

A Chebyshev Pseudo-Spectral Method For The Numerical Solutions Of Distributed Order Fractional Ordinary Differential Equations*

Shina Daniel Olonijju, Sicelo Praisegod Goqo, Precious Sibanda†

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Abstract

Distributed order differential equations offer new perspectives in modelling multi-scale physical problems. In distributed order fractional differential equations (DO-FDEs), the order of the derivatives is distributed over a range of real numbers. In this study, we present a numerical scheme for obtaining solutions of DO-FDEs. The method is a hybridization of the composite trapezoidal rule and the Chebyshev pseudo-spectral method. In the pseudo-spectral method, the premise is that the solution of the DO-FDE may be written as a linear combination of the first kind shifted Chebyshev polynomials, and integrated using the Gauss-Lobatto quadrature. Numerical examples are presented to demonstrate the accuracy and convergence of the method. The accuracy is determined through comparison with exact solutions presented in earlier studies in the literature.

1 Introduction

Evidence of memory retention and anomalous behaviour abound in many complex physical phenomena. Fractional differential and integral equations provide new possibilities for modelling such phenomena. The theory of fractional calculus has been extensively applied to physical problems such as those of viscoelasticity [1], diffusion [2] and growth models [3].

Fractional differential models are usually non-local and defined in terms of classical integrals, these make obtaining closed form solutions difficult. For this reason, numerous studies have been dedicated to developing accurate and efficient numerical schemes for arbitrary order differential equations. Among these schemes are the finite difference method [4], finite volume method [5] and spectral-based methods [6, 7]. Recent studies have been geared towards exploring the global property of spectral methods to approximate the solutions of differential equations of arbitrary real orders. The development of spectral methods such as the tau, Galerkin and collocation methods for fractional differential equations with single order derivatives has been relatively treated in literature. For instance, Doha et al. [8] presented spectral tau method with Chebyshev polynomials for arbitrary real order differential equations. Zayernouri and Karniadakis [9] developed a spectral collocation method using the eigenvalue solution of the fractional Sturm-Liouville problem as basis functions. In the study by Li and Xu [10], a spectral Galerkin method was proposed for time-fractional diffusion equations.

In this study, we consider a distributed order fractional differential equation (DO-FDE) of the form

$$\int_{\alpha_l}^{\alpha_u} \rho(\alpha) {}_0^C D_x^\alpha y(x) d\alpha = f(x), \quad x > 0, \quad (1)$$

where $\alpha \in \mathbb{R}^+$ and $f(x)$ is a real valued function. The Caputo fractional derivative of $y(x)$, a sufficiently smooth function, is defined as [11]

$${}_0^C D_x^\alpha y(x) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{y^n(\tilde{x})}{(x-\tilde{x})^{\alpha+1-n}} d\tilde{x}, & n-1 < \alpha < n, n \in \mathbb{N}, \\ \frac{d^n y(x)}{dx^n}, & \alpha = n \in \mathbb{N}, \end{cases} \quad (2)$$

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†School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01, Scottsville, Pietermaritzburg, 3209, South Africa

where Γ is the Euler gamma function. From the properties of the Caputo fractional differential operator, it is easy to show that the fractional derivative of a power function is

$${}_0^C D_x^\alpha x^j = \begin{cases} 0 & j \in \mathbb{N}_0, j < [\alpha], \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} x^{j-\alpha}, & j \in \mathbb{N}_0, j \geq [\alpha]. \end{cases}$$

This class of fractional differential equations offers new perspectives in the mathematical modelling of multi-scale systems. Unlike in conventional fractional differential equations where fractional orders are single-fixed real values, in DO-FDE, differential orders are distributed over a range of real values. Most published studies on numerical methods for distributed order fractional differential equation are derived from difference schemes. Obtaining the numerical solutions of single order fractional differential equations can be computationally demanding, owing to the global property of the differential operators. Additional computational cost can be incurred for the numerical integration of distributed order fractional differential equations, because differential orders are distributed over a set of real values. Among the few studies on the numerical solutions of DO-FDEs are studies by [13, 12, 14].

