_{10})\cos(\theta)^5\sin(\theta) + (a_1+a_{12}-b_2+b_{11})\cos(\theta)^4\sin(\theta)^2 \\ &\quad + (a_2+a_{13}-b_1+b_{12})\cos(\theta)^3\sin(\theta)^3 + (a_1+a_{14}-b_2+b_{13})\cos(\theta)^2\sin(\theta)^4 \\ &\quad + (a_2+b_{14})\cos(\theta)\sin(\theta)^5, \end{aligned}$$

$$\begin{aligned} E(\theta;a,b) &= a_1\cos(\theta)^2 + (a_2+b_1)\cos(\theta)\sin(\theta) + b_2\sin(\theta)^2 \\ &\quad + (R^2+3\cos(\theta))^{\frac{1}{3}}(a_6\cos(\theta)^4 + (a_7+b_6)\cos(\theta)^3\sin(\theta) + (a_8+b_7)\cos(\theta)^2\sin(\theta)^2 \\ &\quad + (a_9+b_8)\cos(\theta)\sin(\theta)^3 + b_9\sin(\theta)^4) + (R^2+3\cos(\theta))^{\frac{2}{3}}(-b_4\cos(\theta)^5\sin(\theta) \\ &\quad + (a_4-b_3)\cos(\theta)^4\sin(\theta)^2 + (a_3-b_4-b_5)\cos(\theta)^3\sin(\theta)^3 + (a_4+a_5-b_3)\cos(\theta)^2\sin(\theta)^4 \\ &\quad + (a_3-b_5)\cos(\theta)\sin(\theta)^5 + a_5\sin(\theta)^6), \end{aligned}$$

$$\begin{aligned} F(\theta;a,b) &= a_{10}\cos(\theta)^6 + (a_{11}-b_1+b_{10})\cos(\theta)^4\sin(\theta) + (a_1+a_{12}-b_2+b_{11})\cos(\theta)^3\sin(\theta)^2 \\ &\quad + (a_2+a_{13}-b_1+b_{12})\cos(\theta)^2\sin(\theta)^3 + (a_1+a_{14}-b_2+b_{13})\cos(\theta)\sin(\theta)^4 \\ &\quad + (a_2+b_{14})\sin(\theta)^5 + (R^2+3\cos(\theta))^{\frac{1}{3}}(-b_6\cos(\theta)^6\sin(\theta) + (a_6-b_7)\cos(\theta)^5\sin(\theta)^2 \\ &\quad + (a_7-b_6-b_8)\cos(\theta)^4\sin(\theta)^3 + (a_6+a_8-b_7-b_9)\cos(\theta)^3\sin(\theta)^4 \\ &\quad + (a_7+a_9-b_8)\cos(\theta)^2\sin(\theta)^5 + (a_8-b_9)\cos(\theta)\sin(\theta)^6 + a_9\sin(\theta)^7), \end{aligned}$$

$$\begin{aligned} G(\theta;a,b) &= -b_{10}\cos(\theta)^7\sin(\theta) + (a_{10}-b_{11})\cos(\theta)^6\sin(\theta)^2 + (a_{11}-b_{10}-b_{12})\cos(\theta)^5\sin(\theta)^3 \\ &\quad + (a_{10}+a_{12}-b_{11}-b_{13})\cos(\theta)^4\sin(\theta)^4 + (a_{11}+a_{13}-b_{12}-b_{14})\cos(\theta)^3\sin(\theta)^5 \\ &\quad + (a_{12}+a_{14}-b_{13})\cos(\theta)^2\sin(\theta)^6 + (a_3-b_{14})\cos(\theta)\sin(\theta)^7 + a_{14}\sin(\theta)^8, \end{aligned}$$

where $a = (a_1, ..., a_{14}), b = (b_1, ..., b_{14}), c = (c_1, ..., c_{14})$ and $d = (d_1, ..., d_{14})$.

The discontinuous differential system (7) is under the assumptions of Theorem 2, so we must study the zeros of the averaged function $f : (0, 1) \longrightarrow \mathbb{R}$ where

$$\begin{aligned} f(R) &= \frac{3}{2}\int_0^{\pi} \frac{A(\theta;a,b)R + B(\theta;a,b)R^2 + C(\theta;a,b)R^3 + D(\theta;a,b)R^4}{R^2+3\cos(\theta)} \\ &\quad + \frac{E(\theta;a,b)R^5 + F(\theta;a,b)R^6 + G(\theta;a,b)R^7}{R^2+3\cos(\theta)}d\theta \\ &\quad + \frac{3}{2}\int_{\pi}^{2\pi} \frac{A(\theta;c,d)R + B(\theta;c,d)R^2 + C(\theta;c,d)R^3 + D(\theta;c,d)R^4}{R^2+3\cos(\theta)} \\ &\quad + \frac{E(\theta;c,d)R^5 + F(\theta;c,d)R^6 + G(\theta;c,d)R^7}{R^2+3\cos(\theta)}d\theta. \end{aligned}$$

