

# New Integral Inequalities Using Quasi-Convex Functions Via Generalized Integral Operators And Their Applications\*

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## Abstract

In this paper using generalized integral operators, we first obtain new interesting generalized improved Hölder integral inequality. Also, after deriving a new lemma using these operators, we give two results via quasi-convex functions. Some special cases of our results recapture known results. At the end, some error estimates are given to illustrate the applications and efficiency of the obtained results.

## 1 Introduction and Preliminaries

A function  $f : \mathcal{I} \subset \mathfrak{R} \rightarrow \mathfrak{R}$  is said to be convex, if

$$f(t\wp_1 + (1-t)\wp_2) \leq tf(\wp_1) + (1-t)f(\wp_2),$$

holds for all  $\wp_1, \wp_2 \in \mathcal{I}$  and  $t \in [0, 1]$ . Likewise  $f$  is concave if  $(-f)$  is convex.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1** *Let  $f : \mathcal{I} \subset \mathfrak{R} \rightarrow \mathfrak{R}$  be a convex function on  $\mathcal{I}$  and  $\wp_1, \wp_2 \in \mathcal{I}$  with  $\wp_1 < \wp_2$ . Then the following double inequality holds:*

$$f\left(\frac{\wp_1 + \wp_2}{2}\right) \leq \frac{1}{\wp_2 - \wp_1} \int_{\wp_1}^{\wp_2} f(x)dx \leq \frac{f(\wp_1) + f(\wp_2)}{2}. \quad (1)$$

The above inequality first published by Hermite in 1883 [11] and by Hadamard, independently, 10 years later [9]. The full history of this inequality can be found in [19].

Various extensions of this notion have been reported in the literature in recent years, see [1, 4, 6, 12, 13, 14, 16, 17, 22, 26]. Mo et al. in [20], introduced the following generalized convex function.

To facilitate understanding of this work, we present the following sets [28, 29], considering  $0 < \alpha \leq 1$ :

- the fractal set of integers  $\mathfrak{Z}^\alpha$  is defined by  $\mathfrak{Z}^\alpha = \{0^\alpha\} \cup \{\pm m^\alpha : m \in \mathfrak{N}\}$ ;
- the fractal set of rational numbers  $\mathfrak{Q}^\alpha$  is defined by

$$\mathfrak{Q}^\alpha = \{q^\alpha : q \in \mathfrak{Q}\} = \left\{ \left(\frac{m}{n}\right)^\alpha : m \in \mathfrak{Z}, n \in \mathfrak{N} \right\};$$

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- the fractal set of irrational numbers  $\mathfrak{J}^\alpha$  is defined by

$$\mathfrak{J}^\alpha = \{r^\alpha : r \in \mathfrak{J}\} = \left\{r^\alpha \neq \left(\frac{m}{n}\right)^\alpha\right\};$$

- the fractal real set  $\mathfrak{R}^\alpha$  is defined by  $\mathfrak{R}^\alpha = \mathfrak{Q}^\alpha \cup \mathfrak{J}^\alpha$ .

where  $\mathfrak{R}$ ,  $\mathfrak{R}^+$ ,  $\mathfrak{Q}$ ,  $\mathfrak{J}$ ,  $\mathfrak{Z}$  and  $\mathfrak{N}$  are the sets of real and positive real numbers, rational numbers, irrational numbers, integers and positive integers, respectively.

**Definition 1** Let  $f : \mathcal{I} \subset \mathfrak{R} \rightarrow \mathfrak{R}^\alpha$  be a function and  $\alpha \in (0, 1]$ . For any  $\wp_1, \wp_2 \in \mathcal{I}$  and  $t \in [0, 1]$ , if the following inequality holds:

$$f(t\wp_1 + (1-t)\wp_2) \leq t^\alpha f(\wp_1) + (1-t)^\alpha f(\wp_2),$$

then  $f$  is said to be generalized convex on  $\mathcal{I}$ .

The following are two interesting examples of generalized convex functions, see [20]:

1.  $f(t) = t^{\alpha p}$ , where  $t \geq 0$  and  $p > 1$ ;
2.  $g(t) = \mathcal{E}_\alpha(t^\alpha)$ ,  $t \in \mathfrak{R}$ , where  $\mathcal{E}_\alpha(t^\alpha) = \sum_{k=0}^{+\infty} \frac{t^{\alpha k}}{\Gamma(1+k\alpha)}$  is the Mittag-Leffler function and  $\Gamma(\cdot)$  is the well-known gamma function.

Now, we recall some basic and useful definitions and theorems as follows:

**Definition 2** ([12]) A function  $f : [\wp_1, \wp_2] \rightarrow \mathfrak{R}$  is said to be quasi-convex on  $[\wp_1, \wp_2]$ , if

$$f(tx_1 + (1-t)x_2) \leq \sup\{f(x_1), f(x_2)\}$$

for all  $x_1, x_2 \in [\wp_1, \wp_2]$  and  $t \in [0, 1]$ .

It's clear that any convex function is a quasi-convex function but there exist quasi-convex functions which are not convex, see [12]. The following famous inequality is known as Young inequality.

