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Properties Of Harmonic Mappings Associated With Polylogarithm Functions*

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Abstract

The polylogarithm functions, also known as Jonquiére's functions, are important due to its various applications in quantum statistics and quantum electrodynamics. In this article a new variety of their applications have been searched and these functions are connected with the concepts of harmonic analysis. A new subclass of harmonic mappings involving integral operator is defined by polylogarithm functions in the region of circular domain. Furthermore, we investigate necessary and sufficient conditions involving convolution, coefficient bounds, topological properties, radii problems, distortion theorem and integral representation for functions belonging to this class.

1 Introduction

A harmonic mapping f in a simply connected domain $\mathbb{D} \subset \mathbb{C}$ is a complex-valued function of the form $f = h + \overline{g}$, where h and g are analytic functions and normalized by the conditions $h(0) = h'(0) - 1 = 0$, $g(0) = 0$. These functions h and g are also known as analytic and co-analytic parts of f respectively. The Jacobian of f is given by

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = |h'(z)|^2 - |g'(z)|^2 \quad (z \in \mathbb{D}).$$

It can be noted that if f is analytic in $\mathbb{D}$, then $f_{\overline{z}} = 0$ and $f_z(z) = f'(z)$. A very familiar result, in [1], for analytic functions states that an analytic function f is locally univalent at a point z_0 if and only if its Jacobian is never zero at that point in $\mathbb{D}$. In [3] the converse of this theorem proved by Lewy, which is also true for harmonic mappings. Thus, f is sense-preserving and locally univalent if and only if

$$|h'(z)| - |g'(z)| > 0 \quad (z \in \mathbb{D}). \tag{1}$$

We indicate by $\mathcal{H}$ the class of complex-valued harmonic mappings in the unit disc $\mathfrak{A} := \mathfrak{A}(1)$, where $\mathfrak{A}(r) := \{z \in \mathbb{C} : |z| < r\}$. Then $f \in \mathcal{H}$, if $f = h + \overline{g}$, where h and g are analytic functions in $\mathfrak{A}$. Also, by $\mathcal{H}_0$ we denote the class of functions $f \in \mathcal{H}$ having the following series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n} \quad (z \in \mathfrak{A}), \tag{2}$$

and let

$$\mathcal{S}_{\mathcal{H}} := \{f \in \mathcal{H}_0 : f \text{ is univalent and sense-preserving in } \mathfrak{A}\}.$$

If co-analytic part of $f \in \mathcal{S}_{\mathcal{H}}$ vanishes, then the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$, i.e, $g(z) = 0$ in $\mathfrak{A}$. Clunie and Sheil-Small (see [2]) studied the class $\mathcal{S}_{\mathcal{H}}$ as well as some of its geometric subclasses and obtained coefficient bounds. More works on $\mathcal{S}_{\mathcal{H}}$ and its subclasses can be seen in several different papers such as

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Sheil-Small [4], Jahangiri [11], Silverman [9] and Silverman and Silvia [13]. A function $g_1(z)$ is subordinated to function $g_2(z)$ and symbolically represented by $g_1(z) \prec g_2(z)$, if there is complex-valued function $w(z)$ with $|w(z)| \leq 1$ and $w(0) = 0$ in such a way that

$$g_1(z) = g_2(w(z)) \quad (z \in \mathfrak{A}).$$

Further, if $g_2(z)$ is univalent in $\mathfrak{A}$, we have equivalent condition

$$g_1(z) \prec g_2(z) \;\; (z \in \mathfrak{A}) \iff g_1(0) = g_2(0) \text{ and } g_1(\mathfrak{A}) \subset g_2(\mathfrak{A}).$$

Convolution or Hadamard product of two function f_1 , f_2 is symbolized by $f_1 * f_2$ and defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} \left(a_1 a_2 z^n + \overline{b_1 b_2 z^n} \right) \quad (z \in \mathfrak{A}). \tag{3}$$

The familiar generalization of Reimann zeta and polylogarithm functions, or simply the δ^{th} order polylogarithm function is denoted by $\varphi_\delta(c; z)$, and given by

$$\varphi_\delta(c; z) = \sum_{n=1}^{\infty} \frac{z^n}{(n + c)^\delta},$$

where any term $n + c = 0$ is precluded (see [6]). Utilizing the definition of the Gamma function (for more details see [5]), the integral formula for $\varphi_\delta(c; z)$ is obtained by a simple transformation as follows:

$$\varphi_\delta(c; z) = \frac{1}{\Gamma(\delta)} \int_0^1 z \left(\log \frac{1}{t} \right)^{\delta - 1} \frac{t^c}{1 - tz} dt,$$

where $\operatorname{Re}(c) > -1$ and $\operatorname{Re}(\delta) > 1$. For further related work, see Ponnusamy [17] and Ponnusamy and Sabapathy [18] on polylogarithm function.

