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An Integrable Quintic Planar Differential System With Two Explicit Non-Algebraic Limit Cycles*

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Abstract

As a continuation of our recent work [5], we introduce a three-parameter quintic planar differential system and after adequate regular transformations, we show that it is equivalent to a Riccati equation. This equation is shown to be integrable. Then via the Poincaré return-map, we show that the system admits exactly two non-algebraic limit cycles. Moreover, these limit cycles are given in an explicit way. As far as we know, this result has no analogue in the literature.

1 Introduction

It is a fact that most of research papers devoted to the study of limit cycles concerning planar and autonomous differential systems of degree n, of the form

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P_n(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q_n(x,y), \end{aligned} \tag{1}$$

where $P_n(x,y)$ and $Q_n(x,y)$ are coprime polynomials of $\mathbb{R}[x,y]$ and $n = \max\{\deg P_n, \deg Q_n\}$, are focused on their number, stability and location in the phase plane. A limit cycle of system (1) is an isolated periodic orbit in the set of its all periodic orbits and it is said to be algebraic if it is contained in the zero set of an irreducible invariant algebraic curve of the system. We recall that an algebraic curve defined by $U(x,y) = 0$ is an invariant curve for system (1) if there exists a polynomial $K(x,y)$ (called the cofactor) such that

$$P_n(x,y)\frac{\partial U}{\partial x}(x,y) + Q_n(x,y)\frac{\partial U}{\partial x}(x,y) = K(x,y)U(x,y). \tag{2}$$

A natural problem is to express analytically the limit cycles. In addition to the intrinsic theoretical interest of the theory of exact limit cycles, we can apply it to compare the efficiency of the numerical methods dealing with the approximation of the shape of the limit cycle obtained by these methods with the exact one once it is known. Until recently, the only limit cycles known in an explicit way were algebraic (see for instance [3], [4], [9], [10], [14], and references therein). These limit cycles are searched as smooth ovals contained in algebraic invariant curves. In 1998, M. Abdelkadder [1] presented for the first time an example of a Liénard equation with exact algebraic limit cycle when the well known sufficient conditions of existence and uniqueness of the limit cycle are observed. This example was recovered as a particular case by A. Bendjeddou and R. Cheurfa [3] when studying a more general class of planar systems. Although, limit cycles of planar polynomial differential systems are not in general algebraic. For instance, the limit cycle appearing in the van der Pol equation is non-algebraic as it is proved by Odani [12]. In the chronological order, the

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first examples were explicit non-algebraic limit cycles appeared are those of A. Gasull [8], Al-Dossary [2] for $n = 5$ and Benterki and Llibre [6] for $n = 3$. The fist result for the coexistence of algebraic and non-algebraic limit cycles goes back to J. Giné and M. Grau [9] for $n = 9$. In [5], we have extended this last result by means of a different and shorter method to the case $n = 5$. Indeed, in [5] the system under investigation is transformed after two regular change of variables into a generalized Riccati equation for which a particular periodic solution is obtained. It is shown that this solution corresponds to an algebraic limit cycle of the system. Hence, the "complete" first integral of the system follows immediately providing thus the possibility to express the explicit analytical form for the wanted non-algebraic limit cycle, where by complete we mean that the set of all associated level curves to the first integral determines completely the phase portrait of the system. This work aims to give an answer to a question raised among others in this later paper, that is to give an example of a quintic system admitting two non-algebraic limit cycles explicitly obtained. As we can see below, by adopting the same demarche as in [5], a nonlinear differential equation arises but this time with unknown particular solution. Fortunately, a successful guess enables us to overcome this difficulty and to solve completely this equation. The searched limit cycles are detected via the Poincaré return map.

