

# The Poisson-Modified Lindley Distribution\*

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## Abstract

In this paper, we present the Poisson modified-Lindley distribution. It constitutes a new discrete one-parameter distribution obtained by compounding the Poisson and modified Lindley distributions. We describe its main interesting properties, with emphasis on those related to the distribution function, raw moments, variance, coefficient of variation, index of dispersion, factorial moments, skewness, kurtosis, and probability generating function. Among its features, it is revealed to be useful for the analysis of over-dispersed count data sets, motivating a statistical work in this regard. As a first step, the model parameter is estimated by the maximum likelihood method and method of moments. Then, the utility of the new model is illustrated through the analysis of two different data sets: the first is the number of European corn-borer larvae *pyrausta* in the field, and the second one is about the number of times that the computer breaks down in each of the 128 consecutive weeks of operation. The fit of the new model is both satisfactory and competitive, giving better results than those of the Poisson model, as well as the distinguished Poisson Lindley and Poisson Bilal models.

## 1 Introduction

The Poisson distribution is one of the most famous one-parameter distributions used for modeling count data. It satisfies the well-known property of the mean and variance being equal, which remains an obvious handicap for the construction of models from over-dispersed or under-dispersed count data. Despite this, it is widely used in many fields of research, such as environmental, actuarial, biology and economics. The reason for this popularity is its simple form and easy implementation, which is supported in most statistical software. To overcome the drawback of “mean equality of variance”, researchers have shown great interest in introducing mixed-Poisson distributions. In this regard, Shoukri *et al.* [33] introduced the Poisson inverse gaussian regression model, Shmueli *et al.* [32] revived the Conway-Maxwell-Poisson distribution, Rodriguez-Avi *et al.* [21] introduced a regression model for count data, Lord and Geedipally [18] introduced the negative binomial-Lindley distribution for analyzing the crash data in the case of zero-inflation, Deniz [8] proposed the uniform Poisson distribution, Saez-Castillo and Conde-Sanchez [22] developed the hyper Poisson regression model, Cheng *et al.* [6] used the Poisson-Weibull generalized linear model for analyzing motor vehicle crash data and emphasized that the Poisson-Weibull model yields better results than the Poisson-gamma model, Zamani *et al.* [36] introduced the Poisson weighted exponential model, Gencturk and Yigiter [9] proposed the negative binomial gamma distribution for modeling a certain type of claim counts, Bhati *et al.* [4] presented the Poisson-transmuted exponential linear model and applied it to health-care data sets, and Imoto *et al.* [16] introduced the modified Conway-Maxwell-Poisson type binomial distribution. Recently, Altun [2] introduced the Poisson Bilal (PB) distribution and its associated two models for modeling the over-dispersed count data sets.

As a matter of fact, compound distributions are useful for modeling phenomena that show over-dispersion, i.e., a greater amount of variability than would be expected under a certain model. For this reason, some of the notable Poisson distributions have been compounded with different versions of the Lindley distribution.

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Some of them are mentioned below. Sankaran [23] introduced the discrete Poisson-Lindley (PL) distribution by combining the Poisson and Lindley distributions. The statistical aspects of the PL distribution can be found in [12]. The PL distribution itself has been generalized by many researchers. Ghitany *et al.* [10] and Asgharzadeh *et al.* [1] derived the zero-truncated PL and Pareto Poisson-Lindley distributions, respectively, Mahmoudi and Zakerzadeh [19] proposed an extended version of the compound Poisson distribution, which was obtained by compounding the Poisson distribution with the generalized Lindley distribution (established by [37]), to model the over-dispersed count data sets and showed that the generalized Poisson-Lindley distribution provides better results than the Poisson and PL distributions in case of over-dispersion. Gómez-Déniz *et al.* [13] introduced the multivariate discrete PL distributions and studied their extensions as well as its applications in actuarial science. Shanker and Mishra [31] introduced a two-parameter Poisson-Lindley distribution by compounding the Poisson distribution with the two-parameter Lindley distribution introduced by [26]. A quasi Poisson-Lindley distribution was developed by [25], by compounding the Poisson distribution with a quasi Lindley distribution introduced by [27]. Shanker *et al.* [28] proposed a discrete two-parameter PL distribution by mixing the Poisson distribution with a two-parameter Lindley distribution to model waiting and survival time's data introduced by [29]. Further, Shanker and Tekie [30] obtained a new quasi Poisson-Lindley distribution by compounding the Poisson distribution with a new quasi Lindley distribution introduced by [31]. In addition, Nedjar and Zeghdoudi [40] and Zeghdoudi and Nedjar [35] presented two new compound Poisson distributions, named the Poisson gamma Lindley distribution and Poisson pseudo-Lindley distributions, by compounding Poisson with the gamma Lindley and pseudo-Lindley distributions, both proposed by [39] and [38], respectively. Also, Grine and Zeghdoudi [14] studied the Poisson quasi Lindley distribution and its applications, Wongrin and Bodhisuwan [34] investigated the generalized PL linear model for count data and showed that the related linear model provides better modeling ability than the Poisson and negative binomial regression models in the case of over-dispersion, and Mohammadpour *et al.* [20] studied the PL INAR(1) model with applications.