Thus the function $f$ can be written as

$$\begin{aligned} f(R) &= a_1g_1 + a_2g_2 + a_3g_3 + a_4g_4 + a_5g_5 + a_6g_6 + a_7g_7 + a_8g_8 + a_9g_9 \\ &\quad + a_{10}g_{10} + a_{11}g_{11} + a_{12}g_{12} + a_{13}g_{13} + a_{14}g_{14} + b_1g_{15} + b_2g_{16} \\ &\quad + b_3g_{17} + b_4g_{18} + b_5g_{19} + b_6g_{20} + b_7g_{21} + b_8g_{22} + b_9g_{23} + b_{10}g_{24} \\ &\quad + b_{11}g_{25} + b_{12}g_{26} + b_{13}g_{27} + b_{14}g_{28} + c_1g_{29} + c_2g_{30} + c_3g_{31} + c_4g_{32} \\ &\quad + c_5g_{33} + c_6g_{34} + c_7g_{35} + c_8g_{36} + c_9g_{37} + c_{10}g_{38} + c_{11}g_{39} + c_{12}g_{40} \\ &\quad + c_{13}g_{41} + c_{14}g_{42} + d_1g_{43} + d_2g_{44} + d_3g_{45} + d_4g_{46} + d_5g_{47} + d_6g_{48} \\ &\quad + d_7g_{49} + d_8g_{50} + d_9g_{51} + d_{10}g_{52} + d_{11}g_{53} + d_{12}g_{54} + d_{13}g_{55} + d_{14}g_{56}. \end{aligned}$$

The expressions of $g_i(R)$, $i \in \{1, \ldots, 56\}$ are respectively presented in the Appendix 4.

**Lemma 1** *Let $f : \mathbb{R} \to \mathbb{R}$ be a $2\pi$-periodic function having the form*

$$f(\theta) = \cos^{\alpha}\theta \sin^{\beta}\theta (R^2 + 3\cos\theta)^{\gamma} \quad or \quad f(\theta) = \frac{\cos^{\alpha}\theta\sin^{\beta}\theta}{(R^2+3\cos\theta)^{\gamma}},$$

*where $\gamma = 0, \frac{1}{3}, \frac{2}{3}, 1$, $\beta = 2p$ and $\alpha, p \in \mathbb{N}$. We have:*

$$\int_0^{\pi} f(\theta)d\theta = \int_{\pi}^{2\pi} f(\theta)d\theta,$$

*in the both cases.*

**Proof.** Putting $z = 2\pi - \theta$, in both formulas, we get

$$\int_0^{\pi} f(\theta)d\theta = -\int_{2\pi}^{\pi} f(2\pi - z)dz = \int_{\pi}^{2\pi} f(z)dz = \int_{\pi}^{2\pi} f(\theta)d\theta.$$

After using the manipulator Maple, we obtain that

$$g_1 = g_{16} = g_{29} = g_{44}, \qquad g_2 = \frac{5}{2}g_{15} = -g_{30} = -\frac{5}{2}g_{43},$$

$$g_3 = g_{31}, \qquad g_7 = g_{35},$$

$$g_{10} = -g_{25} = g_{38} = -g_{53}, \qquad g_{12} = -g_{27} = g_{40} = -g_{55},$$

$$g_{11} = -g_{39}, \qquad g_{13} = -g_{41}, \qquad g_{14} = g_{42}, \qquad g_{17} = g_{45},$$

$$g_{18} = g_{46}, \qquad g_{20} = g_{48}, \qquad g_{24} = -g_{52}, \qquad g_{26} = -g_{54},$$

$$g_{28} = -g_{56},$$

and applying the Lemma 1, we get

$$g_4 = g_{32}, \quad g_{17} = g_{45},$$
$$g_5 = g_{33}, \quad g_{21} = g_{49},$$
$$g_6 = g_{34}, \quad g_{23} = g_{51},$$
$$g_8 = g_{36}.$$

So the function $f$ becomes

$$\begin{aligned} f(R) = &(a_1 + b_2 + c_1 + d_2)g_1 + (a_2 + \frac{5}{2}b_1 - c_2 - \frac{5}{2}d_1)g_2 + (a_3 + c_3)g_3 + (a_4 + c_4)g_4 \\ &+ (a_5 + c_5)g_5 + (a_6 + c_6)g_6 + (a_7 + c_7)g_7 + (a_8 + c_8)g_8 + a_9 g_9 \\ &+ (a_{10} - b_{11} + c_{10} - d_{11})g_{10} + (a_{11} - c_{11})g_{11} + (a_{12} - b_{13} + c_{12} - d_{13})g_{12} \\ &+ (a_{13} - c_{13})g_{13} + (a_{14} + c_{14})g_{14} + (b_3 + d_3)g_{17} + (b_4 + d_4)g_{18} + b_5 g_{19} \\ &+ (b_6 - d_6)g_{20} + (b_7 + d_7)g_{21} + b_8 g_{22} + (b_9 + d_9)g_{23} + (b_{10} - d_{10})g_{24} \\ &+ (b_{12} - d_{12})g_{25} + (b_{14} - d_{14})g_{28} + c_9 g_{37} + d_5 g_{47} + d_8 g_{50}. \end{aligned}$$

