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Branciari S_b-Metric Space And Related Fixed Point Theorems With An Application*

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Abstract

In this article, we introduce the concept of Branciari S_b-metric space which is a generalization of S-metric space and S_b-metric space. By defining a topology on such spaces some interesting topological properties have been studied herein. We prove fixed point theorems for two contractive type mappings over such spaces. Finally we apply our established theorem to find a unique solution of a system of linear algebraic equations.

1 Introduction and Preliminaries

During the last ninety years, fixed point theory is one of the interesting areas of research in Mathematics. Several generalizations of usual metric structure had been made by a good number of researchers for their need, particularly in order to study fixed point theory, one such generalized metric space known as rectangular metric space was introduced by Branciari [1] in the year 2000, where the triangle inequality had been replaced by a so-called quadrilateral or rectangular inequality. Z. Kadelburg and S. Radenović in their survey article (See [3]) discussed in a nutshell the structure of rectangular metric spaces. The definition of rectangular metric space is given as follows:

Definition 1 ([1]) *Let X be a nonempty set and $\rho : X \times X \to [0, \infty)$ be a mapping. Then ρ is said to be a rectangular metric if it satisfies the following conditions:*

(i) $\rho(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$*;*

(iii) $\rho(x, y) \leq \rho(x, a) + \rho(a, b) + \rho(b, y)$ *for all* $x, y \in X$ *and for all* $a, b(a \neq b) \in X \setminus \{x, y\}$*. The pair* (X, ρ) *is called a rectangular metric space.*

There are several rectangular metric spaces which are not usual metric spaces. Let us recall the following example.

Example 1 ([3]) *Let* $U = \{0, 2\}$, $V = \{\frac{1}{n} : n \geq 1\}$ *and* $X = U \cup V$*. Define* $\rho : X^2 \to [0, \infty)$ *by*

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \text{ and either } x, y \in U \text{ or } x, y \in V, \\ y & \text{if } x \in U \text{ and } y \in V, \\ x & \text{if } x \in V \text{ and } y \in U. \end{cases}$$

Then ρ *is a rectangular metric on* X *but not an usual metric since*

$$\rho(0, 2) = 1 > \frac{2}{3} = \rho(0, \frac{1}{3}) + \rho(\frac{1}{3}, 2).$$

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In 2012, Sedghi et al. [8] introduced a new type of metric structure consisting of three variables known as S-metric. Several authors worked on such type of spaces and proved several fixed point theorems (See [2, 5, 6]). Subsequently in the year 2016, N. Souayah and N. Mlaiki [9] investigated the notion of S_b-metric spaces which generalizes the concept of S-metric spaces.

Definition 2 ([8, 7]) *Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:*

(i) $S(x, y, z) = 0$ *if and only if* $x = y = z$;

(ii) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

The pair (X, S) is called an S-metric space.

There are various types of S-metric spaces, some of them are given below.

Example 2 ([8]) *(1) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.*

(2) Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the usual S-metric on $\mathbb{R}$.

Definition 3 ([4, 9]) *Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $S_b : X^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $x, y, z, t \in X$: the following conditions hold:*

(i) $S_b(x, y, z) = 0$ *if and only if* $x = y = z$;

(ii) $S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]$.

The pair (X, S_b) is called an S_b-metric space.

Example 3 ([9]) *Let X be a nonempty set and $card(X) \geq 5$. Suppose $X = X_1 \cup X_2$ a partition of X such that $card(X_1) \geq 4$. Let $s \geq 1$. Then*

$$S_b(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ 5 & \text{if } x = 1 = y \text{ and } z = 2, \\ \frac{1}{n+1} & \text{if } x = 1 = y \text{ and } z \geq 3, \\ \frac{1}{n+2} & \text{if } x = 2 = y \text{ and } z \geq 3, \\ 3 & \text{otherwise.} \end{cases}$$

for all $x, y, z \in X$. Then S_b is an S_b-metric on X with coefficient s.

2 Introduction to Branciari S_b-Metric Space

In this section we give the definition of Branciari S_b-metric space and discuss some properties of such spaces.