In this study, we present an accurate numerical scheme for the differential equations (1). The method involves approximating the integral using a Newton-Cotes formula. The resulting linear multi-term fractional differential equation is then approximated in terms of the shifted Chebyshev polynomials of the first kind and the Gauss-Lobatto quadrature. We present several examples to illustrate the use of the proposed method and compare numerical solutions with the exact solutions where possible.

2 Method of Solution

In this section, we present the method for solving the DO-FDE (1). The solution process is divided into two part, the first involving approximating the integral as a finite sum using a quadrature rule. This transforms the DO-FDE into a multi-term fractional differential equation. We then approximate the numerical solution of the multi-term fractional differential equations in terms of the first kind shifted Chebyshev polynomials. We present the fractional differentiation matrix in terms of the shifted Chebyshev polynomials and use the Gauss-Lobatto quadrature. We remark here that the numerical method inherits all the properties (accuracy, stability and convergence) of the quadrature rule and the numerical integration method.

2.1 The First Kind Shifted Chebyshev Polynomials

The class of polynomials $\{T_n(x) = \cos(n \cos^{-1} x), n = 0, 1, \dots\}$, called Chebyshev polynomials of the first kind, are eigenvalue solutions of the Sturm-Liouville problem [15]. Consider a mapping of the variable $x : [-1, 1] \mapsto [0, L]$ using the affine mapping $x = 2\tilde{x}/L - 1$, we then have the recurrence relation of the shifted form of the first kind Chebyshev polynomials given as

$$\tilde{T}_{L,n+1}(\tilde{x}) = 2 \left(\frac{2\tilde{x}}{L} - 1 \right) \tilde{T}_{L,n}(\tilde{x}) - \tilde{T}_{L,n-1}(\tilde{x}), \quad 0 \leq \tilde{x} \leq L, \quad n = 1, 2, \dots, \quad (3)$$

where the zeroth and first order polynomials are given respectively as $\tilde{T}_{L,0}(\tilde{x}) = 1$ and $\tilde{T}_{L,1}(\tilde{x}) = 2\tilde{x}/L - 1$. In series form, equation (3) is given as (tilde dropped for brevity) [15]

$$T_{L,n}(x) = n \sum_{j=0}^n \frac{(-1)^{n-j} (n+j-1)! 2^{2j}}{(n-j)! (2j)! L^j} x^j, \quad n = 0, 1, 2, \dots,$$

which satisfy the orthogonality condition

$$\int_0^L T_{L,n}(x) T_{L,m}(x) w_L(x) dx = \delta_{mn} h_n,$$

where the weight function $w_L(x) = 1/\sqrt{Lx - x^2}$, $h_n = c_n \pi/2$, $c_0 = 2$, $c_n = 1$ for $n \geq 1$. For approximations constructed on the Chebyshev-Gauss-Lobatto quadrature, we use the Christoffel number $\varpi_j = \pi/c_j N$, $0 \leq j \leq N$, with $c_0 = c_N = 2$ and $c_j = 1 \forall 1 \leq j \leq N - 1$.

2.2 Quadrature Rule

We approximate the integral in (1) using the composite trapezoidal rule. If we express the integral as a finite sum using the composite trapezoidal rule with Q intervals, we have

$$\frac{\Delta\alpha}{2} \left[\rho_0 {}_0^C D_x^{\alpha_0} y(x) + 2 \sum_{e=1}^{Q-1} \rho_e {}_0^C D_x^{\alpha_e} y(x) + \rho_Q {}_0^C D_x^{\alpha_Q} y(x) \right] + \mathcal{O}((\Delta\alpha)^2) = f(x), \quad (4)$$

where $[\rho_0, \rho_e, \rho_Q] \equiv [\rho(\alpha_0), \rho(\alpha_e), \rho(\alpha_Q)]$, $\Delta\alpha = (\alpha_u - \alpha_l)/Q$, with ${}_0^C D_x^{\alpha_0} \equiv {}_0^C D_x^{\alpha_l}$ and ${}_0^C D_x^{\alpha_Q} \equiv {}_0^C D_x^{\alpha_u}$, provided $y(x)$ is regular with respect to all $\alpha \in [\alpha_0, \alpha_Q]$.