Out of the 27 functions $G_i = g_i : (\sqrt{3}, +\infty) \to \mathbb{R}$, $i \in \{\{1, \ldots, 14\} \cup \{17, \ldots, 25\} \cup \{28, 37, 47, 50\}\}$, we have that 20 are linearly independent. Indeed, using the software *Mathematica 11.0* to calculate the Taylor expansions for those 27 functions in the variable $R$ until its $30^{th}$ power around $R = 2$, which are too long and therefore they are not presented here, we construct a $27 \times 31$ matrix, where in the $k$ row we place the 31 coefficients of $R^0, R^1, \ldots, R^{30}$ of the Taylor expansion of $G_k$,

$$k \in \{\{1, \ldots, 14\} \cup \{17, \ldots, 25\} \cup \{28, 37, 47, 50\}\},$$

and we conclude that the rank of such matrix is 20.

By Proposition 1 since there are 20 linearly independent functions among the 27 previously described, then there exists a linear combination of them with at least 19 zeros, because all the coefficients of these functions are linearly independent, as it is easy to check. Thus there exist $R_1, R_2, \ldots, R_{19} \in (\sqrt{3}, +\infty)$ and coefficients $a_i, b_i, c_i, d_i \in \mathbb{R}$, $i \in \{1, \ldots, 14\}$ such that $f(R_k) = 0$, $k \in \{1, \ldots, 19\}$.

In summary, we conclude that there are planar discontinuous quartic polynomial differential systems (2) having at least 19 limit cycles bifurcating from the periodic orbits of the periodic annulus of the uniform isochronous center located at the origin of the unperturbed differential system (1), using the averaging theory of first order for discontinuous differential systems. This completes the proof of theorem 1. ■

# 4 Appendix: Averaging functions $g_i(R), i \in \{1, \ldots, 56\}$

$$\begin{aligned} g_1 &= \frac{3}{2}\int_0^\pi (\cos(\theta)^2(R^2 + 3\cos(\theta)) + \cos(\theta)^3\sin(\theta)^2 + \cos(\theta)\sin(\theta)^4)d\theta, \\ g_2 &= \frac{3}{2}\int_0^\pi (\cos(\theta)\sin(\theta)(R^2 + 3\cos(\theta)) + \cos(\theta)^2\sin(\theta)^3 + \sin(\theta)^5)d\theta, \\ g_3 &= \frac{3}{2}\int_0^\pi (\cos(\theta)^2\sin(\theta)(R^2 + 3\cos(\theta))^{\frac{2}{3}} + \frac{(\cos(\theta)\sin(\theta)^5 + \cos(\theta)^3\sin(\theta)^3)}{(R^2 + 3\cos(\theta))^{\frac{1}{3}}})d\theta, \\ g_4 &= \frac{3}{2}\int_0^\pi (\cos(\theta)^3(R^2 + 3\cos(\theta))^{\frac{2}{3}} + \frac{(\cos(\theta)^2\sin(\theta)^4 + \cos(\theta)^4\sin(\theta)^2)}{(R^2 + 3\cos(\theta))^{\frac{1}{3}}})d\theta, \\ g_5 &= \frac{3}{2}\int_0^\pi (\cos(\theta)\sin(\theta)^2(R^2 + 3\cos(\theta))^{\frac{2}{3}} + \frac{(\cos(\theta)^2\sin(\theta)^4 + \sin(\theta)^6)}{(R^2 + 3\cos(\theta))^{\frac{1}{3}}})d\theta, \end{aligned}$$