Definition 4 *Let X be a nonempty set and $\sigma : X^3 \to \mathbb{R}_0^+$ be a function. Then σ is said to be Branciari S_b-metric if it satisfies the following conditions:*

(i) $\sigma(x, y, z) = 0$ *if and only if* $x = y = z$;

(ii) *For any $x, y, z \in X$ and for $a, b \in X \setminus \{x, y, z\}$ with $a \neq b$ we have*

$$\sigma(x, y, z) \leq k[\sigma(x, x, a) + \sigma(y, y, a) + \sigma(z, z, b) + \sigma(a, a, b)],$$

where $k \geq 1$. The pair (X, σ) is called Branciari S_b-metric space.

Definition 5 *A Branciari S_b-metric σ on a nonempty set X is said to be symmetric if $\sigma(x, x, y) = \sigma(y, y, x)$ for all $x, y \in X$.*

Proposition 1 (i) *Let (X, S) be an S-metric space (See Definition 2). The X is also a Branciari S_b-metric space for $k = 2$.*

(ii) *Let (X, S_b) be an S_b-metric space with coefficient $s \geq 1$ (See Definition 3). The X is also a Branciari S_b-metric space for $k = 2s^2$.*

Proof. *(i) The first condition of Definition 4 follows trivially for an S-metric. Now let us choose $x, y, z \in X$ and $a, b \in X \setminus \{x, y, z\}$ with $a \neq b$. Then we get*

$$\begin{aligned} S(x, y, z) &\leq S(x, x, a) + S(y, y, a) + S(z, z, a) \\ &\leq S(x, x, a) + S(y, y, a) + 2S(z, z, b) + S(a, a, b) \\ &\leq 2[S(x, x, a) + S(y, y, a) + S(z, z, b) + S(a, a, b)]. \end{aligned}$$

Therefore all the conditions of Definition 4 are satisfied and therefore S is a Branciari S_b-metric for $k = 2$.

(ii) Clearly, S_b satisfies the first condition of Definition 4. Now let us take $x, y, z \in X$ and $a, b \in X \setminus \{x, y, z\}$ with $a \neq b$. Then we get

$$\begin{aligned} S_b(x, y, z) &\leq s[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)] \\ &\leq s[S_b(x, x, a) + S_b(y, y, a)] + s^2[2S_b(z, z, b) + S_b(a, a, b)] \\ &\leq 2s^2[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, b) + S_b(a, a, b)]. \end{aligned}$$

Therefore S_b is a Branciari S_b-metric for $k = 2s^2$. ■

Proposition 1 shows that any S-metric space or an S_b-metric space is also a Branciari S_b-metric space but there are several Branciari S_b-metric spaces which are neither S-metric spaces nor S_b-metric spaces.

Example 4 *Let $X = \mathbb{N}$ and $\sigma : X^3 \to [0, \infty)$ be defined by*

$$\sigma(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ 5 & \text{if } x = 1 = y \text{ and } z = 2, \\ \frac{1}{n+1} & \text{if } x = 1 = y \text{ and } z \geq 3, \\ \frac{1}{n+2} & \text{if } x = 2 = y \text{ and } z \geq 3, \\ 3 & \text{otherwise.} \end{cases}$$

Also we take $\sigma(x, x, y) = \sigma(y, y, x)$ for all $x, y \in X$. Then σ is a symmetric Branciari S_b-metric on X for $k = \frac{5}{3}$ but it is neither an S-metric nor an S_b-metric for any $k \geq 1$.

Definition 6 *Let (X, σ) be a Branciari S_b-metric space. Then*

(i) *A sequence $\{x_n\}$ in X is said to be convergent to some $z \in X$ if $\sigma(x_n, x_n, z) \to 0$ as $n \to \infty$.*

(ii) *A sequence $\{x_n\}$ in X is said to be Cauchy if $\sigma(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.*

(iii) X is said to be complete if every Cauchy sequence in X is convergent to some element in X.

Next we prove some lemmas with respect to symmetric Branciari S_b-metric space which will be useful for our further results.