The essentials of the PL distribution are now discussed. The PL distribution is defined by the following probability mass function (pmf):

$$p_{PL}(x; \theta) = \frac{\theta^2(x + \theta + 2)}{(\theta + 1)^{x+3}}, \quad x \in \mathbb{N},$$

where  $\theta > 0$ . It has been introduced by [23] with the aim of providing a new model for count data, and a suitable alternative to the Poisson model as well. It is the distribution of a random variable  $X$  following the Poisson distribution with parameter  $\lambda$ , assuming that  $\lambda$  is a random variable following the Lindley distribution. In this regard, we recall that the Poisson distribution is defined with the following pmf:

$$p_P(x; \lambda) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x \in \mathbb{N},$$

where  $\lambda > 0$ , and the Lindley (L) distribution is defined by the following probability density function (pdf):

$$f_L(x; \theta) = \frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x}, \quad x, \theta > 0,$$

where  $\theta > 0$ . Formerly, the Lindley distribution was introduced by [17] with a pdf having the feature to be a mixture of exponential distribution with scale parameter  $\theta$  and gamma distribution with shape parameter 2 and scale parameter  $\theta$ , where the mixing proportion is specified as  $\theta/(1 + \theta)$ . In this setting,  $X$  remains with support on  $\mathbb{N}$ , the pmf of the random vector  $(X, \lambda)$  is given as  $p_*(x, \lambda) = p_P(x; \lambda)f_L(\lambda; \theta)$  and, consequently, the pmf of  $X$  is obtained by

$$\begin{aligned} p_{PL}(x; \theta) &= \int_0^{+\infty} p_*(x, \lambda) d\lambda = \int_0^{+\infty} e^{-\lambda} \frac{\lambda^x}{x!} \frac{\theta^2}{\theta + 1} (1 + \lambda) e^{-\theta \lambda} d\lambda \\ &= \frac{\theta^2}{\theta + 1} \frac{1}{x!} \left[ \int_0^{+\infty} \lambda^x e^{-(\theta+1)\lambda} d\lambda + \int_0^{+\infty} \lambda^{x+1} e^{-(\theta+1)\lambda} d\lambda \right] \end{aligned}$$

$$= \frac{\theta^2}{\theta + 1} \frac{1}{x!} \left[ \frac{1}{(\theta + 1)^{x+1}} x! + \frac{1}{(\theta + 1)^{x+2}} (x + 1)! \right] = \frac{\theta^2 (x + \theta + 2)}{(\theta + 1)^{x+3}}.$$

Following this spirit, we aim to introduce a new discrete distribution for count modeling by the use of a new extension of the Lindley distribution developed by [7], called the modified Lindley (ML) distribution. The ML distribution is defined by the pdf given as

$$f_{ML}(x; \theta) = \frac{\theta}{1 + \theta} e^{-2\theta x} [(1 + \theta)e^{\theta x} + 2\theta x - 1], \quad x > 0,$$

where  $\theta > 0$ . The prime motivation for using the ML distribution is that it provides a simple alternative to the exponential and Lindley distributions, with the following remarkable first order stochastic ordering property:  $F_L(x; \theta) \leq F_{ML}(x; \theta) \leq F_E(x; \theta)$ , where  $F_L(x; \theta)$  denotes the cumulative distribution function (cdf) of the Lindley distribution,  $F_{ML}(x; \theta)$  is the one of the ML distribution and  $F_E(x; \theta)$  is the one of the exponential distribution.

Thus, we introduce the Poisson-modified Lindley (P-ML) distribution defined by the distribution of a random variable  $X$  following the Poisson distribution with parameter  $\lambda$ , assuming that  $\lambda$  is a random variable following the ML distribution. Then,  $X$  is with support  $\mathbb{N}$ , the pmf of  $(X, \lambda)$  is given as  $p_{**}(x, \lambda) = p_P(x; \lambda)f_{ML}(\lambda; \theta)$ , from which we derive the pmf of  $X$  by

$$\begin{aligned} p_{P-ML}(x; \theta) &= \int_0^{+\infty} p_{**}(x, \lambda) d\lambda = \int_0^{+\infty} e^{-\lambda} \frac{\lambda^x}{x!} \frac{\theta}{1 + \theta} e^{-2\theta\lambda} [(1 + \theta)e^{\theta\lambda} + 2\theta\lambda - 1] d\lambda \\ &= \frac{\theta}{\theta + 1} \frac{1}{x!} \left[ (1 + \theta) \int_0^{+\infty} \lambda^x e^{-(\theta+1)\lambda} d\lambda + 2\theta \int_0^{+\infty} \lambda^{x+1} e^{-(2\theta+1)\lambda} d\lambda - \int_0^{+\infty} \lambda^x e^{-(2\theta+1)\lambda} d\lambda \right] \\ &= \frac{\theta}{\theta + 1} \frac{1}{x!} \left[ (1 + \theta) \frac{1}{(\theta + 1)^{x+1}} x! + 2\theta \frac{1}{(2\theta + 1)^{x+2}} (x + 1)! - \frac{1}{(2\theta + 1)^{x+1}} x! \right], \quad (1) \\ p_{P-ML}(x; \theta) &= \frac{\theta}{\theta + 1} \left[ \frac{1}{(\theta + 1)^x} + 2\theta \frac{x}{(2\theta + 1)^{x+2}} - \frac{1}{(2\theta + 1)^{x+2}} \right], \quad x \in \mathbb{N}. \end{aligned}$$

Particular attention is paid to the P-ML distribution in the next, examining various of its characteristics from both theoretical and practical sides. To summarize the next developments, the benefits of the P-ML distribution are to (i) enjoy manageable distribution functions, (ii) possess tractable moments measures and functions, (iii) be enough flexible to handle over-dispersed data, (iv) permit simple estimation procedures, and (v) present better fits in comparison to modern discrete distributions for some data sets, such as the Poisson, Poisson Lindley and Poisson Bilal distributions.

The paper is structured as follows. Section 2 presents the properties of the P-ML distribution. Estimation and applications can be found in Section 3. Some final remarks are formulated in Section 4.

## 2 Properties of the P-ML Distribution

In this section, let  $Y$  be a random variable following the P-ML distribution, i.e., with pmf given as (1). Here, we provide some statistical properties of the P-ML distribution through the use of  $Y$ . These include a reformulation of the pmf, distribution function, raw moments, variance, coefficient of variation, index of dispersion, factorial moments, skewness, kurtosis, and probability generating function.