The abscissas are evenly spaced points $\alpha_e = \alpha_l + e\Delta\alpha$, for $e = 0, 1, 2, \dots, Q-1, Q$. The approximation of the integral is an essential part of the numerical integration of (1). If $\alpha_l = 0$, then ${}_0^C D_x^{\alpha_l} y(x) \equiv y(x)$.

2.3 Approximation of $y(x)$ and Its Derivatives

Assume that $y(x)$ is a continuously differentiable function defined on the interval $[0, L]$, and approximate $y(x)$ as series expansion in terms of the shifted form of first kind Chebyshev polynomials as

$$y(x) \approx Y_N(x) = \sum_{n=0}^N Y_n T_{L,n}(x),$$

where the coefficients Y_n satisfy the orthogonality condition written in discrete form as

$$Y_n = \frac{1}{h_n} \sum_{j=0}^N Y(x_j) T_{L,n}(x_j) \varpi_j, \quad n = 0, 1, \dots, N.$$

Therefore, the approximation of $y(x)$ is given as

$$Y_N(x) = \sum_{j=0}^N \left[\varpi_j \frac{1}{h_n} T_{L,n}(x_j) T_{L,n}(x_k) \right] Y(x_j) = Y, \quad k = 0, 1, \dots, N. \quad (5)$$

Theorem 1 *The arbitrary derivative of a continuously differentiable function $y(x)$ is given as*

$${}_0^C D_x^\alpha y(x) \approx D^\alpha Y_N(x) = \sum_{j=0}^N D_{j,l}^\alpha Y(x_j) = D^\alpha Y,$$

where the entries $D_{j,l}^\alpha$ are given as

$$D_{j,l}^\alpha = \varpi_j \sum_{n=0}^N \sum_{k=0}^N \frac{1}{h_n} n \sum_{j=0}^n T_{L,n}(x_j) \frac{(-1)^{n-j} (n+j-1)! 2^{2j}}{(n-j)! (2j)!} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} q_{j,k} T_{L,k}(x_l), \quad j, l = 0, 1, \dots, N,$$

and $q_{j,k}$ is defined as

$$q_{j,k} = \begin{cases} 0 & j = 0, 1, \dots, [\alpha] - 1, \\ \frac{k\sqrt{\pi}}{h_k} \sum_{r=0}^k \frac{(-1)^{k-r} (k+r-1)! 2^{2r}}{(k-r)! (2r)!} \frac{\Gamma(j-\alpha+r+1/2)}{\Gamma(j-\alpha+r+1)} L^{j-\alpha}, & j = [\alpha], [\alpha]+1, \dots, N, \\ & k = 0, 1, \dots, N \end{cases}$$

The proof of this result can be found in Olonijju et al. [16].

Using this theorem, we can easily approximate the derivatives in (4).

2.4 Numerical Integration of (1)

The numerical integration method is a combination of the quadrature rule described in Section 2.2 and the truncated Chebyshev polynomials approximation described in Section 2.3. Equation (1) is solved subject to initial conditions that are dependent on the values of α . If $\alpha_l = 0$ and $\alpha_u = 1$, the equation is solved subject to the condition $y(0) = y_0$, and when $\alpha_u = 2$, an additional condition is required. When the composite trapezoidal rule is used, the numerical integration of eq. (1) is given as

$$\frac{\Delta\alpha}{2} \left[\rho_0 D^{\alpha_0} + 2 \sum_{e=1}^{Q-1} \rho_e D^{\alpha_e} + \rho_Q D^{\alpha_Q} \right] Y = F, \tag{6}$$

where Y and F are defined using eq. (5) and evaluated on the Chebyshev-Gauss-Lobatto points $x_j = L/2(1 - \cos(\pi j/N))$, $0 \leq j \leq N$. If $\alpha_l = 0$ and $\alpha_u = 2$, eq. (6) can be written as a linear algebraic system