Lemma 1 *Let (X, σ) be a symmetric Branciari S_b-metric space and $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ for $n \neq m$. Then $\{x_n\}$ converges to atmost one point.*

Proof. If possible let $\{x_n\}$ converges to two distinct points x and y. Since all the elements of $\{x_n\}$ are distinct, we see that there exists $p \geq 1$ such that $x_n \notin \{x, y\}$ for all $n > p$. Thus

$$\begin{aligned}\sigma(x,x,y) &\leq k[2\sigma(x,x,x_n) + \sigma(y,y,x_{n+1}) + \sigma(x_n,x_n,x_{n+1})] \\ &= k[2\sigma(x_n,x_n,x) + \sigma(x_{n+1},x_{n+1},y) + \sigma(x_n,x_n,x_{n+1})] \text{ for all } n > p.\end{aligned}$$

Taking $n \to \infty$ we get $\sigma(x,x,y) = 0$, a contradiction. Hence $\{x_n\}$ can converge to atmost one point in X. ■

Lemma 2 *Let (X, σ) be a symmetric Branciari S_b-metric space and $\{x_n\}$ be a sequence in X with distinct terms, which is both Cauchy and convergent. Then the limit of $\{x_n\}$ is unique say x and moreover for all $z \in X$ one can get*

$$\frac{1}{k}\sigma(x,x,z) \leq \liminf_{n\to\infty} \sigma(x_n,x_n,z) \leq \limsup_{n\to\infty} \sigma(x_n,x_n,z) \leq k\sigma(x,x,z).$$

Proof. If $z = x$ then the inequality is clearly satisfied. So let $z(\neq x) \in X$. Then there exists $q \geq 1$ such that $x_n \notin \{x, z\}$ for all $n > q$. Therefore

$$\begin{aligned}\sigma(x_n,x_n,z) &\leq k[2\sigma(x_n,x_n,x_{n+1}) + \sigma(z,z,x) + \sigma(x_{n+1},x_{n+1},x)] \\ &= k[2\sigma(x_n,x_n,x_{n+1}) + \sigma(x,x,z) + \sigma(x_{n+1},x_{n+1},x)] \text{ for all } n > q.\end{aligned}$$

Taking $n \to \infty$ we have $\limsup_{n\to\infty} \sigma(x_n,x_n,z) \leq k\sigma(x,x,z)$. Also we get

$$\begin{aligned}\sigma(x,x,z) &\leq k[2\sigma(x,x,x_{n+1}) + \sigma(z,z,x_n) + \sigma(x_{n+1},x_{n+1},x_n)] \\ &= k[2\sigma(x_{n+1},x_{n+1},x) + \sigma(x_n,x_n,z) + \sigma(x_{n+1},x_{n+1},x_n)] \text{ for all } n > q.\end{aligned}$$

Taking $n \to \infty$ we have $\liminf_{n\to\infty} \sigma(x_n,x_n,z) \geq \frac{1}{k}\sigma(x,x,z)$. Hence by combining these two we get the required conclusion. ■

The following remark shows the distinction between some behavioral properties namely convergence and Cauchyness of a sequence in a Branciari S_b-metric space with that in an S-metric space and S_b-metric space.

Remark 1 *(i) In an S-metric space, the limit of a convergent sequence is always unique but from Example 4 it is clear that $\{n\}_{n\geq 3}$ converges to both 1 and 2.*

(ii) Any convergent sequence in an S-metric space is Cauchy but in Example 4 we see that $\{n\}_{n\geq 3}$ is convergent but it is not Cauchy since $\sigma(n,n,m) = 3$ for all $n, m \geq 3$.

(iii) It is also known that an S-metric is continuous that is $\lim_{n\to\infty} S(x_n,x_n,z) = S(x,x,z)$ for all $z \in X$ whenever $\{x_n\}$ converges to x but in Example 4 we see that $\lim_{n\to\infty} \sigma(n,n,1) = 0 \neq 5 = \sigma(2,2,1)$ though $\{n\}_{n\geq 3}$ converges to 2.

We can define open balls in a Branciari S_b-metric space (X, σ) in a usual way. For an $x \in X$ and $\epsilon > 0$ we define $B^\sigma(x, \epsilon) = \{y \in X : \sigma(y,y,x) < \epsilon\}$.

Remark 2 *Let us take*

$$\tau_\sigma = \{\emptyset\} \cup \{U_\sigma(\neq \emptyset) \subset X : \text{ there exists } \epsilon_x > 0 \text{ for every } x \in U_\sigma \text{ such that } B^\sigma(x, \epsilon_x) \subset U_\sigma\}.$$

Then τ_σ forms a topology on X.

The topology τ_σ of X possesses some interesting properties, which are as follows.