For $f \in \mathcal{A}$ expressed in series expansion (1), Al-Shaqsi [19] defined the following integral operator

$$\mathcal{J}_d^\eta f(z) = (1 + d)^\eta \varphi_\eta(d; z) * f(z) = -\frac{(1 + d)^\eta}{\Gamma(\delta)} \int_0^1 t^{d-1} \left(\log \frac{1}{t} \right)^{\eta - 1} f(tz)\, dt, \tag{4}$$

where $d > 0$, $\eta > 1$ and $z \in \mathfrak{A}$.

In [19], Al-Shaqsi realized that the operator defined by (4) can be represented by series expansion

$$\mathcal{J}_d^\eta f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1 + d}{n + d} \right)^\eta a_n z^n. \tag{5}$$

Now we define the operator $\mathcal{J}_d^\eta f(z)$ in (5) for a function $f \in \mathcal{H}$ given by (1) as follows:

$$\mathcal{J}_d^\eta f(z) = \mathcal{J}_d^\eta h(z) + \overline{\mathcal{J}_d^\eta g(z)} \;\; (z \in \mathfrak{A}),$$

where

$$\mathcal{J}_d^\eta h(z) = z + \sum_{n=2}^{\infty} \left(\frac{1 + d}{n + d} \right)^\eta a_n z^n \;\text{ and }\; \mathcal{J}_d^\eta g(z) = \sum_{n=2}^{\infty} \left(\frac{1 + d}{n + d} \right)^\eta b_n z^n,$$

where $d > 0$, $\eta > 1$ and $z \in \mathfrak{A}$.

Motivated from the work of Dziok (see [10]) and using the operator $\mathcal{J}_d^\eta f(z)$, we introduce a new subclass of harmonic univalent mappings below:

For $-Q \leq P < Q \leq 1$ and $0 \leq a < 1$, let $\mathcal{S}_\mathcal{H}^{d,\eta}(P, Q)$ denote the class of functions $f \in \mathcal{S}_\mathcal{H}$ such that

$$\frac{\mathcal{D}_\mathcal{H}\left(\mathcal{J}_d^\eta f(z) \right)}{\mathcal{J}_d^\eta f(z)} \prec \frac{1 + Pz}{1 + Qz}, \tag{6}$$

with

$$\mathcal{D}_\mathcal{H} \mathcal{J}_d^\eta f(z) := \mathcal{D}_\mathcal{H} \mathcal{J}_d^\eta h(z) - \overline{\mathcal{D}_\mathcal{H} \mathcal{J}_d^\eta g(z)}.$$

Note that

1. $\mathcal{S}_{\mathcal{H}}^{d,0}(P,Q)=\mathcal{S}_{\mathcal{H}}^{*}(P,Q)$ studied by Dziok (see [10]),

2. $\mathcal{S}_{\mathcal{H}}^{d,0}(2a-1,1)=\mathcal{S}_{\mathcal{H}}^{*}(a)$ introduced by Jahangiri in [15].

In present article, we obtain analytic condition for a new class $\mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)$ of harmonic mappings. Some results such as subordination conditions, coefficient bounds, distortion bounds, radii problems, integral means and extreme point theorem for this class are also obtained. Let $\mathcal{V}\subset\mathcal{H}_0$, $\mathfrak{A}_0=\mathfrak{A}\{0\}$. Using Ruscheweyh's approach [12] we define the dual set of $\mathcal{V}^*$ by

$$\mathcal{V}^*:=\left\{f\in\mathcal{H}_0:\bigcap\nolimits_{g\in\mathcal{V}}(f*g)(z)\neq 0\quad(z\in\mathfrak{A})\right\}.$$