$$\frac{\Delta\alpha}{2} \sum_{j=2}^N \left[\left[\rho_0 D_{j,l}^{\alpha_0} + 2 \sum_{e=1}^{Q-1} \rho_e D_{j,l}^{\alpha_e} + \rho_Q D_{j,l}^{\alpha_Q} \right] Y(x_j) \right] = F(x_j), \quad l = 0, 1, 2, \dots, N,$$

which, when combined with the initial conditions: $y(x_0) = y_0$ and $y'(x_0) = y'_0$ that are respectively evaluated at the collocation points as

$$Y(x_0) = y_0, \quad \sum_{j=0}^N D_{j,0} Y(x_j) = y'_0,$$

leads to a consistent system.

3 Convergence Analysis

In this section, we demonstrate the convergence of numerical scheme. To do this, we introduce $L^2([0, L])$ norm that is defined as $\|\cdot\|_{L^2_w([0,L])}$. We define the fractional Sobolev space, $H_w^\alpha([0, L])$, $\alpha \geq 0$, as

$$H_w^\alpha([0, L]) = \{y \in L^2_w([0, L]) \text{ s.t. } {}_0^C D_x^\alpha y(x) \in L^2_w([0, L])\},$$

endowed with the semi-norm

$$|y|_{l,\alpha} = \|{}_0^C D_x^\alpha y\|_{L^2_w([0,L])},$$

and the associated norm

$$\|y\|_{l,\alpha} = \left(\|y\|_{L^2_w([0,L])}^2 + |y|_{l,\alpha}^2 \right)^{1/2}.$$

Lemma 1 ([17]) *Let $y \in H_w^\alpha([0, L])$, and $0 < r < \alpha$, then there exists a positive constant such that*

$$\|y\|_{H_w^r([0,L])} \leq C \|y\|_{H_w^\alpha([0,L])}.$$

For the distributed order derivative, we have

$$\int_{\alpha_l}^{\alpha_u} \rho(\alpha) {}_0^C D_x^\alpha y(x) d\alpha = \frac{\Delta\alpha}{2} \left[\rho_0 {}_0^C D_x^{\alpha_0} y(x) + 2 \sum_{e=1}^{Q-1} \rho_e {}_0^C D_x^{\alpha_e} y(x) + \rho_Q {}_0^C D_x^{\alpha_Q} y(x) \right] + \mathcal{O}((\Delta\alpha)^2), \tag{7}$$

so we define the norm $\|y\|_{H_w^m}$ as

$$\|y\|_{H_w^m} = \left(\|y\|_{H_w^{\alpha_0}}^2 + \sum_{e=1}^{Q-1} \|y\|_{H_w^{\alpha_e}}^2 + \|y\|_{H_w^{\alpha_Q}}^2 \right)^{1/2}. \tag{8}$$

Lemma 2 ([18]) *Assume that T_N is $L_w^2([0, L])$ orthogonal projection onto the space of Chebyshev polynomials with degree up to $N + 1$, and k and m are real numbers such that $0 \leq k \leq m$, then for positive constant C and for any function $y \in C([0, L]) \cap H_w^m([0, L])$, the following error estimates hold [18]*

$$\|y - T_N y\|_{H_w^k([0, L])} \leq C N^{2k-m} \|y\|_{H_w^m([0, L])}$$

and

$$\|y - T_N y\|_{L_w^2([0, L])} \leq C N^{-m} \|y\|_{H_w^m([0, L])}.$$

We now present the convergence analysis of the numerical scheme.