### 2.1 Probability Mass Function

The pmf of the P-ML distribution, i.e.,  $p_{P-ML}(x; \theta) = P(Y = x)$  where  $P$  denotes the probability measure, is defined in (1). After simple developments, it can also be reduced to the following ratio:

$$p_{P-ML}(x; \theta) = \frac{\theta[(2\theta + 1)^{x+2} + (2\theta x - 1)(\theta + 1)^x]}{(\theta + 1)^{x+1}(2\theta + 1)^{x+2}}, \quad x \in \mathbb{N}. \quad (2)$$

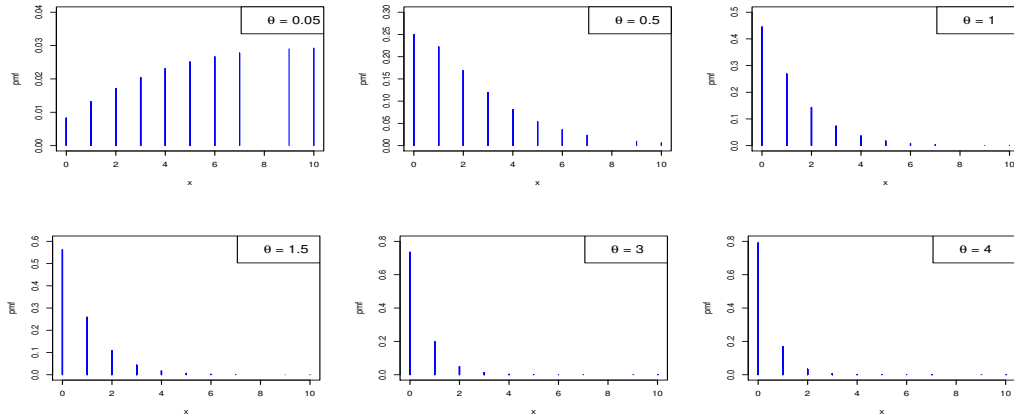


Figure 1: Plots of the pmf of the P-ML distribution for some parameter values of  $\theta$

One can show that  $p_{P-ML}(x; \theta) \in (0, 1)$  for any  $k \in \mathbb{N}$ . Moreover, by using the following geometric formulas:  $\sum_{x=0}^{+\infty} z^x = 1/(1-z)$  and  $\sum_{x=1}^{+\infty} xz^{x-1} = 1/(1-z)^2$  with  $|z| < 1$ , we verify that  $\sum_{x=0}^{+\infty} p_{P-ML}(x; \theta) = 1$ , ensuring the basic pmf property. As the former ML distribution, it provides a one-parameter alternative to the PL distribution and the former Poisson distribution as well.

Figure 1 displays the plots of the pmf of the P-ML distribution for some values of  $\theta$ . From the plots in Figure 1, we observe that, as the value of the parameter  $\theta$  increases, the distribution assigns higher probability to smaller values of the variable, which can be of interest to model rare events.

## 2.2 Distribution Function

The distribution function of  $Y$  is given as  $F_{P-ML}(m; \theta) = P(Y \leq m)$  for any positive integer  $m$ . Based on (1), we have

$$\begin{aligned} F_{P-ML}(m; \theta) &= \sum_{x=0}^m p_{P-ML}(x; \theta) \\ &= \frac{\theta}{\theta+1} \left[ \sum_{x=0}^m \frac{1}{(\theta+1)^x} + \frac{2\theta}{(2\theta+1)^2} \sum_{x=1}^m \frac{x}{(2\theta+1)^x} \right. \\ &\quad \left. - \frac{1}{(2\theta+1)^2} \sum_{x=0}^m \frac{1}{(2\theta+1)^x} \right]. \end{aligned}$$

Now, through the use of the following formulas:

$$\sum_{x=0}^m z^x = (1 - z^{m+1})/(1 - z) \quad \text{and} \quad \sum_{x=1}^m xz^x = z[mz^{m+1} - (m+1)z^m + 1]/(1 - z)^2$$

for  $z \neq 1$ , after some factorizations, we get

$$\begin{aligned} F_{P-ML}(m; \theta) &= \frac{1}{(\theta+1)^{m+1}(2\theta+1)^{m+2}} \\ &\quad \times \left[ 4\theta^3(\theta+1)^m(2\theta+1)^m + 4\theta^2[2(\theta+1)^m - 1](2\theta+1)^m + [(\theta+1)^m - 1](2\theta+1)^m \right] \end{aligned}$$

$$+ \theta[5(2\theta + 1)^m(\theta + 1)^m - m(\theta + 1)^m - (\theta + 1)^m - 4(2\theta + 1)^m] \Big].$$

We can express the quantile function of  $Y$  as  $Q_{P-ML}(q; \theta) = \inf\{m \in \mathbb{N} : F_{P-ML}(m; \theta) \geq q\}$ . Even if we have a closed-form expression for  $F_{P-ML}(m; \theta)$ , an analytical expression for  $Q_{P-ML}(q; \theta)$  seems to be impossible. With the help of mathematical software, however, we can easily have a numerical point-wise evaluation of it.

### 2.3 Raw Moments

The  $r^{th}$  raw moment of  $Y$  is defined by  $\mu'_r = E(Y^r)$  where  $E$  denotes the expectation related to the probability measure  $P$ . Mathematically, for  $r \geq 1$ , we can obtain it as follows:

$$\begin{aligned} \mu'_r &= \sum_{x=0}^{+\infty} x^r p_{P-ML}(x; \theta) \\ &= \frac{\theta}{\theta + 1} \left[ \sum_{x=1}^{+\infty} \frac{x^r}{(\theta + 1)^x} + 2\theta \sum_{x=1}^{+\infty} \frac{x^{r+1}}{(2\theta + 1)^{x+2}} - \sum_{x=1}^{+\infty} \frac{x^r}{(2\theta + 1)^{x+2}} \right] \\ &= \frac{\theta}{\theta + 1} \left[ \text{Li}_{-r} \left( \frac{1}{\theta + 1} \right) + \frac{2\theta}{(2\theta + 1)^2} \text{Li}_{-r-1} \left( \frac{1}{2\theta + 1} \right) - \frac{1}{(2\theta + 1)^2} \text{Li}_{-r} \left( \frac{1}{2\theta + 1} \right) \right], \end{aligned}$$