**Proposition 1** ([13]) If  $x_1, x_2 > 0$  and  $t \in [0, 1]$ , then

$$x_1^t x_2^{1-t} \leq tx_1 + (1-t)x_2.$$

Equality holds if and only if  $x_1 = x_2$ .

One of the areas of greatest development in mathematics today is that of Fractional and Generalized Calculus, the proliferation of integral and differential operators (whether local and global) in various applications and theoretical developments can be seen in many publications and research. A more general idea, on the need to use operators of various kinds as well as their classification, see [2, 3, 23, 27].

The following is the generalized derivative that we will use in our work, it was defined in [30] and independently in [21], where it was studied intensively. In the first work it was called Generalized Conformable Fractional Derivative (GCFD), in the second work it is specified that not only are proper conformables, that is, they "return" the ordinary derivative when  $\alpha \rightarrow 1$ , if not they can be non-conformables, and even improper conformables, i.e.  $F(t, \alpha) \rightarrow 1$  when  $t \rightarrow +\infty$ , see [7].

**Definition 3** Let  $f : [0, +\infty) \rightarrow \mathfrak{R}$ , where  $\mathcal{F}(t, \alpha)$  is some positive function defined on  $[0, +\infty)$  such that the following limit exist. The  $\mathcal{N}_{\mathcal{F}}$ -derivative of function  $f$  of order  $\alpha$  is defined by

$$\mathcal{N}_{\mathcal{F}}^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon \mathcal{F}(t, \alpha)) - f(t)}{\varepsilon} \quad (2)$$

for all  $t > 0$ , where  $\alpha \in (0, 1]$ .

If  $f$  is  $\alpha$ -differentiable in some  $(0, a)$  with  $a > 0$  and  $\lim_{t \rightarrow 0^+} \mathcal{N}_{\mathcal{F}}^{(\alpha)} f(t)$  exists, then we define

$$\mathcal{N}_{\mathcal{F}}^{(\alpha)} f(0) = \lim_{t \rightarrow 0^+} \mathcal{N}_{\mathcal{F}}^{(\alpha)} f(t).$$

Throughout this remaining paper, we will consider that the integral operator kernel  $\mathcal{T}(t, \alpha)$  given below is positive and absolutely continuous function with respect to first variable, more details can be found in [8, 30].

**Definition 4 ([8])** Let  $\mathcal{I} \subset \mathfrak{R}$  for  $\wp_1, \wp_2 \in \mathcal{I}$  and  $\alpha \in (0, 1]$ . The integral operators  $\mathcal{J}_{\mathcal{T}, \wp_1+}^{\alpha}$  and  $\mathcal{J}_{\mathcal{T}, \wp_2-}^{\alpha}$ , respectively, right and left, are defined for every locally integrable function  $f$  on  $\mathcal{I}$  as follows:

$$\mathcal{J}_{\mathcal{T}, \wp_1+}^{\alpha}(f)(t) = \int_{\wp_1}^t \frac{f(s)}{\mathcal{T}(t-s, \alpha)} ds, \quad t > \wp_1 \tag{3}$$

and

$$\mathcal{J}_{\mathcal{T}, \wp_2-}^{\alpha}(f)(t) = \int_t^{\wp_2} \frac{f(s)}{\mathcal{T}(s-t, \alpha)} ds, \quad t < \wp_2. \tag{4}$$

**Remark 1** There is no relation or formula between the functions  $\mathcal{F}$  and  $\mathcal{T}$  but if we know the function  $\mathcal{F}$ , then we can choose  $\mathcal{T}$  in such a way that Definition 4 will be well defined.

**Remark 2** It is easy to see that the case of the  $\mathcal{J}_{\mathcal{T}}^{\alpha}$  operator defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing the cases listed at the beginning of the work. Hence, we have

- 1) If  $\mathcal{F}(t, \alpha) = t^{1-\alpha}$ , then  $\mathcal{T}(t-s, \alpha) = \Gamma(\alpha)\mathcal{F}(t-s, \alpha)$  and from (3), we have the right side Riemann–Liouville fractional integrals  $(\mathcal{R}_{\wp_1+}^{\alpha} f)(t)$ . Similarly, from (4) we obtain the left derivative of Riemann–Liouville. Then its corresponding right differential operator is

$$\left(\mathcal{R}\mathcal{D}_{\wp_1+}^{\alpha} f\right)(t) = \frac{d}{dt}(\mathcal{R}_{\wp_1+}^{1-\alpha} f)(t).$$

Analogously, we obtain the left differential operator.

- 2) With  $\mathcal{F}(t, \alpha) = (1-\alpha)t$ , then  $\mathcal{T}(t-s, \alpha) = \Gamma(\alpha)\mathcal{F}(\ln t - \ln s, \alpha)t$  and from (3), we obtain the right Hadamard integral. The left Hadamard integral is obtained similarly from (4). The right derivative is

$$\left(\mathcal{H}\mathcal{D}_{\wp_1+}^{\alpha} f\right)(t) = t \frac{d}{dt}(\mathcal{H}_{\wp_1+}^{1-\alpha} f)(t).$$

Similarly for the left derivative.