2 Analytic Criteria

Theorem 1 *Let $f\in\mathcal{H}_0$ be of the form (2). Then $f\in\mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)$ if and only if*

$$\mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)=\left\{\mathcal{J}_d^{\eta}\varphi_{\xi}(z):\ |\xi|=1\right\}^*,$$

where

$$\varphi_{\xi}(z)=z\frac{1+Q\xi-(1+P\xi)(1-z)}{(1-z)^2}-\overline{z}\frac{1+Q\xi-(1+P\xi)(1-\overline{z})}{(1-\overline{z})^2}\quad(z\in\mathfrak{A}).$$

Proof. Let $f\in\mathcal{H}_0$.Then $f\in\mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)$ if and only if

$$\frac{\mathcal{D}_{\mathcal{H}}(\mathcal{J}_d^{\eta}f(z))}{\mathcal{J}_d^{\eta}f(z)}\neq\frac{1+P\xi}{1+Q\xi}\quad(\xi\in\mathbb{C},\ |\xi|=1).$$

Since

$$\mathcal{D}_{\mathcal{H}}\mathcal{J}_d^{\eta}h(z)=\mathcal{J}_d^{\eta}h(z)*\frac{z}{(1-z)^2}$$

and

$$\mathcal{J}_d^{\eta}h(z)=\mathcal{J}_d^{\eta}h(z)*\frac{z}{1-z},$$

we have

$$\begin{aligned}
&(1+Q\xi)\mathcal{D}_{\mathcal{H}}(\mathcal{J}_d^{\eta}h(z))-(1+P\xi)\mathcal{J}_d^{\eta}h(z)\\
=\ &(1+Q\xi)\mathcal{D}_{\mathcal{H}}(\mathcal{J}_d^{\eta}h(z))-(1+P\xi)\mathcal{J}_d^{\eta}h(z)-\left[(1+Q\xi)\overline{\mathcal{D}_{\mathcal{H}}(\mathcal{J}_d^{\eta}g(z))}+(1+P\xi)\overline{\mathcal{J}_d^{\eta}g(z)}\right]\\
=\ &\mathcal{J}_d^{\eta}h(z)*\left(\frac{(1+Q\xi)z}{(1-z)^2}-\frac{(1+P\xi)z}{1-z}\right)-\overline{\mathcal{J}_d^{\eta}g(z)}*\left(\frac{(1+Q\xi)\overline{z}}{(1-\overline{z})^2}+\frac{(1+P\xi)\overline{z}}{1-\overline{z}}\right)\\
=\ &f(z)*\mathcal{J}_d^{\eta}\varphi_{\xi}(z)\neq 0\quad(z\in\mathfrak{A}_0,\ |\xi|=1).
\end{aligned}$$

Thus, $f\in\mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)$ if and only if $f(z)*\mathcal{J}_d^{\eta}\varphi_{\xi}(z)\neq 0$ for $z\in\mathfrak{A}_0$, $|\xi|=1$. That is, $\mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)=\left\{\mathcal{J}_d^{\eta}\varphi_{\xi}(z):\ |\xi|=1\right\}^*$. ■

A sufficient coefficient bound for the class $\mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)$ is provided in the following.

Theorem 2 *Let $f\in\mathcal{H}_0$ of the form (2) satisfy the condition*

$$\sum_{n=2}^{\infty}(C_n|a_n|+D_n|b_n|)\leq Q-P,\tag{7}$$

where

$$C_n = \left| \left(\frac{1+d}{n+d} \right)^{\eta} \right| \left[(1+Q)\, n - (1+P) \right] \tag{8}$$

and

$$D_n = \left| \left(\frac{1+d}{n+d} \right)^{\eta} \right| \left[(1+Q)\, n + (1+P) \right]. \tag{9}$$

Then $f \in \mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)$. *The above inequality* (7) *is sharp for function defined by*

$$f(z) = z + \sum_{n=2}^{\infty} x_n \frac{Q-P}{C_n} z^n + \sum_{n=2}^{\infty} y_n \frac{Q-P}{D_n} \overline{z}^n \quad (z \in \mathfrak{A})$$

such that $\sum_{n=2}^{\infty} (|x_n| + |y_n|) = 1$.