where  $\text{Li}_s(z) = \sum_{x=1}^{+\infty} z^x/x^s$ , with  $s \in \mathbb{R}$  and  $|z| < 1$ , is the known Polylogarithm function. There is no synthetic formula for  $\text{Li}_s(z)$  for all  $s$ , but for the first four integer values, the following equalities hold:  $\text{Li}_{-1}(z) = z/(1 - z)^2$ ,  $\text{Li}_{-2}(z) = z(z + 1)/(1 - z)^3$ ,  $\text{Li}_{-3}(z) = z(z^2 + 4z + 1)/(1 - z)^4$  and  $\text{Li}_{-4}(z) = z(z^3 + 11z^2 + 11z + 1)/(1 - z)^5$ . Based on these results, after some developments, we can express the four first raw moment of  $Y$  as

$$\mu = \mu'_1 = \frac{4\theta + 5}{4\theta(\theta + 1)}, \quad \mu'_2 = \frac{(\theta + 2)(4\theta + 5)}{4\theta^2(\theta + 1)}, \quad \mu'_3 = \frac{8\theta^3 + 58\theta^2 + 108\theta + 57}{8\theta^3(\theta + 1)}$$

and

$$\mu'_4 = \frac{4\theta^4 + 61\theta^3 + 214\theta^2 + 267\theta + 108}{4\theta^4(\theta + 1)}.$$

In particular, the variance of  $Y$  is given as

$$\sigma^2 = \mu'_2 - \mu^2 = \frac{(4\theta + 5)(2\theta + 1)(2\theta + 3)}{16\theta^2(\theta + 1)^2}.$$

We are able to prove that  $\sigma^2$  is a decreasing function with respect to  $\theta$  with  $\lim_{\theta \rightarrow +\infty} \sigma^2 = 0$ . Figure 2 illustrates the compartment of  $\sigma^2$  when it is viewed as a function of  $\theta$ . The coefficient of variation ( $CV$ ) follows:

$$CV = \frac{\sigma}{\mu} = \sqrt{\frac{(2\theta + 1)(2\theta + 3)}{4\theta + 5}}$$

and the index of dispersion ( $DI$ ) as well:

$$DI = \frac{\sigma^2}{\mu} = \frac{(2\theta + 1)(2\theta + 3)}{4\theta(\theta + 1)}.$$

Since  $(2\theta + 1)(2\theta + 3) = 4\theta^2 + 8\theta + 3 > 4\theta(\theta + 1)$ , we have the important inequality:  $DI > 1$ . More specifically, we can prove that  $DI$  is a decreasing function with respect to  $\theta$  with  $\lim_{\theta \rightarrow +\infty} DI = 1$ . Thus, the

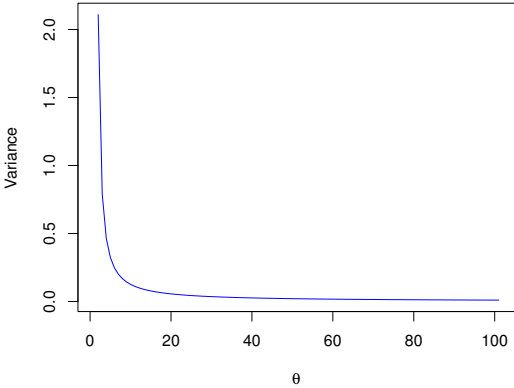


Figure 2: Illustration of the variance of the P-ML distribution defined as a function with respect to  $\theta$

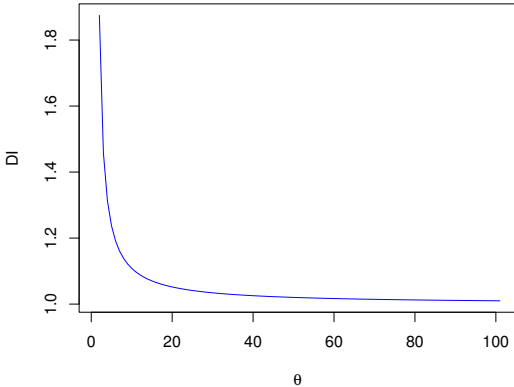


Figure 3: Illustration of the DI of the P-ML distribution defined as a function with respect to  $\theta$

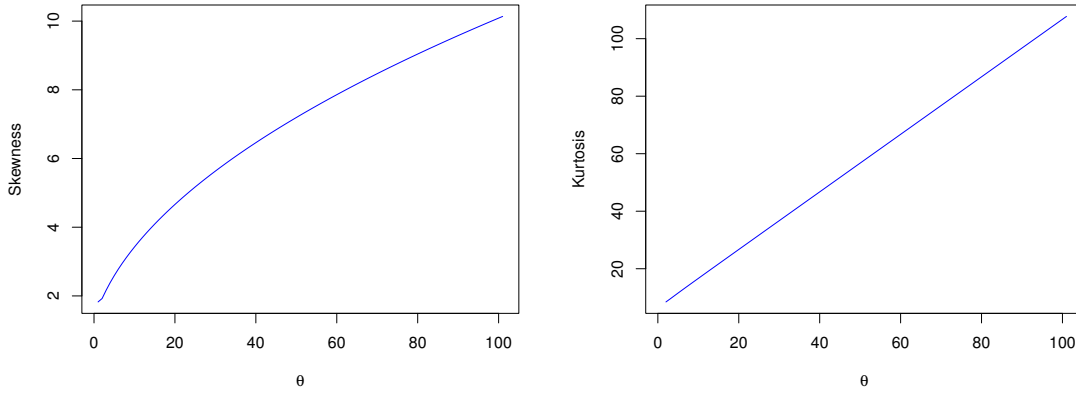


Figure 4: Illustration of skewness and kurtosis of the P-ML distribution for the parameter  $\theta$

P-ML distribution can be used for the modeling of over-dispersed count data sets. The analytical properties of  $DI$  are illustrated in Figure 3.