- 3) The right Katugampola integral is obtained from (3), making

$$\mathcal{F}(t, \alpha) = t^{1-\alpha}, \quad r(t) = t^{\rho}, \quad \mathcal{T}(t-s, \alpha) = \rho^{\alpha-1} \frac{\Gamma(\alpha)}{\mathcal{F}(\rho, \alpha)} \frac{\mathcal{F}(r(t) - r(s), \alpha)}{r'(s)}.$$

Analogously for the left fractional integral. In this case, the right derivative is

$$\left(\mathcal{K}\mathcal{D}_{\wp_1+}^{\alpha, \rho} f\right)(t) = t^{1-\rho} \frac{d}{dt} \mathcal{K}_{\wp_1+}^{1-\alpha, \rho} f(t) = \mathcal{F}(t, \rho) \frac{d}{dt} \mathcal{K}_{\wp_1+}^{1-\alpha, \rho} f(t).$$

We can obtain the left derivative in the same way.

- 4) The solution of equation  $(-\Delta)^{-\frac{\alpha}{2}} \phi(u) = -f(u)$  called Riesz potential, is given by the expression

$$\phi(u) = \mathcal{C}_n^{\alpha} \int_{\mathfrak{R}^n} \frac{f(v)}{|u-v|^{n-\alpha}} dv,$$

where  $\mathcal{C}_n^{\alpha}$  is a constant, see [5, 10, 18]. Obviously, this solution can be expressed in terms of the operator (3) very easily.

- 5) Obviously, we can define the lateral derivative operators, respectively, right and left, in the case of our generalized derivative. One way is in the Definition 3 to consider values of  $\varepsilon$  greater or less than zero and, the other way is sufficient to consider them from the corresponding integral operator. To do this, just make use of the fact that, if  $f$  is differentiable, then  $\mathcal{N}_{\mathcal{F}}^{\alpha} f(t) = \mathcal{F}(t, \alpha) f'(t)$ , where  $f'(t)$  is the ordinary derivative. For the right derivative, we have

$$(\mathcal{N}_{\mathcal{F}, \wp_1+}^{\alpha} f)(t) = \mathcal{N}_{\mathcal{F}}^{\alpha} [\mathcal{J}_{\mathcal{F}, \wp_1+}^{\alpha}(f)(t)] = \frac{d}{du} [\mathcal{J}_{\mathcal{F}, \wp_1+}^{\alpha}(f)(t)] \mathcal{F}(u, \alpha).$$

Similarly to the left derivative.

- 6) It is clear then, that from our definition, new extensions and generalizations of known integral operators can be defined, see [15]. Let  $p : [\wp_1, \wp_2] \rightarrow \mathfrak{R}$  be an increasing and positive monotone function on  $(\wp_1, \wp_2]$  having a continuous derivative  $p'(t)$  on  $(\wp_1, \wp_2)$ . The left side fractional integral of  $f$  with respect to the function  $p$  on  $[\wp_1, \wp_2]$  of order  $\alpha > 0$  is defined by

$$\mathcal{I}_{p, \wp_1+}^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_{\wp_1}^t \frac{p'(s)f(s)}{[p(t) - p(s)]^{1-\alpha}} ds, \quad t > \wp_1. \tag{5}$$

Similarly the right lateral derivative is defined by

$$\mathcal{I}_{p, \wp_2-}^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\wp_2} \frac{p'(s)f(s)}{[p(s) - p(t)]^{1-\alpha}} ds, \quad t < \wp_2. \tag{6}$$

It will be very easy for the reader to build the  $\mathcal{T}$  in this case.

- 7) The  $k$ -analogue of above definition is defined in [6, 16], under the same assumptions on function  $p$  as follows:

$$\mathcal{I}_{p, \wp_1+}^{\alpha, k}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_{\wp_1}^t \frac{p'(s)f(s)}{[p(t) - p(s)]^{1-\frac{\alpha}{k}}} ds, \quad t > \wp_1. \tag{7}$$

Similarly the right lateral derivative is given as

$$\mathcal{I}_{p, \wp_2-}^{\alpha, k}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\wp_2} \frac{p'(s)f(s)}{[p(s) - p(t)]^{1-\frac{\alpha}{k}}} ds, \quad t < \wp_2. \tag{8}$$

The corresponding differential operator is also very easy to obtain.

**Remark 3** We will also use the central integral operator defined by

$$\mathcal{J}_{\mathcal{T}, \wp_1}^{\alpha}(f)(\wp_2) = \int_{\wp_1}^{\wp_2} \frac{f(t)}{\mathcal{T}(t, \alpha)} dt, \quad \wp_1 < \wp_2. \tag{9}$$

**Theorem 2** ([8]) Let  $h$  be  $\mathcal{N}_{\mathcal{T}}$ -differentiable function on  $(x_0, +\infty)$  with  $\alpha \in (0, 1]$ . Then for all  $x > x_0$ , we have

a)  $\mathcal{J}_{\mathcal{T}, x_0}^{\alpha} (\mathcal{N}_{\mathcal{T}}^{\alpha} h(x)) = h(x) - h(x_0);$

b)  $\mathcal{N}_{\mathcal{T}}^{\alpha} (\mathcal{J}_{\mathcal{T}, x_0}^{\alpha} h(x)) = h(x).$

An necessary and important property is established in the following result, its proof is similar to that of the entire case, so we leave the details to the interested reader.