Remark 3 *(i) (X, σ) may not be Hausdorff with respect to the topology τ_σ. If it is, then in Example 4 there exists two open sets U and V such that $1 \in U$, $2 \in V$ and $U \cap V = \emptyset$. Now there exists $r_1, r_2 > 0$ such that $B^\sigma(1, r_1) \subset U$, $B^\sigma(2, r_2) \subset V$ and therefore $B^\sigma(1, r_1) \cap B^\sigma(2, r_2) = \emptyset$, but we see that $B^\sigma(1, r) \cap B^\sigma(2, t) \neq \emptyset$ for any $r, t > 0$, a contradiction. Hence in Example 4 the topology τ_σ is not Hausdorff.*

(ii) Also the open balls in (X, τ_σ) may not be always open sets. In Example 4 we see that $B^\sigma(3, \frac{1}{2}) = \{1, 2, 3\}$ but $B^\sigma(1, r)$ contains all but finitely many elements of $\mathbb{N}$ for any $r > 0$ and therefore $B^\sigma(1, r) \not\subseteq B^\sigma(3, \frac{1}{2})$ for any $r > 0$. Hence $B^\sigma(3, \frac{1}{2})$ is not open in τ_σ.

3 Some Fixed Point Theorems in Symmetric Branciari S_b-Metric Space

Theorem 1 (Analogue to Banach Contraction Theorem) *Let (X, σ) be a complete symmetric Branciari S_b-metric space and $T : X \to X$ satisfies*

$$\sigma(Tx, Tx, Ty) \leq \alpha \sigma(x, x, y) \text{ for all } x, y \in X, \tag{1}$$

where $\alpha \in (0, 1)$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrarily chosen and we construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \geq 1$. If $x_{p-1} = x_p$ for some $p \in \mathbb{N}$, then T has a fixed point in X. So we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. We show that $\{x_n\}$ is Cauchy sequence in X.

Case-I: $\alpha \in (0, \frac{1}{\sqrt{k}})$. Then

$$\begin{aligned} S_n &= \sigma(x_n, x_n, x_{n+1}) \\ &\leq \alpha \sigma(x_{n-1}, x_{n-1}, x_n) \\ &\leq \alpha^2 \sigma(x_{n-2}, x_{n-2}, x_{n-1}) \\ &\dots \\ &\leq \alpha^n \sigma(x_0, x_0, x_1) = \alpha^n S_0 \text{ for all } n \geq 1. \end{aligned}$$

Similarly, we have

$$S_n^* = \sigma(x_n, x_n, x_{n+2}) \leq \alpha^n \sigma(x_0, x_0, x_2) = \alpha^n S_0^*$$

for all $n \in \mathbb{N}$. Now if $x_n = x_m$ for some $m > n$, then we have

$$0 < S_n = \sigma(x_n, x_n, x_{n+1}) = \sigma(x_m, x_m, x_{m+1}) = S_m.$$

Therefore $S_n = S_m \leq \alpha S_{m-1} \leq ... \leq \alpha^{m-n} S_n < S_n$, which is a contradiction. From this it follows that $x_n \neq x_m$ for any $n, m(n \neq m) \in \mathbb{N}$. Now for $p = 2m+1$, we have

$$\begin{aligned}
& \sigma(x_n, x_n, x_{n+p}) \\
\leq\; & k[2\sigma(x_n, x_n, x_{n+1}) + \sigma(x_{n+p}, x_{n+p}, x_{n+2}) + \sigma(x_{n+1}, x_{n+1}, x_{n+2})] \\
=\; & 2k\sigma(x_n, x_n, x_{n+1}) + k\sigma(x_{n+1}, x_{n+1}, x_{n+2}) + k\sigma(x_{n+2}, x_{n+2}, x_{n+p}) \\
\leq\; & 2k\sigma(x_n, x_n, x_{n+1}) + k\sigma(x_{n+1}, x_{n+1}, x_{n+2}) \\
& +k^2[2\sigma(x_{n+2}, x_{n+2}, x_{n+3}) + \sigma(x_{n+p}, x_{n+p}, x_{n+4}) + \sigma(x_{n+3}, x_{n+3}, x_{n+4})] \\
=\; & 2k\sigma(x_n, x_n, x_{n+1}) + k\sigma(x_{n+1}, x_{n+1}, x_{n+2}) \\
& +2k^2\sigma(x_{n+2}, x_{n+2}, x_{n+3}) + k^2\sigma(x_{n+3}, x_{n+3}, x_{n+4}) + k^2\sigma(x_{n+4}, x_{n+4}, x_{n+p}) \\
& \vdots \\
\leq\; & 2k[\sigma(x_n, x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+1}, x_{n+2})] \\
& +2k^2[\sigma(x_{n+2}, x_{n+2}, x_{n+3}) + \sigma(x_{n+3}, x_{n+3}, x_{n+4})] + \cdots \\
& +2k^m[\sigma(x_{n+2m-2}, x_{n+2m-2}, x_{n+2m-1}) + \sigma(x_{n+2m-1}, x_{n+2m-1}, x_{n+2m})] \\
& +k^m\sigma(x_{n+2m}, x_{n+2m}, x_{n+2m+1}) \\
\leq\; & 2[\{k(\alpha^n + \alpha^{n+1}) + k^2(\alpha^{n+2} + \alpha^{n+3}) + \cdots \\
& +k^m(\alpha^{n+2m-2} + \alpha^{n+2m-1})\} + k^m\alpha^{n+2m}]\sigma(x_0, x_0, x_1) \\
=\; & 2k(1+\alpha)\alpha^n[1 + k\alpha^2 + ... + k^m\alpha^{2m}]S_0 \\
\leq\; & \frac{2k(1+\alpha)}{1-k\alpha^2}\alpha^n S_0 \quad \text{for all } n \geq 1. \qquad (2)
\end{aligned}$$