Proof. It is easy to see that the theorem is true for $f(z) = z$. Let $f \in \mathcal{H}_0$ be expressed in series expansion (2) and let there exist $n \geq 2$ such that $a_n \neq 0$ or $b_n \neq 0$. Since

$$C_n \geq n(Q-P), \quad D_n \geq (Q-P)\, n, \quad n = 2, 3, \cdots,$$

from (7) we obtain

$$\sum_{n=2}^{\infty} (n\ |a_n| + n\ |b_n|) \leq 1 \tag{10}$$

and

$$\begin{aligned} |h'(z)| - |g'(z)| &\geq 1 - \sum_{n=2}^{\infty} n\ |a_n|\, |z|^n - \sum_{n=2}^{\infty} n\ |b_n|\, |z|^n \geq 1 - |z| \sum_{n=2}^{\infty} (n\ |a_n| + n\ |b_n|) \\ &\geq 1 - \frac{|z|}{Q-P} \sum_{n=2}^{\infty} (C_n\, |a_n| + D_n\, |b_n|) \geq 1 - |z| > 0 \quad (z \in \mathfrak{A}). \end{aligned}$$

Therefore, by (1), f is locally univalent and sense-preserving in $\mathfrak{A}$. Moreover, if $z_1, z_2 \in \mathfrak{A}$ with $z_1 \neq z_2$, then

$$\left| \frac{z_1^n - z_2^n}{z_1 - z_2} \right| = \left| \sum_{k=1}^{n} z_1^{k-1} z_2^{n-k} \right| \leq \sum_{k=1}^{n} \left| z_1^{k-1} \right| \left| z_2^{n-k} \right| < n \quad (n = 2, 3, \cdots).$$

Hence

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\ &= 1 - \left| \frac{\sum_{n=2}^{\infty} b_n (z_1^n - z_2^n)}{(z_1 - z_2) + \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n)} \right| \\ &> 1 - \frac{\sum_{n=2}^{\infty} n b_n}{1 - \sum_{n=2}^{\infty} n a_n} \geq 1 - \frac{\sum_{n=2}^{\infty} \frac{D_n}{P-Q} b_n}{1 - \sum_{n=2}^{\infty} \frac{C_n}{P-Q} a_n} \geq 0, \end{aligned}$$

which shows that f is univalent. On the other hand, $f \in \mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q)$ if and only if there exists a complex-valued function ω $(\omega(0) = 0,\ |\omega(z)| < 1\ (z \in \mathfrak{A}))$, such that

$$\frac{\mathcal{D}_{\mathcal{H}}\left(\mathcal{J}_d^{\eta} f(z)\right)}{\mathcal{J}_d^{\eta} f(z)} = \frac{1 + P\omega(z)}{1 + Q\omega(z)} \quad (z \in \mathfrak{A}),$$

or alternatively,

$$\left| \frac{\mathcal{D}_{\mathcal{H}}\left(\mathcal{J}_d^{\eta} f(z)\right) - \mathcal{J}_d^{\eta} f(z)}{Q\mathcal{D}_{\mathcal{H}}\left(\mathcal{J}_d^{\eta} f(z)\right) - P\mathcal{J}_d^{\eta} f(z)} \right| < 1 \quad (z \in \mathfrak{A}). \tag{11}$$

Thus, it is sufficient to establish that

$$\left|\mathcal{D}_{\mathcal{H}}\left(\mathcal{J}_d^{\eta} f(z)\right)-\mathcal{J}_d^{\eta} f(z)\right|-\left|Q\mathcal{D}_{\mathcal{H}}\left(\mathcal{J}_d^{\eta} f(z)\right)-P\mathcal{J}_d^{\eta} f(z)\right|<0,$$