**Theorem 3 (Integration by parts)** Let  $p$  and  $q$  are two  $\mathcal{N}_{\mathcal{T}}$ -differentiable functions on  $(x_0, +\infty)$  with  $\alpha \in (0, 1]$ . Then for all  $x > x_0$ , we get

$$\mathcal{J}_{\mathcal{T}, x_0}^{\alpha} ((p\mathcal{N}_{\mathcal{T}}^{\alpha} q)(x)) = [p(x)q(x) - p(x_0)q(x_0)] - \mathcal{J}_{\mathcal{T}, x_0}^{\alpha} ((q\mathcal{N}_{\mathcal{T}}^{\alpha} p)(x)).$$

Motivated by above results and literatures, using generalized fractional integral operators, we will obtain in Section 2 a new interesting generalized improved Hölder integral inequality. Also, after deriving a useful lemma using these operators, we will give two results via quasi-convex functions. Some special cases of our results will deduce known results. In Section 3, some applications of the obtained results for error estimates will be illustrated. In Section 4, a briefly conclusion will be provided as well.

## 2 Main Results

Before we prove our results, let denote, respectively,  $\mathcal{L}[\wp_1, \wp_2]$  the set of all Riemann integrable functions on  $[\wp_1, \wp_2]$  and  $\mathcal{I}^\circ$  the interior of  $\mathcal{I}$ .

**Theorem 4 (Generalized improved Hölder integral inequality)** *Let  $u$  and  $v$  be two real functions defined on  $[\wp_1, \wp_2]$  and  $|u|, |v|^q, |u||v|^q \in \mathcal{L}[\wp_1, \wp_2]$ . Then for  $q \geq 1$ , the following inequalities hold:*

$$\begin{aligned} \mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |uv|(\wp_2) &\leq \frac{1}{\wp_2 - \wp_1} \left[ (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(\wp_2 - t) |u|(\wp_2))^{1-\frac{1}{q}} (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(\wp_2 - t) |v|^q(\wp_2))^{\frac{1}{q}} \right. \\ &\quad \left. + (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(t - \wp_1) |u|(\wp_2))^{1-\frac{1}{q}} (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(t - \wp_1) |v|^q(\wp_2))^{\frac{1}{q}} \right] \\ &\leq (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u|(\wp_2))^{1-\frac{1}{q}} (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u||v|^q(\wp_2))^{\frac{1}{q}}. \end{aligned} \tag{10}$$

**Proof.** We consider only the case  $q > 1$ , where  $\frac{1}{p} + \frac{1}{q} = 1$  since for  $q = 1$ , it is easy to verify that equality in (10) is fulfilled. Using properties of modulus, we have

$$\begin{aligned} |uv| &= \frac{1}{\wp_2 - \wp_1} \left| (\wp_2 - t)^{\frac{1}{p}} u^{\frac{1}{p}} (\wp_2 - t)^{\frac{1}{q}} u^{\frac{1}{q}} v + (t - \wp_1)^{\frac{1}{p}} u^{\frac{1}{p}} (t - \wp_1)^{\frac{1}{q}} u^{\frac{1}{q}} v \right| \\ &\leq \frac{1}{\wp_2 - \wp_1} \left\{ \left| (\wp_2 - t)^{\frac{1}{p}} u^{\frac{1}{p}} (\wp_2 - t)^{\frac{1}{q}} u^{\frac{1}{q}} v \right| + \left| (t - \wp_1)^{\frac{1}{p}} u^{\frac{1}{p}} (t - \wp_1)^{\frac{1}{q}} u^{\frac{1}{q}} v \right| \right\}. \end{aligned}$$

Then, from definition of  $\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha$  and Hölder’s inequality the desired left side inequality of (10) is derived.

For the right side inequality of (10), we will first consider the following case:

$$(\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u|(\wp_2))^{1-\frac{1}{q}} (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u||v|^q(\wp_2))^{\frac{1}{q}} = 0.$$

Then the inequality in the right side of (10) is trivial, if  $|u| = 0$ . The same results can be obtain for  $|u| \neq 0$ . So, we take

$$\mathcal{A} = (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u|(\wp_2))^{1-\frac{1}{q}} (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u||v|^q(\wp_2))^{\frac{1}{q}} \neq 0.$$