2 The Main Result

As a main result, we shall prove:

Theorem 1 *Let the quintic system*

$$\begin{aligned} \dot{x} &= x + x\left(x^2+y^2-2c\right)\left(ax^2-4bxy+ay^2\right) - 4y\left(x^2+y^2\right)\left(x^2+y^2-c\right), \\ \dot{y} &= y + y\left(x^2+y^2-2c\right)\left(ax^2-4bxy+ay^2\right) + 4x\left(x^2+y^2\right)\left(x^2+y^2-c\right), \end{aligned} \tag{3}$$

in which a, b and c are real parameters such that $a > 0$ and $c < 0$. Then, under the hypotheses

(H1) $ac^2 - 1 \neq 0$,

(H2) $\left(ac^2-1\right)^2 - 4c^4b^2 > 0$,

(H3) $\int_0^{2\pi} \dfrac{ac^2-1-2bc^2\sin 2s}{\exp\left(as+b\cos 2s\right)}ds < \dfrac{c^2}{e^b}$,

this system possesses exactly two non-algebraic limit cycles $\left(\gamma_\pm\right)$ enclosing an unstable improper node and explicitly given in polar coordinates $(r;\theta)$ by the equation

$$r\left(\theta; r_*\right) = \sqrt{c \pm \sqrt{c^2 + e^{a\theta+b\cos 2\theta}\left(\frac{\left(r_*^2-c\right)^2}{e^b} + \int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\right)}}, \tag{4}$$

where

$$r_* = \sqrt{c \pm \sqrt{e^b\int_0^{2\pi}\frac{ac^2-1+2bc^2\sin 2s}{\exp\left(as+b\cos 2s\right)}ds}}.$$

Note that the conditions above can be fulfilled, for example for $a = 4, c = 1$ and b ranging from -1 to 3. To prove this theorem, we need the following lemmas. The first one is concerned with the equilibrium points of system (3).

Lemma 1 *The origin is the unique equilibrium point of system (3).*

Proof. Obviously the origin is an equilibrium point. If we remark that any equilibrium point must satisfy

$$x\dot{y} - y\dot{x} = 4\left(x^2+y^2\right)^2\left(x^2+y^2-c\right) = 0,$$

so, any other, if exists, must lie on the circle $x^2+y^2=c$. Let (x_0, y_0) be such a point. Then from the remark above, x_0 and y_0 must satisfy

$$\begin{array}{l} x_0\left(1-c\left(ax_0^2-4bx_0y_0+ay_0^2\right)\right)=0,\\ y_0\left(1-c\left(ax_0^2-4bx_0y_0+ay_0^2\right)\right)=0. \end{array}$$

From **(H1)**, there are no equilibrium point belonging to one of the axes. so this algebraic system reduces to the equation

$$1-c\left(ax_0^2-4bx_0y_0+ay_0^2\right)=0.$$

It remains the possibility to get equilibrium points on the circle $x^2+y^2=c$. But according to **(H2)**, it is not difficult to see that this equality is impossible, and this ends the proof of the lemma. ■

We recall according to the last lines of the introduction that our strategy in the search for analytically given non-algebraic limit cycles requires the integration of the system, in fact, we have

Lemma 2 *The system (3) is integrable and its complete first integral F is given by*

$$F(x,y) = \frac{\left(x^2+y^2-c\right)^2}{\exp\left(b\frac{x^2-y^2}{x^2+y^2}+a\arctan\frac{y}{x}\right)} - \int_0^{\arctan\frac{y}{x}} \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds. \tag{5}$$

Proof. In polar coordinates, this system turns into

$$\begin{array}{l} \dot{r} = (a-2b\sin 2\theta)\,r^5 - 2c\,(a-2b\sin 2\theta)\,r^3 + r,\\ \dot{\theta} = 4r^2\left(r^2-c\right). \end{array} \tag{6}$$

Taking θ as an independent variable, we obtain the equation

$$4r\left(r^2-c\right)\frac{dr}{d\theta} = (a-2b\sin 2\theta)\,r^4 - 2c\,(a-2b\sin 2\theta)\,r^2+1. \tag{7}$$

Via the change of variable $\rho = r^2$, this equation is transformed into the nonlinear differential equation

$$2(\rho-c)\,\rho' = (a-2b\sin 2\theta)\,\rho^2 - 2c\,(a-2b\sin 2\theta)\,\rho+1. \tag{8}$$