Also, the skewness and kurtosis of  $Y$  are, respectively, defined by

$$S = \frac{\mu'_3 - 3\mu'_2\mu + 2\mu^3}{\sigma^3}, \quad K = \frac{\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4}{\sigma^4}.$$

By substituting the expressions of the first four moments of  $Y$ , we arrive at the following expression for  $S$ :

$$S = \frac{106 + 516\theta + 1004\theta^2 + 928\theta^3 + 400\theta^4 + 64\theta^5}{(15 + 52\theta + 52\theta^2 + 16\theta^3)^{3/2}}.$$

We can set a similar expression for  $K$  but we omit it for the sake of place. Then, one can prove that  $S$  and  $K$  are both increasing functions with respect to  $\theta$ , satisfying  $\lim_{\theta \rightarrow +\infty} S = +\infty$  and  $\lim_{\theta \rightarrow +\infty} K = +\infty$ . These aspects can be observed in Figure 4.

## 2.4 Factorial Moments

An alternative way to determine various moments can be done through the use of factorial moments. That is, let  $Y_{(r)} = Y(Y-1)\dots(Y-r+1)$ . Since  $Y$  has the distribution of a random variable  $X$  following the Poisson distribution with parameter  $\lambda$ , assuming that  $\lambda$  is a random variable following the ML distribution, the  $r^{th}$  factorial moment of  $Y$  is given as

$$\mu'_{(r)} = E(Y_{(r)}) = E[E(X_{(r)} | \lambda)] = E(\lambda^r).$$

Hence, owing to the  $r^{th}$  raw moment of  $\lambda$  obtained in [7, Section 3.2], we get

$$\mu'_{(r)} = \frac{1}{\theta^r} \left( 1 + \frac{r}{2^{r+1}(1+\theta)} \right) r!.$$

From this relation, all the obtained measures with the raw moments can be deduced.

## 2.5 Probability Generating Function

Since  $Y$  has the distribution of a random variable  $X$  following the Poisson distribution with parameter  $\lambda$ , assuming that  $\lambda$  is a random variable following the ML distribution, the probability generating function of

$Y$  is given by

$$G(s; \theta) = E(s^Y) = E[E(s^X | \lambda)] = E(e^{(s-1)\lambda}).$$

It follows from the moment generating function of  $\lambda$  determined in [7, Section 3.2] with  $t = s - 1$  that

$$G(s; \theta) = \frac{\theta}{\theta + 1 - s} + \frac{(s - 1)\theta}{(1 + \theta)(2\theta + 1 - s)^2}$$

for  $s < \theta + 1$ . The moment generating and characteristic functions immediately follow; they are respectively given as

$$M(t; \theta) = E(e^{tY}) = G(e^t; \theta) = \frac{\theta}{\theta + 1 - e^t} + \frac{(e^t - 1)\theta}{(1 + \theta)(2\theta + 1 - e^t)^2}$$

for  $t < \log(\theta + 1)$ , and

$$\varphi(t; \theta) = E(e^{itY}) = G(e^{it}; \theta) = \frac{\theta}{\theta + 1 - e^{it}} + \frac{(e^{it} - 1)\theta}{(1 + \theta)(2\theta + 1 - e^{it})^2},$$

with  $i^2 = -1$ , for  $t \in \mathbb{R}$ . From these functions, we can re-find some basic characteristics of the P-ML distribution (raw moments, skewness...), set some probabilistic inequalities, such as, by the Markov inequality, for any  $t \in (0, \log(\theta + 1))$ ,  $P(Y \geq m) \leq e^{-tm}M(t; \theta)$ , and investigate the distribution of linear combinations of random variables involving the P-ML distribution, among others.

### 3 Estimation and Applications

Now, the P-ML model is considered, assuming that  $\theta$  is unknown. We adopt the maximum likelihood method to estimate this parameter, with the method of moments discussed in brief, and we illustrate the applicability of the P-ML distribution by using two real data examples.

#### 3.1 Estimation of Parameters

Let  $x_1, \dots, x_n$  be  $n$  observations of  $Y$ ,  $k$  the greatest integer value among them, and  $n_x$  be the number of observed values  $x$ , hence satisfying the following equality:  $\sum_{x=0}^k n_x = n$ . Then, based on (2), the likelihood function of  $\theta$  is given as

$$L(\theta) = \theta^n \frac{1}{(\theta + 1)^{\sum_{x=0}^k n_x(x+1)} (2\theta + 1)^{\sum_{x=0}^k n_x(x+2)}} \prod_{x=0}^k [(2\theta + 1)^{x+2} + (2\theta x - 1)(\theta + 1)^x]^{n_x}.$$

Therefore, the log-likelihood function, defined by  $\ell(\theta) = \log[L(\theta)]$ , can be expressed as

$$\begin{aligned} \ell(\theta) &= n \log \theta - \log(\theta + 1) \sum_{x=0}^k n_x(x + 1) - \log(2\theta + 1) \sum_{x=0}^k n_x(x + 2) \\ &\quad + \sum_{x=0}^k n_x \log[(2\theta + 1)^{x+2} + (2\theta x - 1)(\theta + 1)^x] \\ &= n \log \theta - n \log(\theta + 1)(\bar{x} + 1) - n \log(2\theta + 1)(\bar{x} + 2) + \sum_{x=0}^k n_x \log[(2\theta + 1)^{x+2} + (2\theta x - 1)(\theta + 1)^x], \end{aligned}$$

where  $\bar{x} = (1/n) \sum_{i=1}^n x_i$ . The maximum likelihood estimate (MLE) of  $\theta$ , say  $\hat{\theta}$ , is the solution of the following non-linear equation:  $\partial \ell(\theta) / \partial \theta = 0$ , with

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{n}{\theta} - \frac{n}{\theta + 1}(\bar{x} + 1) - \frac{2n}{2\theta + 1}(\bar{x} + 2)$$



$$+ \sum_{x=0}^k n_x \frac{2(x+2)(2\theta+1)^{x+1} + 2x(\theta+1)^x + (2\theta x - 1)x(\theta+1)^{x-1}}{(2\theta+1)^{x+2} + (2\theta x - 1)(\theta+1)^x}.$$

Alternatively to the MLE, one can consider the moment estimate (MME) for  $\theta$  which is obtained by solving  $\mu = \bar{x}$  according to  $\theta$ . Mathematically, this MME corresponds exactly to the one of  $\theta$  in the former ML distribution, i.e.,

$$\tilde{\theta} = \frac{1 - \bar{x} + [(\bar{x} - 1)^2 + 5\bar{x}]^{1/2}}{2\bar{x}}.$$

We refer to [7] for more details. Under standard regularity conditions, one can prove that both  $\hat{\theta}$  and  $\tilde{\theta}$  are consistent and asymptotically normal. These properties are useful for constructing diverse statistical objects, allowing a precise evaluation of  $\theta$ , such as confidence intervals and statistical tests.