Since  $\mathcal{A} \neq 0$ , we get

$$\begin{aligned} &\frac{1}{\mathcal{A} \cdot (\wp_2 - \wp_1)} \left[ (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(\wp_2 - t) |u|(\wp_2))^{1-\frac{1}{q}} (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(\wp_2 - t) |v|^q(\wp_2))^{\frac{1}{q}} \right. \\ &\quad \left. + (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(t - \wp_1) |u|(\wp_2))^{1-\frac{1}{q}} (\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(t - \wp_1) |v|^q(\wp_2))^{\frac{1}{q}} \right] \\ &\leq \frac{1}{(\wp_2 - \wp_1)} \left[ \left( \frac{\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(\wp_2 - t) |u|(\wp_2)}{\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u|(\wp_2)} \right)^{1-\frac{1}{q}} \left( \frac{\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(\wp_2 - t) |v|^q(\wp_2)}{\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u||v|^q(\wp_2)} \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left( \frac{\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(t - \wp_1) |u|(\wp_2)}{\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u|(\wp_2)} \right)^{1-\frac{1}{q}} \left( \frac{\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(t - \wp_1) |v|^q(\wp_2)}{\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha |u||v|^q(\wp_2)} \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{11}$$

By applying Young’s inequality given in Proposition 1 on inequality (11), we have

$$\begin{aligned} & \left[ \left( \frac{\mathcal{J}_{T,\varphi_1}^\alpha(\varphi_2 - t)|u|(\varphi_2)}{\mathcal{J}_{T,\varphi_1}^\alpha|u|(\varphi_2)} \right)^{1-\frac{1}{q}} \left( \frac{\mathcal{J}_{T,\varphi_1}^\alpha(\varphi_2 - t)|v|^q(\varphi_2)}{\mathcal{J}_{T,\varphi_1}^\alpha|u||v|^q(\varphi_2)} \right)^{\frac{1}{q}} \right. \\ & \left. + \left( \frac{\mathcal{J}_{T,\varphi_1}^\alpha(t - \varphi_1)|u|(\varphi_2)}{\mathcal{J}_{T,\varphi_1}^\alpha|u|(\varphi_2)} \right)^{1-\frac{1}{q}} \left( \frac{\mathcal{J}_{T,\varphi_1}^\alpha(t - \varphi_1)|v|^q(\varphi_2)}{\mathcal{J}_{T,\varphi_1}^\alpha|u||v|^q(\varphi_2)} \right)^{\frac{1}{q}} \right] \\ \leq & \left[ \frac{(q-1)\mathcal{J}_{T,\varphi_1}^\alpha(\varphi_2 - t)|u|(\varphi_2)}{q\mathcal{J}_{T,\varphi_1}^\alpha|u|(\varphi_2)} + \frac{\mathcal{J}_{T,\varphi_1}^\alpha(\varphi_2 - t)|v|^q(\varphi_2)}{q\mathcal{J}_{T,\varphi_1}^\alpha|u||v|^q(\varphi_2)} \right. \\ & \left. + \frac{(q-1)\mathcal{J}_{T,\varphi_1}^\alpha(t - \varphi_1)|u|(\varphi_2)}{q\mathcal{J}_{T,\varphi_1}^\alpha|u|(\varphi_2)} + \frac{\mathcal{J}_{T,\varphi_1}^\alpha(t - \varphi_1)|v|^q(\varphi_2)}{q\mathcal{J}_{T,\varphi_1}^\alpha|u||v|^q(\varphi_2)} \right] = 1. \end{aligned}$$

The proof of Theorem 4 is completed. ■

**Corollary 1** Choosing  $\mathcal{T}(t, \alpha) \equiv 1$  in Theorem 4, then we get ([13], Theorem 2.1).

**Remark 4** If we consider other specific kernels, for example,  $\mathcal{T}(t, \alpha) = t^{1-\alpha}$ , the Lemma 1 of [24] and Lemma 2.8 of [25] are recaptured. Obviously, using other kernels, we can obtain different “weighted” versions of Hölder’s classical inequality.

The following Lemma 1 is very important to derive our following results.

**Lemma 1** Let  $f : \mathcal{I} \subset \mathfrak{R} \rightarrow \mathfrak{R}$  be a differentiable function on  $\mathcal{I}^\circ$  and let  $\varphi_1, \varphi_2 \in \mathcal{I}$ , where  $\varphi_1 < \varphi_2$ . If  $\mathcal{N}_T^\alpha f \in \mathcal{L}[\varphi_1, \varphi_2]$ , then we have the following identity:

$$\frac{f(\varphi_1) + f(\varphi_2)}{2} - \frac{1}{\varphi_2 - \varphi_1} \mathcal{J}_{T,\varphi_1}^\alpha(f)(\varphi_2) = \frac{(\varphi_2 - \varphi_1)}{4} (\mathcal{J}_1 + \mathcal{J}_2), \tag{12}$$

where

$$\mathcal{J}_1 = \mathcal{J}_{T,0}^\alpha \left[ -t \mathcal{N}_T^\alpha f \left( \frac{1+t}{2} \varphi_1 + \frac{1-t}{2} \varphi_2 \right) \right] \tag{1}$$

and

$$\mathcal{J}_2 = \mathcal{J}_{T,0}^\alpha \left[ t \mathcal{N}_T^\alpha f \left( \frac{1-t}{2} \varphi_1 + \frac{1+t}{2} \varphi_2 \right) \right] \tag{1}.$$