Theorem 2 *Assume that y and Y_N are solutions of eqs. (4) and (6) respectively, satisfying $y \in C([0, L]) \cap H_w^m([0, L])$. Then, there exists a non-negative constant C independent of $\Delta\alpha$ and N such that*

$$\|y - Y_N\|_{L_w^2([0, L])} \leq C_1 \left(\sum_{e=0}^Q N^{2\alpha_e - m} \|y\|_{H_w^m} + (\Delta\alpha)^2 \right) + C_2 N^{-m} \|y\|_{H_w^m}.$$

Proof. To prove this let's define $\tilde{y} = T_N y$. Noting that $f = F$ and subtracting eq. (6) from eq. (4) gives the following error equation

$$y - Y_N = \frac{\Delta\alpha}{2} \rho_0 D^{\alpha_0} (y - \tilde{y}) + \Delta\alpha \sum_{e=1}^{Q-1} \rho_e D^{\alpha_e} (y - \tilde{y}) + \frac{\Delta\alpha}{2} \rho_Q D^{\alpha_Q} (y - \tilde{y}) + \mathcal{O}((\Delta\alpha)^2) + \epsilon,$$

where

$$\epsilon = \frac{\Delta\alpha}{2} \rho_0 D^{\alpha_0} (\tilde{y} - Y_N) + \Delta\alpha \sum_{e=1}^{Q-1} \rho_e D^{\alpha_e} (\tilde{y} - Y_N) + \frac{\Delta\alpha}{2} \rho_Q D^{\alpha_Q} (\tilde{y} - Y_N)$$

is the residual. To estimate $\|y - Y_N\|_{L_w^2([0, L])}$, we have

$$\begin{aligned} \|y - Y_N\|_{L_w^2([0, L])} &\leq \left\| \frac{\Delta\alpha}{2} \rho_0 D^{\alpha_0} (y - \tilde{y}) \right\|_{L_w^2([0, L])} + \left\| \Delta\alpha \sum_{e=1}^{Q-1} \rho_e D^{\alpha_e} (y - \tilde{y}) \right\|_{L_w^2([0, L])} \\ &\quad + \left\| \frac{\Delta\alpha}{2} \rho_Q D^{\alpha_Q} (y - \tilde{y}) \right\|_{L_w^2([0, L])} + (\Delta\alpha)^2 + \|\epsilon\|_{L_w^2([0, L])} \\ &\leq C_1 \left(N^{2\alpha_0 - m} \|y\|_{H_w^m} + \sum_{e=1}^{Q-1} N^{2\alpha_e - m} \|y\|_{H_w^m} + N^{2\alpha_Q - m} \|y\|_{H_w^m} + (\Delta\alpha)^2 \right) \\ &\quad + C_2 N^{-m} \|y\|_{H_w^m} \\ &\leq C_1 \left(\sum_{e=0}^Q N^{2\alpha_e - m} \|y\|_{H_w^m} + (\Delta\alpha)^2 \right) + C_2 N^{-m} \|y\|_{H_w^m}, \end{aligned}$$

where C_1 and C_2 are independent of N . ■

4 Numerical Examples

In this section, we demonstrate the efficiency and accuracy of the method on selected DO-FDEs. The numerical results, where possible, are compared with exact solutions. These examples are motivated by the range of applications of distributed order models studied in Atanackovic et al. [19]. Numerical computations are performed using the *PYTHON* programming language on the *SPYDER IDE*.

Example 1 Consider the DO-FDE defined on the domain $[0, 0.99]$

$$\int_0^2 \frac{\Gamma(6 - \alpha)}{120} {}_0^C D_x^\alpha y(x) d\alpha = \frac{x^5 - x^3}{\log x}, \tag{9}$$

with initial conditions $y(0) = y'(0) = 0$, and whose unique solution is given as $y(x) = x^5$ [12, 14].

We approximate (9) using the numerical scheme described in Section 2 to obtain the following system of algebraic equations

$$\frac{1}{Q} \sum_{j=2}^N \left[\left[1 + 2 \sum_{e=1}^{Q-1} \frac{\Gamma(6 - 2e/Q)}{120} D_{j,l}^{2e/Q} + \frac{1}{20} D_{j,l}^2 \right] Y(x_j) \right] = F(x_j),$$

$$Y(x_0) = 0, \quad \sum_{j=0}^N D_{j,0} Y(x_j) = 0, \quad l = 0, 1, 2, \dots, N,$$

where the function on the right hand side is evaluated at the Chebyshev-Gauss-Lobatto points.