Also for $p = 2m$ we get

$$\begin{aligned}
& \sigma(x_n, x_n, x_{n+p}) \\
\leq\; & k[2\sigma(x_n, x_n, x_{n+1}) + \sigma(x_{n+p}, x_{n+p}, x_{n+2}) + \sigma(x_{n+1}, x_{n+1}, x_{n+2})] \\
=\; & 2k\sigma(x_n, x_n, x_{n+1}) + k\sigma(x_{n+1}, x_{n+1}, x_{n+2}) + k\sigma(x_{n+2}, x_{n+2}, x_{n+p}) \\
\leq\; & 2k\sigma(x_n, x_n, x_{n+1}) + k\sigma(x_{n+1}, x_{n+1}, x_{n+2}) \\
& +k^2[2\sigma(x_{n+2}, x_{n+2}, x_{n+3}) + \sigma(x_{n+p}, x_{n+p}, x_{n+4}) + \sigma(x_{n+3}, x_{n+3}, x_{n+4})] \\
=\; & 2k\sigma(x_n, x_n, x_{n+1}) + k\sigma(x_{n+1}, x_{n+1}, x_{n+2}) \\
& +2k^2\sigma(x_{n+2}, x_{n+2}, x_{n+3}) + k^2\sigma(x_{n+3}, x_{n+3}, x_{n+4}) + k^2\sigma(x_{n+4}, x_{n+4}, x_{n+p}) \\
& \vdots \\
\leq\; & 2k[\sigma(x_n, x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+1}, x_{n+2})] \\
& +2k^2[\sigma(x_{n+2}, x_{n+2}, x_{n+3}) + \sigma(x_{n+3}, x_{n+3}, x_{n+4})] + \cdots \\
& +2k^{m-1}[\sigma(x_{n+2m-4}, x_{n+2m-4}, x_{n+2m-3}) + \sigma(x_{n+2m-3}, x_{n+2m-3}, x_{n+2m-2})] \\
& +k^{m-1}\sigma(x_{n+2m-2}, x_{n+2m-2}, x_{n+2m}) \\
\leq\; & 2[\{k(\alpha^n + \alpha^{n+1}) + k^2(\alpha^{n+2} + \alpha^{n+3}) + \cdots \\
& +k^{m-1}(\alpha^{n+2m-4} + \alpha^{n+2m-3})\}\sigma(x_0, x_0, x_1) + k^{m-1}\alpha^{n+2m-2}\sigma(x_0, x_0, x_2)] \\
=\; & 2k(1+\alpha)\alpha^n[1 + k\alpha^2 + ... + k^{m-2}\alpha^{2m-4}]S_0 + k^{m-1}\alpha^{n+2m-2}S_0^* \\
\leq\; & \frac{2k(1+\alpha)}{1-k\alpha^2}\alpha^n S_0 + \alpha^n(k\alpha^2)^{m-1}S_0^* = \frac{2k(1+\alpha)}{1-k\alpha^2}\alpha^n S_0 + \alpha^n S_0^* \quad \text{for all } n \geq 1. \qquad (3)
\end{aligned}$$