where $z\in\mathfrak{A}\setminus\{0\}$. Now by setting $|z|=r,\ (0<r<1)$ we get

$$\begin{aligned}
&\left|\mathcal{D}_{\mathcal{H}}\left(\mathcal{J}_d^{\eta} f(z)\right)-\mathcal{J}_d^{\eta} f(z)\right|-\left|Q\mathcal{D}_{\mathcal{H}}\left(\mathcal{J}_d^{\eta} f(z)\right)-P\mathcal{J}_d^{\eta} f(z)\right| \\
=\ & \left|\sum_{n=2}^{\infty}\left(\frac{1+d}{n+d}\right)^{\eta}(n-1)\,a_n z^n-\sum_{n=2}^{\infty}\left(\frac{1+d}{n+d}\right)^{\eta}(n+1)\,\overline{b_n z^n}\right| \\
& -\left|(Q-P)\,z+\sum_{n=2}^{\infty}\left(\frac{1+d}{n+d}\right)^{\eta}(Qn-P)\,a_n z^n-\left(\frac{1+d}{n+d}\right)^{\eta}(Qn+P)\,\overline{b_n z^n}\right| \\
\leq\ & \sum_{n=2}^{\infty}\left|\left(\frac{1+d}{n+d}\right)^{\eta}\right|(n-1)\,|a_n|\,r^n+\sum_{n=2}^{\infty}\left|\left(\frac{1+d}{n+d}\right)^{\eta}\right|(n+1)\,|b_n|\,r^n \\
& -(Q-P)\,r+\sum_{n=2}^{\infty}\left|\left(\frac{1+d}{n+d}\right)^{\eta}\right|(Qn-P)\,|a_n|\,r^n+\sum_{n=2}^{\infty}\left|\left(\frac{1+d}{n+d}\right)^{\eta}\right|(Qn+P)\,|b_n|\,r^n \\
\leq\ & r\left\{\sum_{n=2}^{\infty}\left(C_n|a_n|+D_n|b_n|\right)r^n-(Q-P)\right\}<0,
\end{aligned}$$

which completes the proof. ■

Motivated from the work of Silverman [9], we now denote by $\tau^k\ (k\in\mathbb{N}_0=\{0,1,2,\cdots\})$ the class of functions $f\in\mathcal{H}_0$ of the form (2) such that

$$f(z)=h(z)+\overline{g(z)}=z-\sum_{n=2}^{\infty}|a_n|\,z^n+(-1)^k\sum_{n=2}^{\infty}|b_n|\,\overline{z}^n\ \ (z\in\mathfrak{A}). \tag{12}$$

Further, we define

$$\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)=\tau^k\cap\mathcal{S}_{\mathcal{H}}^{d,\eta}(P,Q).$$

We note that for $\eta=0$ and $k=0$, the class $\mathcal{S}_{\tau}^{d,0,0}(P,Q)$ was studied by Dziok (see [10]).

Theorem 3 *Let $f\in\tau^k$ has series expansion of the form (12). Then $f\in\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ if and only if condition (7) holds true.*

Proof. From Theorem 2 it is enough to show that each function $f\in\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ satisfies the relation (7). If $f\in\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$, then it is of the form (12) and must satisfy (11) or equivalently

$$\left|\frac{-\sum_{n=2}^{\infty}\left(\frac{1+d}{n+d}\right)^{\eta}(n-1)\,a_n z^n-\sum_{n=2}^{\infty}\left(\frac{1+d}{n+d}\right)^{\eta}(n+1)\,\overline{b_n z^n}}{(Q-P)\,z-\sum_{n=2}^{\infty}\left(\frac{1+d}{n+d}\right)^{\eta}(Qn-P)\,a_n z^n-\sum_{n=2}^{\infty}\left(\frac{1+d}{n+d}\right)^{\eta}(Qn+P)\,\overline{b_n z^n}}\right|<1$$

for $z\in\mathfrak{A}$. Therefore by putting $z=r\ (r\in[0,1))$, we get

$$\frac{\sum_{n=2}^{\infty}\left|\left(\frac{1+d}{n+d}\right)^{\eta}\right|\left[(n-1)\,|a_n|+(n+1)\,|b_n|\right]r^{n-1}}{(Q-P)+\sum_{n=2}^{\infty}\left|\left(\frac{1+d}{n+d}\right)^{\eta}\right|\left\{(Qn-P)\,|a_n|+(Qn+P)\,|b_n|\right\}}<1. \tag{13}$$

It is clear that the denominator of the left hand side cannot vanish for $r\in(0,1)$. Moreover, it is positive for $r=0$, and in consequence for $r\in(0,1)$. Thus, by (13) we have

$$\sum_{n=2}^{\infty}\left(C_n|a_n|+D_n|b_n|\right)r^{n-1}\leq Q-P\quad r\in[0,1). \tag{14}$$