$$g_6 = \frac{3}{2}\int_0^\pi (\cos(\theta)^4(R^2+3\cos(\theta))^{\frac{1}{3}} + \frac{(\cos(\theta)^3\sin(\theta)^4+\cos(\theta)^5\sin(\theta)^2)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_7 = \frac{3}{2}\int_0^\pi (\cos(\theta)^3\sin(\theta)(R^2+3\cos(\theta))^{\frac{1}{3}} + \frac{(\cos(\theta)^4\sin(\theta)^3+\cos(\theta)^2\sin(\theta)^5)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_8 = \frac{3}{2}\int_0^\pi (\cos(\theta)^2\sin(\theta)^2(R^2+3\cos(\theta))^{\frac{1}{3}} + \frac{(\cos(\theta)^3\sin(\theta)^4+\cos(\theta)\sin(\theta)^6)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_9 = \frac{3}{2}\int_0^\pi (\cos(\theta)\sin(\theta)^3(R^2+3\cos(\theta))^{\frac{1}{3}} + \frac{(\cos(\theta)^2\sin(\theta)^5+\sin(\theta)^7)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{10} = \frac{3}{2}\int_0^\pi (\cos(\theta)^5 + \frac{(\cos(\theta)^4\sin(\theta)^4+\cos(\theta)^6\sin(\theta)^2)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{11} = \frac{3}{2}\int_0^\pi (\cos(\theta)^4\sin(\theta) + \frac{(\cos(\theta)^3\sin(\theta)^5+\cos(\theta)^5\sin(\theta)^3)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{12} = \frac{3}{2}\int_0^\pi (\cos(\theta)^3\sin(\theta)^2 + \frac{(\cos(\theta)^4\sin(\theta)^4+\cos(\theta)^2\sin(\theta)^6)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{13} = \frac{3}{2}\int_0^\pi (\cos(\theta)^2\sin(\theta)^3 + \frac{(\cos(\theta)\sin(\theta)^7+\cos(\theta)^3\sin(\theta)^5)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{14} = \frac{3}{2}\int_0^\pi (\cos(\theta)\sin(\theta)^4 + \frac{(\cos(\theta)^2\sin(\theta)^6+\sin(\theta)^8)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{15} = \frac{3}{2}\int_0^\pi (\cos(\theta)\sin(\theta)(R^2+3\cos(\theta)) - (\cos(\theta)^2\sin(\theta)^3+\cos(\theta)^4\sin(\theta)))d\theta,$$
$$g_{16} = \frac{3}{2}\int_0^\pi (\sin(\theta)^2(R^2+3\cos(\theta)) - (\cos(\theta)\sin(\theta)^4+\cos(\theta)^3\sin(\theta)^2))d\theta,$$
$$g_{17} = \frac{3}{2}\int_0^\pi (\cos(\theta)\sin(\theta)^2(R^2+3\cos(\theta))^{\frac{2}{3}} - \frac{(\cos(\theta)^2\sin(\theta)^4+\cos(\theta)^4\sin(\theta)^2)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$
$$g_{18} = \frac{3}{2}\int_0^\pi (\cos(\theta)^2\sin(\theta)(R^2+3\cos(\theta))^{\frac{2}{3}} - \frac{(\cos(\theta)^5\sin(\theta)+\cos(\theta)^3\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$
$$g_{19} = \frac{3}{2}\int_0^\pi (\sin(\theta)^3(R^2+3\cos(\theta))^{\frac{2}{3}} - \frac{(\cos(\theta)\sin(\theta)^5+\cos(\theta)^3\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$
$$g_{20} = \frac{3}{2}\int_0^\pi (\cos(\theta)^3\sin(\theta)(R^2+3\cos(\theta))^{\frac{1}{3}} - \frac{(\cos(\theta)^6\sin(\theta)+\cos(\theta)^4\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$

$$g_{21} = \frac{3}{2}\int_0^\pi (\cos(\theta)^2\sin(\theta)^2(R^2+3\cos(\theta))^{\frac{1}{3}} - \frac{(\cos(\theta)^5\sin(\theta)^2+\cos(\theta)^3\sin(\theta)^4)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{22} = \frac{3}{2}\int_0^\pi (\cos(\theta)\sin(\theta)^3(R^2+3\cos(\theta))^{\frac{1}{3}} - \frac{(\cos(\theta)^2\sin(\theta)^5+\cos(\theta)^4\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{23} = \frac{3}{2}\int_0^\pi (\sin(\theta)^4(R^2+3\cos(\theta))^{\frac{1}{3}} - \frac{(\cos(\theta)^3\sin(\theta)^4+\cos(\theta)\sin(\theta)^6)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{24} = \frac{3}{2}\int_0^\pi (\cos(\theta)^4\sin(\theta) + \frac{(\cos(\theta)^5\sin(\theta)^3+\cos(\theta)^7\sin(\theta))}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{25} = \frac{3}{2}\int_0^\pi (\cos(\theta)^3\sin(\theta)^2 - \frac{(\cos(\theta)^4\sin(\theta)^4+\cos(\theta)^6\sin(\theta)^2)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{26} = \frac{3}{2}\int_0^\pi (\cos(\theta)^2\sin(\theta)^3 - \frac{(\cos(\theta)^3\sin(\theta)^5+\cos(\theta)^5\sin(\theta)^3)}{(R^2+3\cos(\theta))})d\theta,$$