Therefore from (2) and (3) we conclude that $\{x_n\}$ is Cauchy in X. Since X is complete it follows from Lemma 1 that $\{x_n\}$ converges to a unique point $z \in X$. Now,

$$\sigma(x_{n+1}, x_{n+1}, Tz) \leq \alpha\sigma(x_n, x_n, z) \to 0 \text{ as } n \to \infty.$$

Hence $Tz = z$ and z is a fixed point of T in X. The uniqueness of fixed point is very much clear.
Case-II: $\alpha \in [\frac{1}{\sqrt{k}}, 1)$. Then there exists $N \in \mathbb{N}$ such that $\alpha^N \in (0, \frac{1}{\sqrt{k}})$. Now due to the contractive condition (1) we see that T^N also satisfies the contractive condition (1) for the Lipschitz constant α^N. Therefore by Case-I, T^N has a unique fixed point in X and thus in this case also T has a unique fixed point. ■

Example 5 *Let $X = \mathbb{N}$ and $\sigma : X^3 \to [0, \infty)$ be defined by $\sigma(x, x, x) = 0$ and $\sigma(x, x, y) = \sigma(y, y, x)$ for all $x, y \in X$ with*

$$\sigma(x, y, z) = \begin{cases} 10 & \text{if } x = 1 = y \text{ and } z = 2, \\ \frac{1}{2(n+1)} & \text{if } x = 1 = y \text{ and } z \geq 3, \\ \frac{1}{n+2} & \text{if } x = 2 = y \text{ and } z \geq 3, \\ 5 & \text{otherwise.} \end{cases}$$

Then σ is a complete symmetric Branciari S_b-metric space for $k = 4$ but it is neither an S-metric nor an S_b-metric space. Let $T : X \to X$ be given by

$$T(x) = \begin{cases} 4 & \text{if } x = 1, \\ 5 & \text{if } x \neq 1. \end{cases}$$

Then T^2 satisfies the contractive condition (1) for any $\alpha \in (0, 1)$ and thus T^2 has a unique fixed point in X. Therefore T has a unique fixed point $x = 5$ in X.

Theorem 2 *(Analogue to Kannan fixed point theorem) Let (X, σ) be a complete symmetric Branciari S_b-metric space and $T : X \to X$ satisfies*

$$\sigma(Tx, Tx, Ty) \leq \beta[\sigma(x, x, Tx) + \sigma(y, y, Ty)] \text{ for all } x, y \in X, \tag{4}$$

where $\beta \in (0, \frac{1}{2})$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be taken as arbitrary and let us construct the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \geq 1$. If $x_{i-1} = x_i$ for some $i \in \mathbb{N}$ then T has a fixed point in X. So we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Here we show that $\{x_n\}$ is Cauchy sequence in X.
Case-I: $\beta \in (0, \frac{1}{k+1})$. From the contractive condition (4) we get

$$\begin{aligned} \sigma(x_n, x_n, x_{n+1}) &= \sigma(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \beta[\sigma(x_{n-1}, x_{n-1}, x_n) + \sigma(x_n, x_n, x_{n+1})] \text{ for all } n \geq 1. \end{aligned} \tag{5}$$

From which we get $S_n = \sigma(x_n, x_n, x_{n+1}) \leq \frac{\beta}{1-\beta}\sigma(x_{n-1}, x_{n-1}, x_n) = \gamma\sigma(x_{n-1}, x_{n-1}, x_n) = \gamma S_{n-1} \leq ... \leq \gamma^n S_0$ for all $n \in \mathbb{N}$, where $\gamma = \frac{\beta}{1-\beta} < \frac{1}{k}$. Also we have,

$$\begin{aligned} S_n^* = \sigma(x_n, x_n, x_{n+2}) &= \sigma(Tx_{n-1}, Tx_{n-1}, Tx_{n+1}) \\ &\leq \beta[\sigma(x_{n-1}, x_{n-1}, x_n) + \sigma(x_{n+1}, x_{n+1}, x_{n+2})] \\ &\leq \beta[\gamma^{n-1} + \gamma^{n+1}]S_0 \\ &= \beta(1 + \gamma)\gamma^{n-1}S_0 \text{ for all } n \geq 1. \end{aligned} \tag{6}$$