Let $\Phi:\mathbb{R}\to\mathbb{R}$ be an arbitrary C^1 function. We seek a solution of (8) quadratic in ρ of the form $(\rho(\theta)-c)^2 + \Phi(\theta) = 0$, or under the explicit form $\rho(\theta) = c+\varepsilon\sqrt{\Phi(\theta)}$, with $\varepsilon = \pm 1$. This choice leads to the following linear ordinary differential equation in term of the unknown function Φ

$$\frac{d\Phi}{d\theta} - (a-2b\sin 2\theta)\,\Phi - 2bc^2\sin 2\theta + ac^2 - 1 = 0, \tag{9}$$

we deduce that

$$\Phi = e^{a\theta+b\cos 2\theta}\left(k+\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\right), \tag{10}$$

where k is an arbitrary real constant. We recall that ρ is related to Φ by $\rho(\theta) = c\pm\sqrt{\Phi(\theta)}$. Thus we get the solution of eq. (8) as follows

$$\rho(\theta;k) = c+\varepsilon\sqrt{e^{a\theta+b\cos 2\theta}\left(k+\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\right)}. \tag{11}$$

Hence the general solution of eq. (7) is

$$r(\theta;k)=\sqrt{c+\varepsilon\sqrt{e^{a\theta+b\cos 2\theta}\left(k+\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\right)}}. \quad (12)$$

Consequently, the system (3) is integrable and the passage to the Cartesian coordinates in (12) allow us to write down the expression (5) of the fist integral $F(x,y)$. ■

For our purpose, let us introduce the function g defined on $[0,2\pi]$ by

$$g(\theta)=c^2-2e^{a\theta+b\cos 2\theta}\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds.$$

Then we have

Lemma 3 $\forall\theta\in[0,2\pi]: g(\theta)>0.$

Proof. For any $\theta\in[0,2\pi]$, the hypothesis **(H2)** gives

$$\frac{1-ac^2+2bc^2\sin 2\theta}{e^{a\theta+b\cos 2\theta}}\geq 0$$

and then

$$0\leq\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\leq\int_0^{2\pi} \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds,$$

from which, we deduce that

$$e^{a\theta+b\cos 2\theta}\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\leq e^{a\theta+b}\int_0^{2\pi} \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds.$$

So

$$c^2-2e^{a\theta+b\cos 2\theta}\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\geq c^2-2e^{a\theta+b}\int_0^{2\pi} \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds.$$

But the function $\theta\longmapsto c^2-2e^{a\theta+b}\int_0^{2\pi}\frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds$ is decreasing on $[0,2\pi]$, we deduce the inequalities

$$c^2-2e^{2\pi a+b}\int_0^{2\pi} \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\leq c^2-2e^{a\theta+b\cos 2\theta}\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds.$$

Thus

$$c^2-2e^{a\theta+b\cos 2\theta}\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\geq c^2-2e^{2\pi a+b}\int_0^{2\pi} \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds,$$

i.e.,

$$g(\theta)\geq c^2-2e^{2\pi a+b}\int_0^{2\pi} \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds,$$

and using **(H3)**, we conclude our proof. ■

Lemma 4 *Equation (8) admits exactly two periodic solutions.*

Proof. In (11), we put $\theta=0$. Let $\rho_0=\rho(0;k)=c\pm\sqrt{ke^b}$ so $k=\frac{(\rho_0-c)^2}{e^b}$. A necessary condition of the positivity of this solution is $\rho_0>0$, which requires that $0<k<\frac{c^2}{e^b}$. In function of ρ_0, the solution (11) becomes

$$\rho(\theta;\rho_0)=c\pm\sqrt{e^{a\theta+b\cos 2\theta}\left(\frac{(\rho_0-c)^2}{e^b}+\int_0^\theta \frac{1-ac^2+2bc^2\sin 2s}{\exp(as+b\cos 2s)}ds\right)}. \quad (13)$$

To be a periodic solution, it must satisfy the condition

$$\rho\left(2\pi;\rho_0\right)=\rho\left(0;\rho_0\right)=\rho_0>0, \tag{14}$$

providing the value of $k_*$of k :

$$k_*=\int_0^{2\pi}\frac{ac^2-1+2bc^2\sin 2s}{\exp\left(as+b\cos 2s\right)}ds. \tag{15}$$