### 3.2 Applications

In this subsection, two applications to real data are developed to emphasize the importance of the P-ML model. The MLE of the parameter  $\theta$  is computed and goodness-of-fit statistics for the new model are compared with those of other competing models.

The first real data set described in Table 1 is obtained from [5]. It contains biological experiment data which represent the number of European corn-borer larvae pyrausta in the field. It was an experiment conducted randomly on eight hills in 15 replications, where the experimenter counted the number of borers per hill of corn.

The second real data set presented in Table 2 is available in [3] and Data set 141 of [15]. It represents the number of times that the computer breaks down in each of the 128 consecutive weeks of operation.

Table 1: Number of European corn-borer larvae pyrausta in the field with the obtained statistical results

Y	Observed frequency	Expected frequency			
		Poisson	PL	PB	P-ML
0	43	44.145534184	44.9999986	39.8862899	44.1157918
1	35	44.145532941	29.9999997	33.6138756	31.2144367
2	17	22.072765849	18.7500002	21.3426248	19.2709893
3	11	7.357588409	11.2500004	12.0995966	11.2005392
4	5	1.839397050	6.5625003	6.4591469	6.3208825
5	4	0.367879400	3.7500003	3.3244158	3.5170072
6	1	0.061313232	2.1093752	1.6704471	1.9454977
7	2	0.008759033	1.1718751	0.8255575	1.0748488
8	2	0.001094879	0.6445313	0.4031898	0.5945558
Total	60	60	60	60	
$\hat{\lambda}$	-	1	-	-	-
$\hat{\theta}$	-	-	1	0.5624285	0.7700874
$-2\log L$	-	462.7437	400.8745	402.1484	400.6870
$\chi^2$	-	4162.608	5.4990496	10.0146132	5.6789650
$p$ -value	-	0.000	0.5992985	0.1877469	0.5776942
AIC	-	464.7437	402.8745	404.1484	402.6870
AICc	-	464.7776	402.9084	404.1823	402.7209
BIC	-	467.5312	405.6620	406.9358	405.4745

In the two applications, we shall compare the P-ML model with the Poisson, PL and PB models. The measures of goodness-of-fit include the Akaike information criterion (AIC), Akaike information criterion corrected (AICc), Bayesian information criterion (BIC), and Chi-square statistics ( $\chi^2$ ). In general, the

Table 2: Distribution of the number of times that computer break down in each of the 128 consecutive week of operation with the obtained statistical results

No. of times the computer broke down	Observed frequency	Expected frequency			
		Poisson	PL	PB	P-ML
0	15	4.708857e+01	19.27560383	14.42977774	17.7259203
1	19	4.708857e+01	19.13170881	19.08231193	19.6220324
2	23	2.354428e+01	17.36852330	18.95570258	18.2549947
3	14	7.848094e+00	14.96193268	16.76368107	15.6530792
4	15	1.962024e+00	12.45058537	13.91999018	12.8183756
5	10	3.924047e-01	10.10964793	11.11329656	10.2024619
6	8	6.540078e-02	8.05988261	8.63910859	7.9718041
7	4	9.342968e-03	6.33514960	6.58853505	6.1531345
8	6	1.167871e-03	4.92341554	4.95346750	4.7110926
9	2	1.297634e-04	3.79105622	3.68353642	3.5881866
10	3	1.297634e-05	2.89676320	2.71566028	2.7241949
11	3	1.179668e-06	2.19908843	1.98833716	2.0646421
12	2	9.830563e-08	1.66018086	1.44768579	1.5636915
13	1	7.561971e-09	1.24731297	1.04921528	1.1843565
16	1	2.250586e-12	0.51675851	0.39135270	0.5170605
17	1	1.323874e-13	0.38278119	0.28028170	0.3931708
22	1	4.189369e-20	0.08240787	0.05157894	0.1018824
$\hat{\lambda}$	-	1	-	-	-
$\hat{\theta}$	-	-	0.4232876	0.2077904	0.2963483
$-2\log L$	-	1427	634.2476	633.2061	632.9591
$\chi^2$	-	2.386995e+19	17.3813201	24.58311211	13.9381964
$p$ -value	-	0.0000e+00	0.2965847	0.05583001	0.5302216
AIC	-	1429.0671	636.2476	635.2061	634.9591
AICc	-	1429.0988	636.2793	635.2378	634.9908
BIC	-	1431.9191	639.0996	638.0581	637.8111

smaller the values of these statistics, the better the fit to the data. In addition, the  $p$ -values of the Chi-square test are communicated. The required computations are carried out using a script in the R-language.

The numerical values of  $-2\log L$  i.e., the  $-2$  times the estimated log-likelihood function,  $\chi^2$  and  $p$ -values, AIC, AICc and BIC are listed in Tables 1 and 2 for the fitted Poisson, PL, PB and P-ML models for the first and second data set, respectively.

From Tables 1 and 2, we see that the smallest  $\chi^2$ , AIC, AICc, and BIC statistics are achieved for the P-ML model, except for the first data set where the PL model is better to the other in terms of  $\chi^2$  only. Therefore, from these results, we can modestly say that the new P-ML distribution is a better model than the others; it can be preferred for fitting the current data sets.

Figures 5 and 6 illustrate the plots of the estimated pmfs for the two data sets, respectively.