The solution was obtained numerically in Diethelm and Ford [12] using a combination of a weighted quadrature formula and the fractional Adams' method, and in Katsikadelis [14] using the trapezoidal rule and an analogue equation method. Table 1 shows the convergence of the solution using the infinity error norm. The convergence is given in terms of the number of collocations points N and the number of intervals in the quadrature formula Q . It is expected that as both N and Q grow, the maximum error vanishes. In comparison with the absolute errors presented in Diethelm and Ford [12] and Katsikadelis [14], the maximum errors in this study are smaller. Figure 1 depicts the logscale L -infinity and L^2 error norms for different values of N . The solution errors decay geometrically with an increase in the number of collocation points N .

Example 2 Consider the DO-FDE

$$\int_{0.2}^{1.5} \Gamma(3 - \alpha) {}_0^C D_x^\alpha y(x) d\alpha = 2 \frac{x^{1.8} - x^{0.5}}{\ln x}, \tag{10}$$

whose exact solution is given as $y(x) = x^2$ in Katsikadelis [14]. The equation is solved on the domain $[0, 0.9]$ with the initial conditions $y(0) = y'(0) = 0$.

Like Example 1, the approximation of (10) leads to the linear algebraic system

$$\frac{13}{20Q} \sum_{j=2}^N \left[\left[\Gamma(2.8) D_{j,l}^{0.2} + 2 \sum_{e=1}^{Q-1} \Gamma(3 - \alpha_e) D_{j,l}^{\alpha_e} + \Gamma(1.5) D_{j,l}^{1.5} \right] Y(x_j) \right] = F(x_j),$$

$$Y(x_0) = 0, \quad \sum_{j=0}^N D_{j,0} Y(x_j) = 0, \quad l = 0, 1, 2, \dots, N,$$

where $\alpha_e = 0.2 + 13e/10Q$, $e = 1, \dots, Q - 1$.

Numerical results were presented for this example using fractional Adams' method and analogue equation method by Diethelm and Ford [12] and Katsikadelis [14] respectively. In Table 2, we show the maximum absolute errors in the solution. One distinguishing feature of DO-FDE is the distributed order of the derivatives, and for this reason, we do not expect to obtain the same level of accuracy as in the case of FDEs with fixed order. However, the results presented in Table 2 show that the numerical solution improves as the number of terms in the Chebyshev expansion and the number of intervals in the trapezoidal rule increase. In Diethelm and Ford [12] and Katsikadelis [14], 640 grid points were required to achieve the same level of accuracy in Table 2.

Table 1: Maximum absolute error norms for the approximation of Example 1 using different values of Q and N .

N	Q					
	2	4	8	16	32	64
2	0.99691	1.10720	1.14586	1.16029	1.16621	1.16884
4	0.03859	0.04828	0.05183	0.05312	0.05363	0.05384
6	0.00776	0.00189	0.00047	0.00012	3.14562×10^{-5}	9.41815×10^{-6}
8	0.00764	0.00187	0.00047	0.00012	2.97335×10^{-5}	7.94795×10^{-6}
10	0.00778	0.00189	0.00047	0.00012	2.98145×10^{-5}	7.82399×10^{-6}

Table 2: Absolute error norms in L^∞ for the approximation of Example 2 using different values of Q and N .

N	Q					
	2	4	8	16	32	64
2	0.10225	0.10181	0.10171	0.10168	0.10168	0.10168
4	0.01058	0.00178	0.00110	0.00143	0.00152	0.00154
6	0.01493	0.00324	0.00047	0.00064	0.00072	0.00072
8	0.01618	0.00377	0.00073	0.00035	0.00039	0.00040
10	0.01642	0.00393	0.00084	0.00023	0.00025	0.00025

Example 3 Consider the equation [14]

$$\int_0^2 \Gamma(4 - \alpha) \sinh(\alpha) {}_0^C D_x^\alpha y(x) d\alpha = \frac{6x(x^2 - \cosh 2 - \sinh 2 \ln x)}{(\ln x)^2 - 1}, \quad (11)$$

with the initial conditions $y(0) = y'(0) = 0$, whose exact solution is given as $y(x) = x^3$. The equation is defined in the domain $[0, 1]$. The numerical solution was obtained in Katsikadelis [14] using a hybrid composite trapezoidal rule and analogue equation method.