**Proof.** Integrating by parts and using a change of variable on  $\mathcal{J}_1$ , we have

$$\mathcal{J}_1 = \frac{2}{\varphi_2 - \varphi_1} f(\varphi_1) - \frac{4}{(\varphi_2 - \varphi_1)^2} \mathcal{J}_{T,\varphi_1}^\alpha(f) \left( \frac{\varphi_1 + \varphi_2}{2} \right). \tag{13}$$

Similarly working on  $\mathcal{J}_2$ , we get

$$\mathcal{J}_2 = \frac{2}{\varphi_2 - \varphi_1} f(\varphi_2) - \frac{4}{(\varphi_2 - \varphi_1)^2} \mathcal{J}_{T,\frac{\varphi_1 + \varphi_2}{2}}^\alpha(f)(\varphi_2). \tag{14}$$

After adding equalities (13) and (14), and multiplying both sides by the factor  $\frac{\varphi_2 - \varphi_1}{4}$ , we obtain the desired equality (12). ■

**Remark 5** Making  $\mathcal{T}(t, \alpha) \equiv 1$  in Lemma 1, we have ([1], Lemma 2.1).

**Theorem 5** Let  $f : \mathcal{I} \subset \mathfrak{R} \rightarrow \mathfrak{R}$  be a differentiable function on  $\mathcal{I}^\circ$  and let  $\wp_1, \wp_2 \in \mathcal{I}$ , where  $\wp_1 < \wp_2$ . If  $\mathcal{N}_T^\alpha f \in \mathcal{L}[\wp_1, \wp_2]$  and  $|\mathcal{N}_T^\alpha f|$  is quasi-convex function on  $[\wp_1, \wp_2]$ , then we have

$$\begin{aligned} & \left| \frac{f(\wp_1) + f(\wp_2)}{2} - \frac{1}{\wp_2 - \wp_1} \mathcal{J}_{T, \wp_1}^\alpha (f)(\wp_2) \right| \\ & \leq \frac{(\wp_2 - \wp_1)}{4} \mathbb{T} \times \left[ \sup \left\{ \left| \mathcal{N}_T^\alpha f(\wp_1) \right|, \left| \mathcal{N}_T^\alpha f \left( \frac{\wp_1 + \wp_2}{2} \right) \right| \right\} \right. \\ & \quad \left. + \sup \left\{ \left| \mathcal{N}_T^\alpha f \left( \frac{\wp_1 + \wp_2}{2} \right) \right|, \left| \mathcal{N}_T^\alpha f(\wp_2) \right| \right\} \right], \end{aligned} \tag{15}$$

where  $\mathcal{J}_{T,0}^\alpha(t)(1) = \mathbb{T}$ .

**Proof.** Using Lemma 1 and property of modulus, we have

$$\left| \frac{f(\wp_1) + f(\wp_2)}{2} - \frac{1}{\wp_2 - \wp_1} \mathcal{J}_{T, \wp_1}^\alpha (f)(\wp_2) \right| \leq \frac{(\wp_2 - \wp_1)}{4} (|\mathcal{J}_1| + |\mathcal{J}_2|).$$

From the quasi-convexity of  $|\mathcal{N}_T^\alpha f|$ , we get

$$|\mathcal{J}_1| \leq \mathbb{T} \left( \sup \left\{ \left| \mathcal{N}_T^\alpha f(\wp_1) \right|, \left| \mathcal{N}_T^\alpha f \left( \frac{\wp_1 + \wp_2}{2} \right) \right| \right\} \right) \tag{16}$$

and

$$|\mathcal{J}_2| \leq \mathbb{T} \left( \sup \left\{ \left| \mathcal{N}_T^\alpha f \left( \frac{\wp_1 + \wp_2}{2} \right) \right|, \left| \mathcal{N}_T^\alpha f(\wp_2) \right| \right\} \right). \tag{17}$$

Adding inequalities (16) and (17), and taking out common factor  $\mathbb{T}$ , is obtained inequality (15). ■

**Remark 6** Considering  $\mathcal{T}(t, \alpha) \equiv 1$  in Theorem 5, we have  $\mathbb{T} = \frac{1}{2}$ . In this way, we obtain ([1], Theorem 2.2).

**Theorem 6** Let  $f : \mathcal{I} \subset \mathfrak{R} \rightarrow \mathfrak{R}$  be a differentiable function on  $\mathcal{I}^\circ$  and let  $\wp_1, \wp_2 \in \mathcal{I}$ , where  $\wp_1 < \wp_2$ . If  $\mathcal{N}_T^\alpha f \in \mathcal{L}[\wp_1, \wp_2]$  and  $|\mathcal{N}_T^\alpha f|^q$  is quasi-convex function on  $[\wp_1, \wp_2]$ , then for  $q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$\begin{aligned} & \left| \frac{f(\wp_1) + f(\wp_2)}{2} - \frac{1}{\wp_2 - \wp_1} \mathcal{J}_{T, \wp_1}^\alpha (f)(\wp_2) \right| \\ & \leq \frac{(\wp_2 - \wp_1)}{4} \mathbb{L}(p) \times \left[ \left( \sup \left\{ \left| \mathcal{N}_T^\alpha f(\wp_1) \right|^q, \left| \mathcal{N}_T^\alpha f \left( \frac{\wp_1 + \wp_2}{2} \right) \right|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \sup \left\{ \left| \mathcal{N}_T^\alpha f \left( \frac{\wp_1 + \wp_2}{2} \right) \right|^q, \left| \mathcal{N}_T^\alpha f(\wp_2) \right|^q \right\} \right)^{\frac{1}{q}} \right] \end{aligned} \tag{18}$$

where  $\mathbb{L}(p) = (\mathcal{J}_{T,0}^\alpha(t^p)(1))^{\frac{1}{p}}$ .