## 4 Conclusions

In this paper, a new one-parameter lifetime distribution is introduced, defined as the distribution of a random variable  $X$  following the Poisson distribution with a parameter  $\lambda$ , assuming that  $\lambda$  is a random variable following the modified Lindley distribution. It is called the P-ML distribution. Its properties and applications are studied. Explicit mathematical expressions for some of its basic statistical properties, such as the distribution function, mean, variance, coefficient of variation, index of dispersion, factorial moments, skewness, kurtosis, and probability generating function are discussed. The method of maximum likelihood estimation is used in estimating the sole parameter of the P-ML distribution. The goodness-of-fits of the P-ML distribution over other competitive distributions is evaluated for two real datasets. It is seen that the

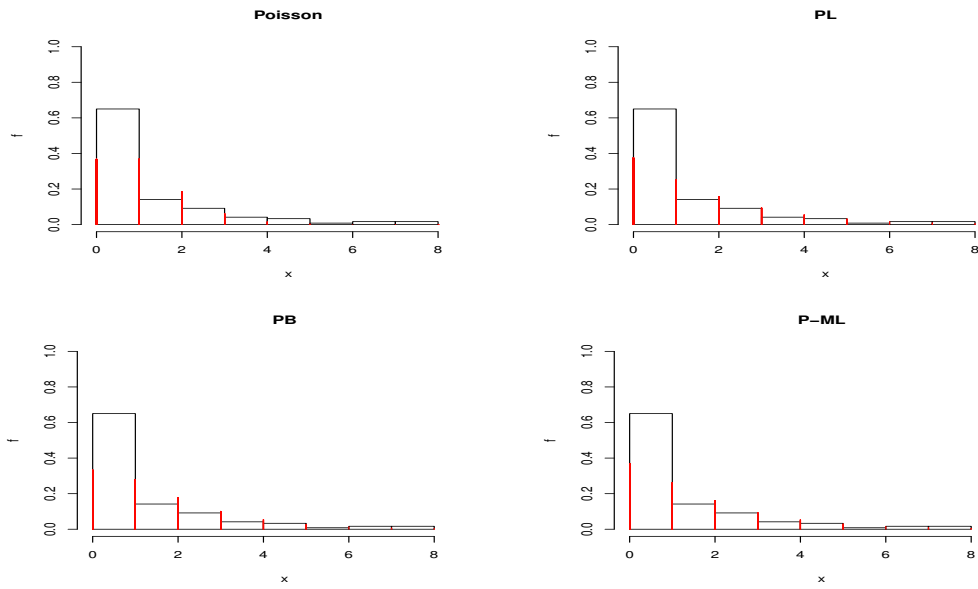


Figure 5: Plots of estimated pmfs for the first data set

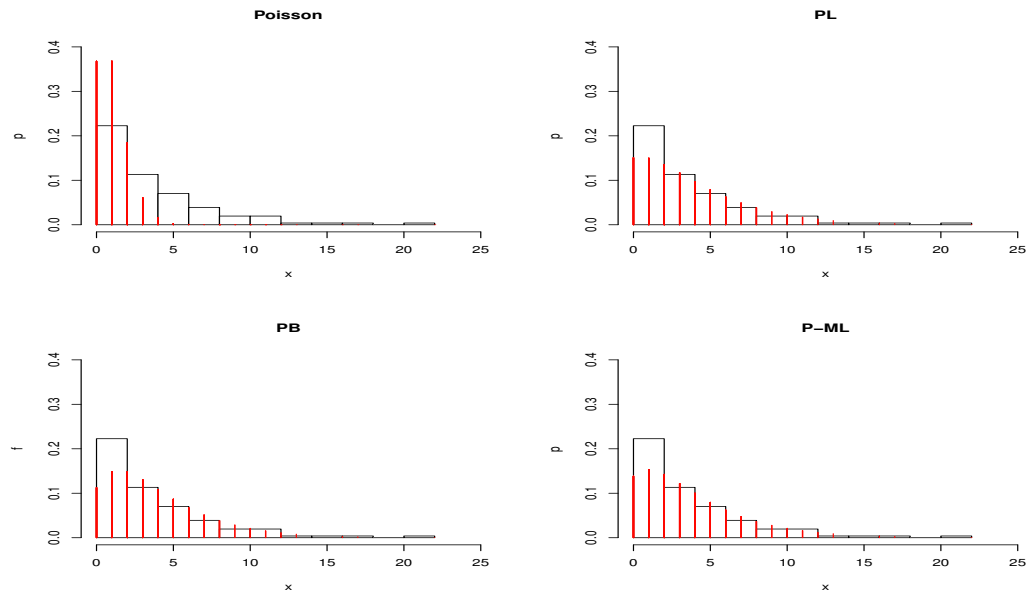


Figure 6: Plots of estimated pmfs for the second data set

P-ML distribution gives a competitive fit, and thus, it can be considered as an important discrete distribution for modeling count data.