By a similar calculation as the previous theorem we can show that $\{x_n\}$ is Cauchy in X and therefore due to the completeness of X there exists a unique $u \in X$ such that $x_n \to u$ as $n \to \infty$. Now,

$$\begin{aligned} \sigma(x_{n+1}, x_{n+1}, Tu) &= \sigma(Tx_n, Tx_n, Tu) \\ &\leq \beta[\sigma(x_n, x_n, x_{n+1}) + \sigma(u, u, Tu)] \\ &\leq \beta\sigma(x_n, x_n, x_{n+1}) + \beta k[2\sigma(u, u, x_n) + \sigma(Tu, Tu, x_{n+1}) + \\ &\quad \sigma(x_n, x_n, x_{n+1})] \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{7}$$

Therefore $\sigma(x_{n+1}, x_{n+1}, Tu) \leq \frac{\beta(1+k)\sigma(x_n,x_n,x_{n+1})+2k\beta\sigma(x_n,x_n,u)}{1-\beta k} \to 0$ as $n \to \infty$. Hence $Tu = u$ and u is a fixed point of T. The uniqueness of fixed point is evident.
Case-II: $\beta \in [\frac{1}{k+1}, \frac{1}{2})$. Then there exists $N \in \mathbb{N}$ such that $\beta\gamma^{N-1} \in (0, \frac{1}{k+1})$, where $\gamma = \frac{\beta}{1-\beta} < 1$. Now due to the contractive condition (4) we see that T^N satisfies the contractive condition (4) for the Lipschitz constant $\beta\gamma^{N-1}$. Therefore by Case-I, T^N has a unique fixed point in X and hence T has a unique fixed point. ■

Example 6 *Let $X = \{\frac{1}{2}, \frac{1}{3}, ...\}$ and $\sigma : X^3 \to [0, \infty)$ be defined by $\sigma(x, x, x) = 0$ and $\sigma(x, x, y) = \sigma(y, y, x)$ for all $x, y \in X$ with*

$$\sigma(x,y,z) = \begin{cases} |n-m| & \text{if } x = \frac{1}{n} = y,\ z = \frac{1}{m} \text{ and } |n-m| > 1, \\ \frac{1}{3} & \text{if } x = \frac{1}{n} = y,\ z = \frac{1}{m} \text{ and } |n-m| = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then σ is a complete symmetric Branciari S_b-metric space for $k = 3$ but not an S-metric, since

$$\sigma(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}) = 2 > 2\sigma(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}) + \sigma(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}) = 1.$$

Let $T : X \to X$ be given by

$$T(x) = \begin{cases} \frac{1}{4} & \text{if } x = \frac{1}{2}, \\ \frac{1}{5} & \text{if } x \leq \frac{1}{3}. \end{cases}$$

Then T satisfies the contractive condition (4) for $\beta = \frac{1}{6}$ and thus T has a unique fixed point $x = \frac{1}{5}$ in X.

4 An Application to the System of Linear Algebraic Equations

In this section we give an application of Theorem 1 for solving a system of linear algebraic equations.

Let us consider the system of n linear algebraic equations in n unknowns

$$\begin{cases} p_{11}x_1 + p_{12}x_2 + ... + p_{1n}x_n + c_1 = 0, \\ p_{21}x_1 + p_{22}x_2 + ... + p_{2n}x_n + c_2 = 0, \\ \vdots \\ p_{n1}x_1 + p_{n2}x_2 + ... + p_{nn}x_n + c_n = 0, \end{cases} \tag{8}$$

where $p_{ij}, c_i \in \mathbb{R}$ for all $1 \leq i, j \leq n$. We can write the system of linear equations in matrix notation as $PX + C = O$, where $P = (p_{ij})_{n\times n}$, $X = (x_1, x_2, ..., x_n)$, $C = (c_1, c_2, ..., c_n)$ and $O = (0, 0, ..., 0)$. To find a solution of the system of linear equations (8) we have to find a fixed point of the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $TX = QX + C$, where $Q = P + I_n$ that is $Q = (q_{ij})_{n\times n}$ with $q_{ij} = p_{ij}$ if $i \neq j$ and $q_{ii} = p_{ii} + 1$ for all $i = 1, ..., n$.

Now we define $\sigma : (\mathbb{R}^n)^3 \to [0, \infty)$ by

$$\sigma(x, y, z) = \max_{1\leq i\leq n} [(x_i - y_i)^2 + (y_i - z_i)^2], \text{ where } x = (x_i), y = (y_i) \text{ and } z = (z_i). \tag{9}$$

Then σ is a symmetric S_b-metric space that is a symmetric Branciari S_b-metric space for $k = 4$.