Let $\{S_n\}$ be a sequence of partial sums connected with the series $\sum_{n=2}^{\infty}(C_n|a_n|+D_n|b_n|)$. Then $\{S_n\}$ is a non-decreasing sequence and by (14), it is bounded by $Q-P$. So, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}(C_n|a_n|+D_n|b_n|)\,r^{n-1}=\lim_{n\to\infty}S_n\leq Q-P,$$

which yields assertion (7). ■

Theorem 4 *Let the functions $f_t(z)$ $(t=1,2,3,\cdots,l)$ defined by (2) be in the class $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$. Then the function $F\in\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$, where*

$$F(z)=\sum_{k=1}^{l}\lambda_k f_k(z)\,;\quad\left(\lambda_k\geq 0,\ \sum_{k=1}^{l}\lambda_k=1\right). \tag{15}$$

Proof. The proof is straight forward and so omitted for details. ■

3 Topological Properties

We consider the usual topology on $\mathcal{H}$ in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathfrak{A}$. The metric induces the usual topology on $\mathcal{H}$. It is easy to verify that the obtained topological space is complete.

Let $\mathcal{F}$ be a subclass of the class $\mathcal{H}$. A function $f\in\mathcal{F}$ is called an extreme point of $\mathcal{F}$ if the condition

$$f=uf_1+(1-u)f_2\qquad(f_1,f_2\in\mathcal{F},\ 0<u<1)$$

implies $f_1=f_2=f$. We shall indicate $E\mathcal{F}$ to represent the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F}\subset\mathcal{F}$.

We say that $\mathcal{F}$ is locally uniformly bounded if for each r $(0<r<1)$, there is a real constant $M=M(r)$ so that

$$|f(z)|\leq M\quad(f\in\mathcal{F},\ |z|\leq r).$$

We say that a class $\mathcal{F}$ is convex if

$$uf+(1-u)g\in\mathcal{F}\quad(f,g\in\mathcal{F},\ 0\leq u\leq 1).$$

Moreover, the closed convex hull of $\mathcal{F}$ is denoted by $\overline{co}\mathcal{F}$ and defined as the intersection of all closed convex subsets of $\mathcal{H}$ that contain $\mathcal{F}$.

A real-valued function $\mathfrak{J}:\mathcal{H}\to\mathbb{R}$ is called convex on a convex class $\mathcal{F}\subset\mathcal{H}$ if

$$\mathfrak{J}(uf+(1-u)g)\leq u\mathfrak{J}(f)+(1-u)\mathfrak{J}(g)\quad(f,g\in\mathcal{F},\ 0\leq u\leq 1).$$

The Krein-Milman theorem (see [7]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

Lemma 1 *Let $\mathcal{F}$ be a non-empty convex compact subclass of the class $\mathcal{H}$ and let $\mathfrak{J}:\mathcal{H}\to\mathbb{R}$ be a real-valued, continuous and convex function on $\mathcal{F}$. Then*

$$\max\{\mathfrak{J}(f):f\in\mathcal{F}\}=\max\{\mathfrak{J}(f):f\in E\mathcal{F}\}.$$

Lemma 2 *A class $\mathcal{F}\subset\mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Since $\mathcal{H}$ is complete metric space, Montel's theorem (see [14]) implies the following lemma.

Lemma 3 *Let $\mathcal{F}$ be a non-empty compact subclass of the class $\mathcal{H}$. Then $E\mathcal{F}$ is non-empty and $\overline{co}E\mathcal{F}=\overline{co}\mathcal{F}$.*

Theorem 5 *The class* $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ *is a convex and compact subset of* $\mathcal{H}$.