From (**H2**), k_* is non negative and taking **(H3)** in consideration, we obtain two positive values of ρ_0: $\rho_0=\rho_*=c\pm\sqrt{k_*e^b}$, thus

$$\rho_*=c\pm\sqrt{e^b\int_0^{2\pi}\frac{ac^2-1+2bc^2\sin 2s}{\exp\left(as+b\cos 2s\right)}ds}, \tag{16}$$

and consequently the two values r_* as in the theorem. Thus a rewriting of the two possible periodic solutions of eq.(9) is

$$\rho_\pm\left(\theta;\rho_*\right)=c\pm\sqrt{e^{a\theta+b\cos 2\theta}\left(\frac{\left(\rho_*-c\right)^2}{e^b}+\int_0^\theta\frac{1-ac^2+2bc^2\sin 2s}{\exp\left(as+b\cos 2s\right)}ds\right)}. \tag{17}$$

It is clear that $\rho_+\left(\theta;\rho_*\right)>0$ for all $\theta\in[0,2\pi[$. But $\rho_-\left(\theta;\rho_*\right)+\rho_+\left(\theta;\rho_*\right)=2c>0$ and

$$\rho_-\left(\theta;\rho_*\right)\rho_+\left(\theta;\rho_*\right)=c^2-2e^{a\theta+b\cos 2\theta}\int_0^\theta\frac{1-ac^2+2bc^2\sin 2s}{\exp\left(as+b\cos 2s\right)}ds=g\left(\theta\right).$$

Using the preceding lemma, we deduce that $\rho_-\left(\theta;\rho_*\right)>0$ for all $\theta\in[0,2\pi[$ and this ends the proof. ■

Proof of Theorem 1. It is obvious that to any periodic solution of eq. (7) corresponds a periodic solution of eq. (8) via the relation $\rho=r^2$ providing the two periodic solutions of eq.(7) under the form

$$r_\pm\left(\theta;r_*\right)=\sqrt{c\pm\sqrt{e^{a\theta+b\cos 2\theta}\left(\frac{\left(r_*^2-c\right)^2}{e^b}+\int_0^\theta\frac{1-ac^2+2bc^2\sin 2s}{\exp\left(as+b\cos 2s\right)}ds\right)}}. \tag{18}$$

As a consequence of the lemma above, the two solutions given by (4) are periodic. The inequality $r_\pm\left(\theta;\rho_*\right)>0$ means that the solution curves $\left(\gamma_\pm\right)$ do not pass through the origin. Finally, it remains to show that $\left(\gamma_\pm\right)$ are in fact limit cycles. For that, it is more advantageous to consider the solutions (13), and introduce the two Poincaré return-maps $\Pi_\pm$ with the positive x-axis as Poincaré section by

$$\rho_0\to\Pi_\pm(2\pi;\rho_0)=\rho_\pm\left(2\pi;\rho_0\right), \tag{19}$$

where

$$\rho_\pm\left(2\pi;\rho_0\right)=c\pm\sqrt{e^{2\pi a+b}\left(\frac{\left(\rho_0-c\right)^2}{e^b}+\int_0^{2\pi}\frac{1-ac^2+2bc^2\sin 2s}{\exp\left(as+b\cos 2s\right)}ds\right)}.$$

We compute $\frac{d\Pi_\pm}{d\rho_0}(2\pi;\rho_0)$ at the values $\rho_0=\rho_*$ where ρ_* are given by (16). Few calculations lead to

$$\frac{d\Pi_\pm}{d\rho_0}(2\pi;\rho_0)=\frac{e^{\pi a}}{\sqrt{\left(c^2-2c\rho_0+\rho_0^2+\int_0^{2\pi}\frac{1-ac^2+2bc^2\sin 2s}{\exp\left(\frac{1}{2}as^2-bs+\frac{b}{2}\sin 2s\right)}ds\right)}}.$$

For $\rho_0=\rho_*$, we find

$$\left.\frac{d\Pi_\pm}{d\rho_0}(2\pi;\rho_0)\right|_{\rho_0=\rho_*}=\pm\frac{e^{\pi a}}{\sqrt{2}}. \tag{20}$$