$$g_{27} = \frac{3}{2}\int_0^{\pi}(\cos(\theta)\sin(\theta)^4 - \frac{(\cos(\theta)^2\sin(\theta)^6 + \cos(\theta)^4\sin(\theta)^4)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{28} = \frac{3}{2}\int_0^{\pi}(\sin(\theta)^5 - \frac{(\cos(\theta)^3\sin(\theta)^5 + \cos(\theta)\sin(\theta)^7)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{29} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^2(R^2+3\cos(\theta)) + \cos(\theta)^3\sin(\theta)^2 + \cos(\theta)\sin(\theta)^4)d\theta,$$
$$g_{30} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)\sin(\theta)(R^2+3\cos(\theta)) + \cos(\theta)^2\sin(\theta)^3 + \sin(\theta)^5)d\theta,$$
$$g_{31} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^2\sin(\theta)(R^2+3\cos(\theta))^{\frac{2}{3}} + \frac{(\cos(\theta)\sin(\theta)^5 + \cos(\theta)^3\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$
$$g_{32} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^3(R^2+3\cos(\theta))^{\frac{2}{3}} + \frac{(\cos(\theta)^2\sin(\theta)^4 + \cos(\theta)^4\sin(\theta)^2)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$
$$g_{33} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)\sin(\theta)^2(R^2+3\cos(\theta))^{\frac{2}{3}} + \frac{(\cos(\theta)^2\sin(\theta)^4 + \sin(\theta)^6)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$
$$g_{34} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^4(R^2+3\cos(\theta))^{\frac{1}{3}} + \frac{(\cos(\theta)^3\sin(\theta)^4 + \cos(\theta)^5\sin(\theta)^2)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{35} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^3\sin(\theta)(R^2+3\cos(\theta))^{\frac{1}{3}} + \frac{(\cos(\theta)^4\sin(\theta)^3 + \cos(\theta)^2\sin(\theta)^5)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{36} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^2\sin(\theta)^2(R^2+3\cos(\theta))^{\frac{1}{3}} + \frac{(\cos(\theta)^3\sin(\theta)^4 + \cos(\theta)\sin(\theta)^6)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{37} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)\sin(\theta)^3(R^2+3\cos(\theta))^{\frac{1}{3}} + \frac{(\cos(\theta)^2\sin(\theta)^5 + \sin(\theta)^7)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{38} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^5 + \frac{(\cos(\theta)^4\sin(\theta)^4 + \cos(\theta)^6\sin(\theta)^2)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{39} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^4\sin(\theta) + \frac{(\cos(\theta)^3\sin(\theta)^5 + \cos(\theta)^5\sin(\theta)^3)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{40} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^3\sin(\theta)^2 + \frac{(\cos(\theta)^4\sin(\theta)^4 + \cos(\theta)^2\sin(\theta)^6)}{(R^2+3\cos(\theta))})d\theta,$$

$$g_{41} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)^2\sin(\theta)^3 + \frac{(\cos(\theta)\sin(\theta)^7 + \cos(\theta)^3\sin(\theta)^5)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{42} = \frac{3}{2}\int_{\pi}^{2\pi}(\cos(\theta)\sin(\theta)^4 + \frac{(\cos(\theta)^2\sin(\theta)^6 + \sin(\theta)^8)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{43} = \frac{3}{2}\int_0^{\pi}(\cos(\theta)\sin(\theta)(R^2+3\cos(\theta)) - (\cos(\theta)^2\sin(\theta)^3 + \cos(\theta)^4\sin(\theta)))d\theta,$$
$$g_{44} = \frac{3}{2}\int_0^{\pi}(\sin(\theta)^2(R^2+3\cos(\theta)) - (\cos(\theta)\sin(\theta)^4 + \cos(\theta)^3\sin(\theta)^2))d\theta,$$
$$g_{45} = \frac{3}{2}\int_0^{\pi}(\cos(\theta)\sin(\theta)^2(R^2+3\cos(\theta))^{\frac{2}{3}} - \frac{(\cos(\theta)^2\sin(\theta)^4 + \cos(\theta)^4\sin(\theta)^2)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$
$$g_{46} = \frac{3}{2}\int_0^{\pi}(\cos(\theta)^2\sin(\theta)(R^2+3\cos(\theta))^{\frac{2}{3}} - \frac{(\cos(\theta)^5\sin(\theta) + \cos(\theta)^3\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$
$$g_{47} = \frac{3}{2}\int_0^{\pi}(\sin(\theta)^3(R^2+3\cos(\theta))^{\frac{2}{3}} - \frac{(\cos(\theta)\sin(\theta)^5 + \cos(\theta)^3\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{1}{3}}})d\theta,$$

$$g_{48}=\frac{3}{2}\int_0^{\pi}(\cos(\theta)^3\sin(\theta)(R^2+3\cos(\theta))^{\frac{1}{3}}-\frac{(\cos(\theta)^6\sin(\theta)+\cos(\theta)^4\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{49}=\frac{3}{2}\int_0^{\pi}(\cos(\theta)^2\sin(\theta)^2(R^2+3\cos(\theta))^{\frac{1}{3}}-\frac{(\cos(\theta)^5\sin(\theta)^2+\cos(\theta)^3\sin(\theta)^4)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{50}=\frac{3}{2}\int_0^{\pi}(\cos(\theta)\sin(\theta)^3(R^2+3\cos(\theta))^{\frac{1}{3}}-\frac{(\cos(\theta)^2\sin(\theta)^5+\cos(\theta)^4\sin(\theta)^3)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{51}=\frac{3}{2}\int_0^{\pi}(\sin(\theta)^4(R^2+3\cos(\theta))^{\frac{1}{3}}-\frac{(\cos(\theta)^3\sin(\theta)^4+\cos(\theta)\sin(\theta)^6)}{(R^2+3\cos(\theta))^{\frac{2}{3}}})d\theta,$$
$$g_{52}=\frac{3}{2}\int_0^{\pi}(\cos(\theta)^4\sin(\theta)+\frac{(\cos(\theta)^5\sin(\theta)^3+\cos(\theta)^7\sin(\theta))}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{53}=\frac{3}{2}\int_0^{\pi}(\cos(\theta)^3\sin(\theta)^2-\frac{(\cos(\theta)^4\sin(\theta)^4+\cos(\theta)^6\sin(\theta)^2)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{54}=\frac{3}{2}\int_0^{\pi}(\cos(\theta)^2\sin(\theta)^3-\frac{(\cos(\theta)^3\sin(\theta)^5+\cos(\theta)^5\sin(\theta)^3)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{55}=\frac{3}{2}\int_0^{\pi}(\cos(\theta)\sin(\theta)^4-\frac{(\cos(\theta)^2\sin(\theta)^6+\cos(\theta)^4\sin(\theta)^4)}{(R^2+3\cos(\theta))})d\theta,$$
$$g_{56}=\frac{3}{2}\int_0^{\pi}(\sin(\theta)^5-\frac{(\cos(\theta)^3\sin(\theta)^5+\cos(\theta)\sin(\theta)^7)}{(R^2+3\cos(\theta))})d\theta.$$