Theorem 3 *If*

$$\sum_{j=1}^{n} |q_{ij}| \leq \sqrt{\alpha} < 1 \text{ for all } 1 \leq i \leq n,$$

then the system of linear equations (8) has a unique solution in $(\mathbb{R}^n, \sigma)$.

Proof. To find a unique solution of (8) we show that the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $Tx = Qx + C$ for all $x \in \mathbb{R}^n$, where $Q = P + I_n$ that is $Q = (q_{ij})_{n\times n}$ with $q_{ij} = p_{ij}$ if $i \neq j$ and $q_{ii} = p_{ii} + 1$ for all $i = 1(1)n$, satisfies the contractive condition (1). Now for any $x = (x_i)$ and $y = (y_i)$ in $\mathbb{R}^n$ we have

$$\begin{aligned}
\sigma(Tx, Tx, Ty) &= \max_{1\leq i\leq n}\left[\sum_{j=1}^{n} q_{ij}(x_j - y_j)\right]^2 \\
&= \left[\max_{1\leq i\leq n}\left|\sum_{j=1}^{n} q_{ij}(x_j - y_j)\right|\right]^2 \\
&\leq \left[\max_{1\leq i\leq n}\sum_{j=1}^{n} |q_{ij}(x_j - y_j)|\right]^2 \\
&\leq \left[\max_{1\leq i\leq n}\sum_{j=1}^{n} |q_{ij}|\sqrt{\sigma(x, x, y)}\right]^2 \\
&= \left[\max_{1\leq i\leq n}\sum_{j=1}^{n} |q_{ij}|\right]^2 \sigma(x, x, y) \\
&\leq \alpha\sigma(x, x, y).
\end{aligned}$$

Since $(\mathbb{R}^n, \sigma)$ is complete, and by Theorem 1, T has a unique fixed point that is the system of linear equations (8) has a unique solution in $\mathbb{R}^n$. ■

We now give a numerical example in support of Theorem 3.

Example 7 *Let us consider the following system of linear algebraic equations in three variables*

$$\begin{cases} 0.7x_1 + 0.2x_2 + 0.1x_3 + 1 = 0, \\ 0.1x_1 + 0.9x_2 + 0.4x_3 + 2 = 0, \\ 0.3x_1 + 0.1x_2 + 0.8x_3 + 3 = 0. \end{cases} \tag{10}$$

Then the system of linear algebraic equations (10) has a unique solution.

Let $X = \mathbb{R}^3$ be the symmetric Branciari S_b-metric space endowed with the metric $\sigma : X^3 \to [0, \infty)$ defined by

$$\sigma(x, y, z) = \max_{1\leq i\leq 3}[(x_i - y_i)^2 + (y_i - z_i)^2], \text{ for all } x = (x_i), y = (y_i) \text{ and } z = (z_i) \text{ in } X.$$

We can write the above system of linear algebraic equations (10) as

$$\begin{aligned}
-0.7x_1 - 0.2x_2 - 0.1x_3 - 1 = 0, \\
-0.1x_1 - 0.9x_2 - 0.4x_3 - 2 = 0, \\
-0.3x_1 - 0.1x_2 - 0.8x_3 - 3 = 0.
\end{aligned}$$

Here $p_{11} = -0.7$, $p_{12} = -0.2$, $p_{13} = -0.1$; $p_{21} = -0.1$, $p_{22} = -0.9$, $p_{23} = -0.4$; $p_{31} = -0.3$, $p_{32} = -0.1$, $p_{33} = -0.8$; $c_1 = -1$, $c_2 = -2$ and $c_3 = -3$. Thus $q_{11} = 0.3$, $q_{12} = -0.2$, $q_{13} = -0.1$; $q_{21} = -0.1$, $q_{22} = 0.1$, $q_{23} = -0.4$; $q_{31} = -0.3$, $q_{32} = -0.1$ and $q_{33} = 0.2$. Here we see that

$$\sum_{j=1}^{3} |q_{ij}| = 0.6 \text{ for all } 1 \leq i \leq 3.$$

Hence from the Theorem 3 it follows that the system of linear algebraic equations (10) has a unique solution in $\mathbb{R}^3$, which is given by $x_1 \backsimeq -0.792$, $x_2 \backsimeq -0.656$ and $x_3 \backsimeq -3.507$.

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