**Proof.** Using Lemma 1 and property of modulus, we have

$$\left| \frac{f(\wp_1) + f(\wp_2)}{2} - \frac{1}{\wp_2 - \wp_1} \mathcal{J}_{T, \wp_1}^\alpha (f)(\wp_2) \right| \leq \frac{(\wp_2 - \wp_1)}{4} (|\mathcal{J}_1| + |\mathcal{J}_2|).$$

Also, by quasi-convexity of  $|\mathcal{N}_T^\alpha f|^q$  and Hölder's inequality, we get

$$|\mathcal{J}_1| \leq \mathbb{L}(p) \left( \sup \left\{ \left| \mathcal{N}_T^\alpha f(\wp_1) \right|^q, \left| \mathcal{N}_T^\alpha f \left( \frac{\wp_1 + \wp_2}{2} \right) \right|^q \right\} \right)^{\frac{1}{q}} \tag{19}$$

and

$$|\mathcal{J}_2| \leq \mathbb{L}(p) \left( \sup \left\{ \left| \mathcal{N}_{\mathcal{T}}^\alpha f \left( \frac{\wp_1 + \wp_2}{2} \right) \right|^q, |\mathcal{N}_{\mathcal{T}}^\alpha f(\wp_2)|^q \right\} \right)^{\frac{1}{q}}. \tag{20}$$

Adding inequalities (19) and (20), and taking out common factor  $\mathbb{L}(p)$ , is obtained inequality (18). ■

**Corollary 2** Taking  $\mathcal{T}(t, \alpha) \equiv 1$  in Theorem 6, we obtain ([1], Theorem 2.3). Interested reader can see also ([13], Theorem 2.4).

### 3 Applications

In this last section, we consider applications of the integral inequalities given in Section 2, to find some error estimates of quadrature rules, which in turn are variants of the well-known method of rectangles.

First, let  $\mathcal{U} : \wp_1 = x_0 < x_1 < \dots < x_{n-1} < x_n = \wp_2$  be a partition of  $[\wp_1, \wp_2]$ . We denote, respectively,

$$\mathcal{P}(\mathcal{U}, f, \mathcal{T}) = \sum_{i=0}^{n-1} \left( \frac{f(x_i) + f(x_{i+1})}{2} \right) h_i$$

and

$$\mathcal{J}_{\mathcal{T}, \wp_1}^\alpha(f)(\wp_2) = \mathcal{P}(\mathcal{U}, f, \mathcal{T}) + \mathcal{R}(\mathcal{U}, f, \mathcal{T}),$$

where  $\mathcal{R}(\mathcal{U}, f, \mathcal{T})$  is the remainder term (error estimation) and  $h_i = x_{i+1} - x_i$  for  $i = 0, 1, 2, \dots, n - 1$ . Using above notations, we are in position to prove the following results.

**Theorem 7** Let  $f : \mathcal{I} \subset \mathfrak{R} \rightarrow \mathfrak{R}$  be a differentiable function on  $\mathcal{I}^\circ$ , where  $\wp_1, \wp_2 \in \mathcal{I}$  and  $\wp_1 < \wp_2$ . If  $\mathcal{N}_{\mathcal{T}}^\alpha f \in \mathcal{L}[\wp_1, \wp_2]$  and  $|\mathcal{N}_{\mathcal{T}}^\alpha f|$  is quasi-convex function on  $[\wp_1, \wp_2]$ , then the remainder term satisfies the following error estimation:

$$|\mathcal{R}(\mathcal{U}, f, \mathcal{T})| \leq \frac{\mathbb{T}}{4} \sum_{i=0}^{n-1} [\mathcal{A}(i) + \mathcal{B}(i)] h_i^2, \tag{21}$$

where

$$\begin{aligned} \mathcal{A}(i) &= \sup \left\{ |\mathcal{N}_{\mathcal{T}}^\alpha f(x_i)|, \left| \mathcal{N}_{\mathcal{T}}^\alpha f \left( \frac{x_i + x_{i+1}}{2} \right) \right| \right\}, \\ \mathcal{B}(i) &= \sup \left\{ \left| \mathcal{N}_{\mathcal{T}}^\alpha f \left( \frac{x_i + x_{i+1}}{2} \right) \right|, |\mathcal{N}_{\mathcal{T}}^\alpha f(x_{i+1})| \right\} \end{aligned}$$

and  $\mathbb{T}$  is defined as in Theorem 5.