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## References

- [1] A. Asgharzadeh, H. S. Bakouch and L. Esmaeili, Pareto Poisson-Lindley distribution with applications, *J. Appl. Stat.*, 40(2013), 1717–1734.
- [2] E. Altun, A new one-parameter discrete distribution with associated regression and integer-valued autoregressive models, *Math. Slovaca*, 70(2020), 979–994.
- [3] A. Bertie and K. McConway, M345 Statistical Methods, Unit 11: Statistical Computing III, Milton Keynes: Open University, 16(1987).
- [4] D. Bhati, P. Kumawat and E. Gómez-Déniz, A new count model generated from mixed Poisson transmuted exponential family with an application to health care data, *Comm. Statist. Theory Methods*, 46(2017), 11060–11076.
- [5] W. Bodhisuwan and S. Sangpoom, The discrete weighted Lindley distribution. In *Proceedings of the International Conference on Mathematics, Statistics, and Their Applications*, Banda Aceh, Indonesia, (2016), 4–6.
- [6] L. Cheng, S. R. Geedipally and D. Lord, The Poisson-Weibull generalized linear model for analyzing motor vehicle crash data, *Safety Science*, 54(2013), 38–42.
- [7] C. Chesneau, L. Tomy and J. Gillariose, A new modified Lindley distribution with properties and applications, *Int. j. stat. manag. syst.*, 24(2021), 1383-1403.
- [8] E. G. Déniz, A new discrete distribution: properties and applications in medical care, *J. Appl. Stat.*, 40(2013), 2760–2770.
- [9] Y. Gencturk and A. Yigiter, Modelling claim number using a new mixture model: negative binomial gamma distribution, *J. Stat. Comput. Simul.*, 86(2016), 1829–1839.
- [10] M. E. Ghitany, D. K. Al-Mutairi and S. Nadarajah, Zero-truncated Poisson-Lindley distribution and its application, *Mathematics and Computers in Simulation*, 79(2008), 279–287.
- [11] M. E. Ghitany, B. Atieh and S. Nadarajah, Lindley distribution and its application, *Math. Comput. Simulation*, 78(2008), 493–506.
- [12] M. E. Ghitany and D. K. Al-Mutairi, Estimation methods for the discrete Poisson-Lindley distribution, *J. Stat. Comput. Simul.*, 79(2009), 1–9.
- [13] E. Gómez-Déniz, J. M. Sarabia and N. Balakrishnan, A multivariate discrete Poisson-Lindley distribution: extensions and actuarial applications, *ASTIN Bulletin: The Journal of the IAA*, 42(2012), 655–678.
- [14] R. Grine and H. Zeghdoudi, On Poisson quasi-Lindley distribution and its applications, *Journal of modern applied statistical methods*, 16(2017), 21.
- [15] D. J. Hand, F. Daly, A. D. Lunn, K. J. McConway and E. O. Ostrowski, *A Hand Book of Small Data Sets*, cRc Press, London: Chapman and Hall, 1993.
- [16] T. Imoto, C. M. Ng, S. H. Ong and S. Chakraborty, A modified Conway-Maxwell-Poisson type binomial distribution and its applications, *Communication in Statistics-Theory and Methods*, 46(2017), 12210–12225.

- [17] D. V. Lindley, Fiducial distributions and Bayes-theorem, *Journal of the Royal Statistical Society, Series B (Methodological)*, 20(1958), 102–107.
- [18] D. Lord and S. R. Geedipally, The negative binomial-Lindley distribution as a tool for analyzing crash data characterized by a large amount of zeros, *Accident Analysis and Prevention*, 43(2011), 1738–1742.
- [19] E. Mahmoudi and H. Zakerzadeh, Generalized Poisson-lindley distribution, *Communications in Statistics-Theory and Methods*, 39(2010), 1785–1798.
- [20] M. Mohammadpour, H. S. Bakouch and M. Shirozhan, Poisson-Lindley INAR (1) model with applications, *Braz. J. Probab. Stat.*, 32(2018), 262–280.
- [21] J. Rodríguez-Avi, A. Conde-Sánchez, A. J. Sáez-Castillo, M. J. OlmoJiménez and A. M. Martínez-Rodríguez, A generalized Waring regression model for count data, *Comput. Statist. Data Anal.*, 53(2009), 3717–3725.
- [22] A. J. Sáez-Castillo and A. Conde-Sánchez, A hyper-Poisson regression model for over-dispersed and under-dispersed count data, *Comput. Statist. Data Anal.*, 61(2013), 148–157.
- [23] M. Sankaran, The discrete Poisson-Lindley distribution, *Biometrics*, 26(1970), 145–149.
- [24] R. Shanker and A. Mishra, A two-parameter Poisson-Lindley distribution, *International journal of Statistics and Systems*, 9(2014), 79–85.
- [25] R. Shanker and A. Mishra, A quasi Poisson-Lindley distribution, *J. Indian Statist. Assoc.*, 54(2016), 113–125.
- [26] R. Shanker and A. Mishra, A two-parameter Lindley distribution, *Statistics in Transition new Series*, 14(2013), 45–56.
- [27] R. Shanker and A. Mishra, A quasi Lindley distribution, *African Journal of Mathematics and Computer Science Research*, 6(2013), 64–71.
- [28] R. Shanker, S. Sharma and R. Shanker, A Discrete two-Parameter Poisson Lindley distribution, *Journal of Ethiopian Statistical Association*, 21(2012), 15–22.
- [29] R. Shanker, S. Sharma and R. Shanker, A two-parameter Lindley distribution for modeling waiting and survival times data, *Appl. Math.*, 4(2013), 363–368.
- [30] R. Shanker and A. L. Tekie, A new quasi Poisson-Lindley distribution, *International Journal of Statistics and Systems*, 9(2014), 87–94.
- [31] R. Shanker and A. G. Amanuel, A new quasi Lindley distribution, *International Journal of Statistics and Systems*, 8(2013), 143–156.
- [32] G. Shmueli, T. P. Minka, J. B. Kadane, S. Borle and P. Boatwright, A useful distribution for fitting discrete data: revival of the Conway-Maxwell-Poisson distribution, *J. Roy. Statist. Soc. Ser. C*, 54(2005), 127–142.
- [33] M. M. Shoukri, M. H. Asyali, R. VanDorp and D. Kelton, The Poisson inverse Gaussian regression model in the analysis of clustered counts data, *Journal of Data Science*, 2(2014), 17–32.
- [34] W. Wongrin and W. Bodhisuwan, Generalized Poisson-Lindley linear model for count data, *J. Appl. Stat.*, 44(2017), 2659–2671.
- [35] H. Zeghdoudi and S. Nedjar, On Poisson pseudo Lindley distribution: Properties and applications, *Journal of probability and statistical science*, 15(2017), 19–28.

- [36] H. Zamani, N. Ismail and P. Faroughi, Poisson-weighted exponential univariate version and regression model with applications, *J. Appl. Math. Stat.*, 10(2014), 148–154.
- [37] H. Zakerzadeh and A. Dolati, Generalized Lindley distribution, *J. Math. Ext.*, 3(2009), 13–25.
- [38] H. Zeghdoudi and S. Nedjar, A pseudo Lindley distribution and its application, *Afr. Stat.*, 11(2016), 923–932.
- [39] H. Zeghdoudi and S. Nedjar, On gamma Lindley distribution: Properties and simulations, *J. Comput. Appl. Math.*, 298(2016), 167–174.
- [40] S. Nedjar and H. Zeghdoudi, New compound distribution: properties, inflated distribution, and applications, *J. Agric. Stat. Sci.*, 16(2020), 519-526.