Using the software Mathematica 11.0, we obtain

$$g_1=\frac{3\pi R^2}{4};$$

$$g_2=5;$$

$$g_3=\tfrac{3}{4}\left(\tfrac{1}{440\sqrt[3]{R^2-3}}(10R^6+222R^2+10\left(\sqrt[3]{1-\tfrac{3}{R^2}}-1\right)R^8+\left(98-63\sqrt[3]{1-\tfrac{3}{R^2}}\right)R^4-1008\right)$$
$$-\left(\tfrac{1}{440\sqrt[3]{R^2+3}}(-10R^6-222R^2+10\left(\sqrt[3]{\tfrac{3}{R^2}+1}-1\right)R^8+\left(98-63\sqrt[3]{\tfrac{3}{R^2}+1}\right)R^4-1008)\right)\Bigg);$$

$$g_4=\tfrac{1}{1760(R^2-3)^{\frac{1}{3}}}\Bigg(-\pi\left(-4320+138R^2+19R^4+20R^6+10R^8\right)\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{1}{3};1;-\tfrac{6}{R^2-3}\right)$$
$$+\pi(3+R^2)\Bigg(-576+169R^2-40R^4+10R^6\Big)\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{4}{3};1;-\tfrac{6}{R^2-3}\right)\Bigg);$$

$$g_5=\tfrac{1}{352(-3+R^2)^{\frac{1}{3}}}\pi\left((720-78R^2-49R^4+4R^6+2R^8\right)\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{1}{3};1;-\tfrac{6}{R^2-3}\right)$$
$$-(3+R^2)(96-19R^2-8R^4+2R^6\Big)\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{4}{3};1;-\tfrac{6}{R^2-3}\right)\Bigg);$$

$$g_6=\tfrac{3\pi}{2912(R^2-3)^{2/3}}\Bigg(\left(R^2+3\right)\left(6R^8-12R^6+17R^4-58R^2-352\right)\,{}_2F_1\left(\tfrac{1}{2},\tfrac{2}{3};1;-\tfrac{6}{R^2-3}\right)$$
$$-\left(R^2-3\right)\left(6R^8-7R^4-1056\right)\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{2}{3};1;-\tfrac{6}{R^2-3}\right)\Bigg);$$

$$g_7=\frac{3}{2}\Bigg(\frac{1}{728(R^2-3)^{2/3}}\Bigg(-12R^8+96R^4+\Big(-179\left(1-\tfrac{3}{R^2}\right)^{2/3}-144\Big)R^2+\Big(6-6\left(1-\tfrac{3}{R^2}\right)^{2/3}\Big)R^{10}$$
$$+\Big(137\left(1-\tfrac{3}{R^2}\right)^{2/3}-58\Big)R^6+648\Bigg)-\frac{1}{728(R^2+3)^{2/3}}\Bigg(12R^8-96R^4+\Big(-179\left(\tfrac{3}{R^2}+1\right)^{2/3}-144\Big)R^2$$
$$+\Big(6-6\left(\tfrac{3}{R^2}+1\right)^{2/3}\Big)R^{10}+\Big(137\left(\tfrac{3}{R^2}+1\right)^{2/3}-58\Big)R^6-648\Bigg)\Bigg);$$

$$g_8=\frac{3}{2912(R^2-3)^{2/3}}\Bigg(\pi\left(R^2-3\right)\left(6R^8-85R^4+504\right){}_2F_1\left(-\tfrac{1}{2},\tfrac{2}{3};1;-\tfrac{6}{R^2-3}\right)$$
$$-\pi\left(R^2+3\right)\left(6R^8-12R^6-61R^4+98R^2+168\right){}_2F_1\Big(\tfrac{1}{2},\tfrac{2}{3};1;-\tfrac{6}{R^2-3}\Big)\Bigg);$$