The discretization of (11) yields the system of linear algebraic equations

$$\frac{1}{Q} \sum_{j=2}^N \left[\left[2 \sum_{e=1}^{Q-1} \Gamma(4 - 2e/Q) \sinh(2e/Q) D_{j,l}^{2e/Q} + \sinh(2) D_{j,l}^2 \right] Y(x_j) \right] = F(x_j),$$

$$Y(x_0) = 0, \quad \sum_{j=0}^N D_{j,0} Y(x_j) = 0, \quad l = 0, 1, 2, \dots, N.$$

Table 3 shows the maximum absolute error in the solution for (11). Again the rate of convergence is typical of spectral methods, although the solution is less accurate when compared with fractional differential equations with fixed or single real orders. Figure 2 shows the error $|y(x) - Y_N(x)|$ in the domain $x \in [0, 1]$ for $N = 15$ and $Q = 64$, while Figure 3 shows the convergence of the numerical solution in terms of L^∞ and L^2 errors with respect to N . Although, the exact solution may not be continuously differentiable in $[0, 1]$, the numerical approximation is accurate. We note that it is possible that the accuracy may deteriorate as N increases if the solution $y(x)$ is no longer in the space of smooth functions $C^N([0, 1])$.

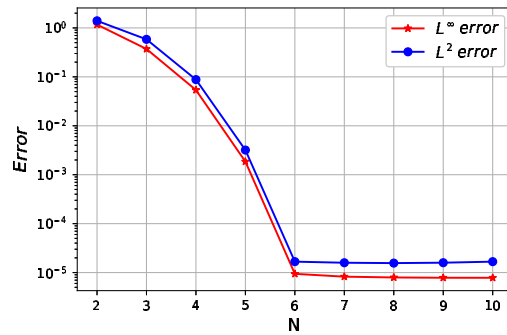


Figure 1: L^∞ and L^2 errors for Example 1 different values of N and $Q = 64$.

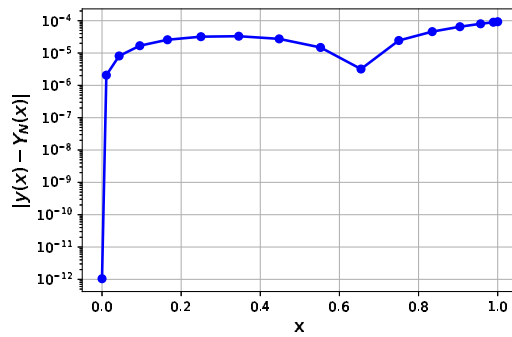


Figure 2: Error $|y(x) - Y_N(x)|$ for Example 3 using $N = 15, Q = 64$.

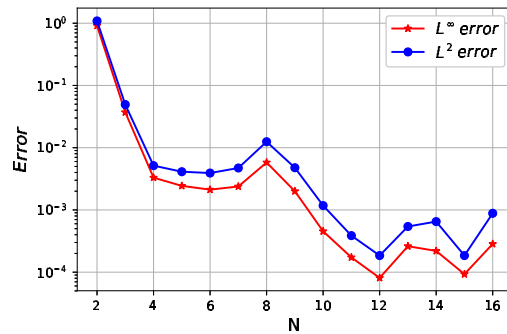


Figure 3: Convergence error norms for Example 3 in both L^∞ and L^2 norms for different number of terms in the Chebyshev expansion. $Q = 64, x \in [0, 1]$.

Table 3: L^∞ error norms for the numerical approximation of Example 3 using different values of N and Q .