**Proof.** Using the Theorem 5 on subinterval  $[x_i, x_{i+1}]$  of closed interval  $[\wp_1, \wp_2]$ , for all  $i = 0, 1, 2, \dots, n - 1$ , we have

$$\left| \left( \frac{f(x_i) + f(x_{i+1})}{2} \right) h_i - \mathcal{J}_{\mathcal{T}, x_i}^\alpha(f)(x_{i+1}) \right| \leq \frac{\mathbb{T}}{4} [\mathcal{A}(i) + \mathcal{B}(i)] h_i^2. \tag{22}$$

Summing inequality (22) over  $i$  from 0 to  $n - 1$  and using the properties of modulus, we obtain the desired inequality (21). ■

**Theorem 8** Let  $f : \mathcal{I} \subset \mathfrak{R} \rightarrow \mathfrak{R}$  be a differentiable function on  $\mathcal{I}^\circ$ , where  $\wp_1, \wp_2 \in \mathcal{I}$  and  $\wp_1 < \wp_2$ . If  $\mathcal{N}_{\mathcal{T}}^\alpha f \in L[\wp_1, \wp_2]$  and  $|\mathcal{N}_{\mathcal{T}}^\alpha f|^q$  is quasi-convex function on  $[\wp_1, \wp_2]$ , then for  $q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , the remainder term satisfies the following error estimation:

$$|\mathcal{R}(\mathcal{U}, f, \mathcal{T})| \leq \frac{\mathbb{L}(p)}{4} \sum_{i=0}^{n-1} \left[ (\mathcal{C}(i, q))^{\frac{1}{q}} + (\mathcal{D}(i, q))^{\frac{1}{q}} \right] h_i^2, \tag{23}$$



where

$$\mathcal{C}(i, q) = \sup \left\{ |\mathcal{N}_{\mathcal{T}}^{\alpha} f(x_i)|^q, \left| \mathcal{N}_{\mathcal{T}}^{\alpha} f\left(\frac{x_i + x_{i+1}}{2}\right) \right|^q \right\},$$

$$\mathcal{D}(i, q) = \sup \left\{ \left| \mathcal{N}_{\mathcal{T}}^{\alpha} f\left(\frac{x_i + x_{i+1}}{2}\right) \right|^q, |\mathcal{N}_{\mathcal{T}}^{\alpha} f(x_{i+1})|^q \right\}$$

and  $\mathbb{L}(p)$  is defined as in Theorem 6.

**Proof.** Using the Theorem 6 on subinterval  $[x_i, x_{i+1}]$  of closed interval  $[\wp_1, \wp_2]$ , for all  $i = 0, 1, 2, \dots, n - 1$ , we have

$$\left| \left( \frac{f(x_i) + f(x_{i+1})}{2} \right) h_i - \mathcal{J}_{\mathcal{T}, x_i}^{\alpha}(f)(x_{i+1}) \right| \leq \frac{\mathbb{L}(p)}{4} \left[ (\mathcal{C}(i, q))^{\frac{1}{q}} + (\mathcal{D}(i, q))^{\frac{1}{q}} \right] h_i^2. \tag{24}$$

Summing inequality (24) over  $i$  from 0 to  $n - 1$  and using the properties of modulus, we get the desired inequality (23). ■

Next, we will show the advantages of the results obtained, in particular the Theorem 7. For this, consider in the Definition 4,  $\alpha = 0.5$ ,  $f(t) = t^3$ ,  $\wp_1 = 0$ ,  $\wp_2 = 1$  and the kernels  $\mathcal{T} = 1, t^{1-\alpha}$  and we will use the partition of the interval  $[0, 1]$  given by  $\mathcal{U} : \wp_1 = x_0 = 0 < 0.25 < 0.5 < 0.75 < 1 = x_4 = \wp_2$ , so  $h = 0.25$ .

For the kernel  $\mathcal{T} \equiv 1$ , we have  $\mathbb{T} = 1$  and  $\mathcal{N}_{\mathcal{F}}f(t) = f'(t)$ . Hence,

$$|\mathcal{R}(\mathcal{U}, t^3, 1)| \leq 0.1494. \tag{25}$$

In the case of kernel  $\mathcal{T} = t^{1-\alpha}$ , we get  $\mathbb{T} = 2$  and  $\mathcal{N}_{\mathcal{F}}f(t) = f'^{\frac{1}{2}}$ . Therefore,

$$|\mathcal{R}(\mathcal{U}, t^3, t^{\frac{1}{2}})| \leq 0.0376. \tag{26}$$

The exact value of the ordinary integral  $\int_0^1 t^3 dt$  is 0.25. If we calculate its approximate value by the method of the rectangles, we will get  $\int_0^1 t^3 dt \approx 0.3905$ .

Comparing with inequalities (25) and (26), we see that in the first case they are practically equivalent results, while in the case of the conformable kernel, a better approximation is obtained.

## 4 Conclusion

In this paper using generalized integral operators, we first obtained new interesting generalized improved Hölder integral inequality. By means of a new lemma using these operators, we given two results via quasi-convex functions. Some known results are recaptured as special cases from our results. Finally, some error estimates are given. Since convexity and (quasi-convexity) have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

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