$$g_9=\frac{3}{2}\Bigg(\frac{1}{364(R^2-3)^{2/3}}\Bigg(6R^8-126R^4+\Big(369\left(1-\tfrac{3}{R^2}\right)^{2/3}-513\Big)R^2+\Big(3\left(1-\tfrac{3}{R^2}\right)^{2/3}-3\Big)R^{10}$$
$$+\Big(68-62\left(1-\tfrac{3}{R^2}\right)^{2/3}\Big)R^6+1080\Bigg)-\frac{1}{364(R^2+3)^{2/3}}\Bigg(-6R^8+126R^4+\Big(369\left(\tfrac{3}{R^2}+1\right)^{2/3}-513\Big)R^2$$
$$+\Big(3\left(\tfrac{3}{R^2}+1\right)^{2/3}-3\Big)R^{10}+\Big(68-62\left(\tfrac{3}{R^2}+1\right)^{2/3}\Big)R^6-1080\Bigg)\Bigg);$$

$$g_{10}=\frac{1}{3888}\left(-\pi R^2\left(36R^4+81+24R^6\sqrt{\tfrac{-3+R^2}{3+R^2}}+8R^8(-1+\sqrt{\tfrac{-3+R^2}{3+R^2}})\right)\right);$$

$$g_{11}=\frac{1}{2430}\Bigg(6\left(-5R^8+30R^4+297\right)+5\left(R^4-9\right)R^6\left(\log\left(R^2+3\right)-\log\left(R^2-3\right)\right)\Bigg);$$

$$g_{12}=\frac{\pi R^2}{3888}\left(-8R^8+108R^4+8\left(R^2-3\right)^{3/2}\left(R^2+3\right)^{3/2}R^2-243\right);$$

$$g_{13}=\frac{1}{2430}\Bigg(6\left(5R^8-75R^4+378\right)+5\left(R^5-9R\right)^2\left(\log\left(R^2-3\right)-\log\left(R^2+3\right)\right)\Bigg);$$

$$g_{14}=\frac{1}{3888}\Bigg(8\left(9-R^4\right)^{5/2}\log\left(\tfrac{3-R^2}{\sqrt{9-R^4}}\right)-8\left(9-R^4\right)^{5/2}\log\left(\tfrac{R^2-3}{\sqrt{9-R^4}}\right)+\pi R^2\left(8R^8-180R^4+1215\right)\Bigg);$$

$$g_{17}=\frac{1}{1760(R^2-3)^{1/3}}\,\pi\Bigg((960-214R^2-157R^4+20R^6+10R^8)\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{1}{3};1;-\tfrac{6}{R^2-3}\right)$$
$$-(3+R^2)(128-7R^2-40R^4+10R^6)\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{4}{3};1;-\tfrac{6}{R^2-3}\right)\Bigg);$$

$$g_{18}=\frac{3}{2}\Bigg(\frac{1}{880\sqrt[3]{R^2-3}}\left(10R^6+310R^2+10\left(\sqrt[3]{1-\tfrac{3}{R^2}}-1\right)R^8+\left(3\sqrt[3]{1-\tfrac{3}{R^2}}+10\right)R^4-480\right)\Bigg)$$
$$-\left(\frac{1}{880\sqrt[3]{R^2+3}}\left(-10R^6-310R^2+10\left(\sqrt[3]{\tfrac{3}{R^2}+1}-1\right)R^8+\left(3\sqrt[3]{\tfrac{3}{R^2}+1}+10\right)R^4-480\right)\right);$$

$$g_{19}=\frac{3}{2}\Bigg(\frac{1}{880\sqrt[3]{R^2-3}}\Big(-10R^6+834R^2+\left(10-10\sqrt[3]{1-\tfrac{3}{R^2}}\right)R^8+\left(239\sqrt[3]{1-\tfrac{3}{R^2}}-274\right)$$
$$R^4-576\Big)-\frac{1}{880\sqrt[3]{R^2+3}}\Big(10R^6-834R^2+\left(10-10\sqrt[3]{\tfrac{3}{R^2}+1}\right)R^8+\left(239\sqrt[3]{\tfrac{3}{R^2}+1}-274\right)$$
$$R^4-576\Big)\Bigg);$$

$$g_{20}=\frac{3}{2}\Bigg(\frac{1}{364(R^2-3)^{2/3}}\Bigg(168+(-98-70\left(1-\frac{3}{R^2}\right)^{2/3})R^2-4R^4-3\left(1+(1-\frac{3}{R^2}\right)^{2/3})R^6$$
$$-6R^8+(3-3\left(1-\frac{3}{R^2}\right)^{2/3})R^{10}\Bigg)-\frac{1}{364(R^2+3)^{2/3}}\Bigg(-168+(-98-70\left(1+\frac{3}{R^2}\right)^{2/3})R^2$$
$$+4R^4-3\left(1+(1+\frac{3}{R^2}\right)^{2/3})R^6+6R^8+(3-3\left(1+\frac{3}{R^2}\right)^{2/3})R^{10}\Bigg)\Bigg);$$

$$g_{21}=\frac{3}{2912(R^2-3)^{2/3}}\Bigg(3\pi\left(R^2-3\right)\left(2R^8-11R^4+64\right)\,{}_2F_1\left(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}\right)$$
$$-\pi\left(R^2+3\right)\left(6R^8-12R^6-9R^4-6R^2+64\right)\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}\right)\Bigg);$$