N	Q					
	2	4	8	16	32	64
2	0.99013	0.92874	0.91644	0.91374	0.91310	0.91294
4	0.22315	0.06628	0.01935	0.00717	0.00408	0.00331
6	0.20182	0.06035	0.01708	0.00572	0.00284	0.00212
8	0.19946	0.05899	0.01727	0.00774	0.00606	0.00579
10	0.19901	0.05864	0.01325	0.00300	0.00032	0.00045

5 Conclusion

In this study, we presented a numerical method for solving a general distributed order fractional differential equation. The distributed order fractional differential equation was first approximated using a quadrature formula, which transforms it to a multi-term fractional differential equation. The resulting multi-term fractional differential equation is then solved using a pseudo-spectral collocation method. The method assumed that the solution of the distributed order fractional differential equation could be written as linear combination of the shifted first kind Chebyshev polynomials integrated using the Gauss-Lobatto quadrature. We note that the scheme inherits the properties of the quadrature and numerical integration formulas. We demonstrated the accuracy and convergence of the method. The scheme has the advantage that it is easy to use and code. In future, the numerical scheme would be extended to higher dimensional distributed order fractional differential equations.

References

- [1] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific, Singapore, 2010.
- [2] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.*, 339(2000), 1–77.
- [3] G. Jumarie, New stochastic fractional models for Malthusian growth, the Poissonian birth process and optimal management of populations, *Math. Comput. Model.*, 44(2006), 231–254.
- [4] N. H. Sweilam, M. M. Khader and A. Nagy, Numerical solution of two-sided space-fractional wave equation using finite difference method, *J. Comput. Appl. Math.*, 235(2011), 2832–2841.
- [5] L. Feng, P. Zhuang, F. Liu and I. Turner, Stability and convergence of a new finite volume method for a two-sided space-fractional diffusion equation, *Appl. Math. Comput.*, 257(2015), 52–65.
- [6] Z. Mao and J. Shen, Spectral element method with geometric mesh for two-sided fractional differential equations, *Adv. Comput. Math.*, 44(2018), 745–771.
- [7] A. Bhrawy, M. A. Zaky and R. A. van Gorder, A space-time Legendre spectral tau method for the two-sided space-time Caputo fractional diffusion-wave equation, *Numer. Algorithms*, 71(2016), 151–180.
- [8] E. H. Doha, A. Bhrawy and S. S. Ezz-Eldien, A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order, *Comput. Math. Appl.*, 62(2011), 2364–2373.
- [9] M. Zayernouri and G. E. Karniadakis, Fractional spectral collocation method, *SIAM J. Sci. Comput.*, 36 (2014), A40–A62.

- [10] X. Li and C. Xu, A space-time spectral method for the time fractional diffusion equation, *SIAM J. Numer. Anal.*, 47(2009), 2108–2131.
- [11] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications*, volume 198, Elsevier, Amsterdam, 1998.
- [12] K. Diethelm and N. J. Ford, Numerical analysis for distributed-order differential equations, *J. Comput. Appl. Math.*, 225(2009), 96–104.
- [13] G. Gao and Z. Sun, Two alternating direction implicit difference schemes with the extrapolation method for the two-dimensional distributed-order differential equations, *Comput. Math. Appl.*, 69(2015), 926–948.
- [14] J. T. Katsikadelis, Numerical solution of distributed order fractional differential equations, *J. Comput. Phys.*, 259(2014), 11–22.
- [15] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, volume 55, Courier Corporation, Massachusetts, 1965.
- [16] S. Oloniju, S. Goqo and P. Sibanda, A Chebyshev spectral method for heat and mass transfer in MHD nanofluid flow with space fractional constitutive model, *Frontiers Heat Mass Transf.*, 13(2019).
- [17] V. J. Ervin and J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Methods Partial Diff. Eq.*, 22(2006), 558–576.
- [18] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods: Fundamentals in Single Somains*, Springer Science & Business Media, Berlin, 2007.
- [19] T. Atanackovic, M. Budincevic and S. Pilipovic, On a fractional distributed-order oscillator, *J. Phys. A*, 38(2005), 6703.