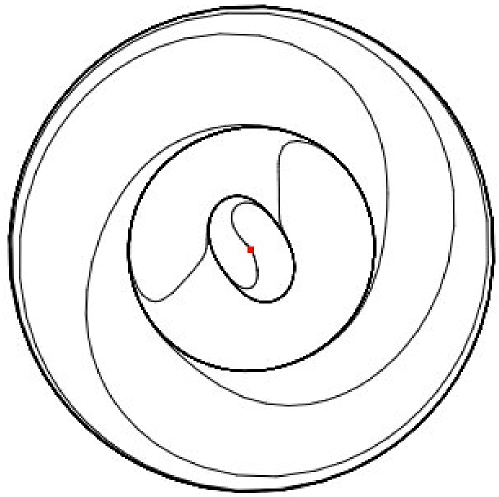

Figure 1: The phase portrait in the Poincaré disk of the polynomial differential system (21).

We conclude that $(\gamma_{\pm})$ are in fact limit cycles. It is not difficult to see that for any θ, we have $r_{-}(\theta; r_0) < c$ and $r_{+}(\theta; r_0) > c$, which means that the circle $x^2 + y^2 - c = 0$ is located between the two limit cycles. Furthermore, inside the circle $x^2 + y^2 - c = 0$ we have $\dot{\theta} < 0$, hence the inner limit cycle (γ_{-}) is stable. Outside this circle, we have $\dot{\theta} > 0$, thus the outer limit cycle (γ_{+}) is unstable (see [6], section 1.6 for more details). Finally from the fact that the Poincaré return-map (19) do not possess other fixed points, the system (3) admits exactly two limit cycles. It remains to show the limit cycles $(\gamma_{\pm})$ are non-algebraic. In the phase plane, all the trajectories of system (3) are represented by the level curves associated to the first integral $F(x, y)$ and particularly, the limit cycles $(\gamma_{\pm})$ are defined by the equations $F(x, y) = k_*$ were k_* is given by (15). If we suppose that $(\gamma_{\pm})$ are algebraic, then $F(x, y)$ should be a polynomial, so there exists an integer n such that $\frac{\partial^n F}{(\partial x)^n}$ vanishes identically, but this is not the case since transcendental terms such as $\exp\left(b\frac{x^2-y^2}{x^2+y^2} + a\arctan\frac{y}{x}\right)$ already exists in $F(x, y)$ and still reappears when partial derivatives of arbitrary order are performed, which means that $F(x, y)$ is not a polynomial and that the limit cycles $(\gamma_{\pm})$ are non-algebraic. ■

Example 1 *As an example, let us take $a = 4$, $b = -1$ and $c = 1$ in system (3). In the standard form, this system reads*

$$\begin{aligned} \dot{x} &= x - 8x^3 - 4x^2y - 8xy^2 + 4y^3 + 4x^5 + 8x^3y^2 - 4x^2y^3 + 4xy^4 - 4y^5, \\ \dot{y} &= y - 4x^3 - 8x^2y - 12xy^2 - 8y^3 + 4x^5 + 4x^4y + 12x^3y^2 + 8x^2y^3 + 8xy^4 + 4y^5. \end{aligned} \tag{21}$$

It is easy to verify that all conditions of the theorem are satisfied with $r_* \simeq 0.15018$ or $r_* \simeq 1.406\,2$. We conclude that system (19) possesses exactly two limit cycles surrounding both the origin as an unstable improper node, plotted on the Poincaré disc as shown in Figure 1.

Concluding remark. By integrating analytically the quintic planar system under investigation, it is shown that it possesses exactly two explicit non-algebraic limit cycles, each one correspond to a closed isolated level curve of the complete first integral. Obtaining interesting results of this kind becomes more and more difficult for systems of lower degrees. The following problems can be addressed:

- The coexistence of explicit algebraic and non-algebraic limit cycles for cubic systems.

- Obtain a quadratic system with exact non-algebraic limit cycle (this question has been raised before by Benterki and Llibre [5]).
- Can one obtain explicit non-algebraic limit cycles for systems arising from applications (such as Kolmogorov system, Kaldor system, or Liénard system)?

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