$$g_{22}=\frac{3}{2}\Bigg(\frac{1}{364(R^2-3)^{2/3}}\Bigg(6R^8-74R^4+\left(-60\left(1-\frac{3}{R^2}\right)^{2/3}-123\right)R^2+\left(3\left(1-\frac{3}{R^2}\right)^{2/3}-3\right)R^{10}$$
$$+\left(42-36\left(1-\frac{3}{R^2}\right)^{2/3}\right)R^6+144)-\frac{1}{364(R^2+3)^{2/3}}\Bigg(-6R^8+74R^4+\Bigg(-60\left(\frac{3}{R^2}+1\right)^{2/3}$$
$$-123\Bigg)R^2+\left(3\left(\frac{3}{R^2}+1\right)^{2/3}-3\right)R^{10}+\left(42-36\left(\frac{3}{R^2}+1\right)^{2/3}\right)R^6-144\Bigg)\Bigg);$$

$$g_{23}=\frac{9\pi}{2912(R^2-3)^{2/3}}\Bigg(\left(R^2+3\right)\left(2R^8-4R^6-29R^4+50R^2-48\right)\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}\right)$$
$$-\left(R^2-3\right)\left(2R^8-37R^4-144\right)\,{}_2F_1\left(-\frac{1}{2},\frac{2}{3};1;-\frac{6}{R^2-3}\right)\Bigg)\Bigg);$$

$$g_{24}=\frac{1}{2430}\Bigg(6\left(5\left(R^4+3\right)R^4+324\right)+5R^{10}\left(\log\left(R^2-3\right)-\log\left(R^2+3\right)\right)\Bigg);$$

$$g_{25}=\frac{1}{2430}\Bigg(6\left(5\left(R^4-6\right)R^4+108\right)+5\left(R^4-9\right)R^6\left(\log\left(R^2-3\right)-\log\left(R^2+3\right)\right)\Bigg);$$

$$g_{28}=\frac{1}{2430}\Bigg(6\left(-5R^8+75R^4+432\right)-5\left(R^5-9R\right)^2\left(\log\left(R^2-3\right)-\log\left(R^2+3\right)\right)\Bigg);$$

$$g_{37}=\frac{3}{2}\Bigg(\frac{1}{364(R^2+3)^{2/3}}\Bigg(-6R^8+126R^4+\left(369\left(\frac{3}{R^2}+1\right)^{2/3}-513\right)R^2+\left(3\left(\frac{3}{R^2}+1\right)^{2/3}-3\right)R^{10}$$
$$+\left(68-62\left(\frac{3}{R^2}+1\right)^{2/3}\right)R^6-1080\Bigg)-\frac{1}{364(R^2-3)^{2/3}}\Bigg(6R^8-126R^4+\left(369\left(1-\frac{3}{R^2}\right)^{2/3}-513\right)R^2$$
$$+\left(3\left(1-\frac{3}{R^2}\right)^{2/3}-3\right)R^{10}+\left(68-62\left(1-\frac{3}{R^2}\right)^{2/3}\right)R^6+1080\Bigg)\Bigg);$$

$$g_{47}=\frac{3}{2}\Bigg(\frac{1}{880\sqrt[3]{R^2+3}}(10R^6-834R^2+\left(10-10\sqrt[3]{\frac{3}{R^2}+1}\right)R^8+\left(239\sqrt[3]{\frac{3}{R^2}+1}-274\right)$$
$$R^4-576)-\frac{1}{880\sqrt[3]{R^2-3}}(-10R^6+834R^2+\left(10-10\sqrt[3]{1-\frac{3}{R^2}}\right)R^8+\left(239\sqrt[3]{1-\frac{3}{R^2}}-274\right)$$
$$R^4-576)\Bigg);$$

$$g_{50}=\frac{3}{2}\Bigg(\frac{1}{364(R^2+3)^{2/3}}(-6R^8+74R^4+\left(-60\left(\frac{3}{R^2}+1\right)^{2/3}-123\right)R^2+\left(3\left(\frac{3}{R^2}+1\right)^{2/3}-3\right)R^{10}$$

$$+\left(42-36\left(\frac{3}{R^2}+1\right)^{2/3}\right)R^6-144\right)-\frac{1}{364(R^2-3)^{2/3}}\left(6R^8-74R^4+\left(-60\left(1-\frac{3}{R^2}\right)^{2/3}-123\right)R^2\right.$$

$$\left.+\left(3\left(1-\frac{3}{R^2}\right)^{2/3}-3\right)R^{10}+\left(42-36\left(1-\frac{3}{R^2}\right)^{2/3}\right)R^6+144\right)\Bigg);$$

where ${}_2F_1\,(a,b,c,z)$ is the hypergeometric function which has the series expansion

$$\sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

with

$$(a)_k=\begin{cases}1, & \text{if } k=0,\\ a(a+1)(a+2)...(a+k-1), & \text{if } k>0.\end{cases}$$

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