Proof. Let $f_l \in \mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ be functions of the form

$$f_l(z) = z - \sum_{n=2}^{\infty}\left(|a_{l,n}| z^n - (-1)^k |b_{l,n}| \overline{z}^n\right) \quad (z \in \mathfrak{A},\ l \in \mathbb{N} = \{1, 2, \cdots\}), \tag{16}$$

and $0 \le u \le 1$. Letting

$$uf_1(z) + (1-u) f_2(z) = z - \sum_{n=2}^{\infty}\left\{(u|a_{1,n}| + (1-u)|a_{2,n}|) z^n + (-1)^k (u|b_{1,n}| + (1-u)|b_{2,n}|) \overline{z}^n\right\}$$

and using Theorem 3, we have

$$\begin{aligned} &\sum_{n=2}^{\infty}\{C_n(u|a_{1,n}| + (1-u)|a_{2,n}|) z^n + D_n(u|b_{1,n}| + (1-u)|b_{2,n}|)\} \\ = \;& u\sum_{n=2}^{\infty}\{C_n|a_{1,n}| + D_n|b_{1,n}|\} + (1-u)\sum_{n=2}^{\infty}\{C_n|a_{2,n}| + D_n|b_{2,n}|\} \\ \le \;& u(Q-P) + (1-u)(Q-P) = Q-P, \end{aligned}$$

which implies that the function $\Psi = \upsilon f_1 + (1-\upsilon) f_2 \in \mathcal{S}_{\tau}^{d,\eta}(P,Q)$. Hence, the class $\mathcal{S}_{\tau}^{d,\eta}(P,Q)$ is convex. Furthermore, for $f \in \mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$, $|z| \le r$ $(r \in (0,1))$, we have

$$|f(z)| \le r + \sum_{n=2}^{\infty}(|a_n| + |b_n|) r^n \le r + \sum_{n=2}^{\infty}(C_n|a_n| + D_n|b_n|) \le r + (Q-P). \tag{17}$$

Thus, we conclude that the class $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ is locally uniformly bounded. By Lemma 2, we need only to show that it is closed, i.e. if $f_l \to f$, then $f \in \mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$. Let f_l and f be given by (16) and (12), respectively. Using Theorem 3, we have

$$\sum_{n=2}^{\infty}(C_n|a_{i,n}| + D_n|b_{i,n}|) \le Q-P \quad (i \in \mathbb{N}). \tag{18}$$

Since $f_i \to f$, we conclude that $|a_{i,n}| \to |a_n|$ and $|b_{i,n}| \to |b_n|$ as $i \to \infty$ $(n \in \mathbb{N})$. The sequence $\{S_n\}$ of partial sums associated with the series $\sum_{n=2}^{\infty}(C_n|a_{i,n}| + D_n|b_{i,n}|)$ is non-decreasing sequence. Moreover, by (18) it is bounded by $Q-P$. Therefore, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}(C_n|a_{i,n}| + D_n|b_{i,n}|) = \lim_{n\to\infty}\{S_n\} \le Q-P.$$

This gives condition (7) and in consequence, $f \in \mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$, which completes the proof. ∎

Theorem 6 *Let* $f_k(z) \in \mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ *of the form* (16) *for* $k = 1, 2$. *Then their weighted mean* $w_j(z)$ *for any real number* j *is also in the class* $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$, *where* $w_j(z)$ *is given by*

$$w_j(z) = \left\{\frac{(1-j) f_1(z) + (1+j) f_2(z)}{2}\right\}. \tag{19}$$

Proof. From (19) one can easily write

$$w_j(z) = z + \sum_{n=2}^{\infty}\left[\frac{(1-j) a_{n,1} + (1+j) a_{n,2}}{2} z^n + (-1)^k \frac{(1-j)\overline{b_{n,1}} + (1+j)\overline{b_{n,2}}}{2}\overline{z^n}\right].$$

To prove $w_j(z) \in \mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$, we consider

$$\begin{aligned}
& \sum_{n=2}^{\infty}\left(C_n\left|\frac{(1-j)\,a_{n,1}+(1+j)\,a_{n,2}}{2}\right|+D_n\left|(-1)^k\,\frac{(1-j)\,\overline{b_{n,1}}+(1+j)\,\overline{b_{n,2}}}{2}\right|\right)\\
= & \sum_{n=2}^{\infty}\left(C_n\frac{(1-j)}{2}\left|a_{n,1}\right|+D_n\frac{(1-j)}{2}\left|b_{n,1}\right|\right)+\sum_{n=2}^{\infty}\left(C_n\frac{(1+j)}{2}\left|a_{n,2}\right|+D_n\frac{(1-j)}{2}\left|b_{n,2}\right|\right)\\
= & \frac{(1-j)}{2}\sum_{n=2}^{\infty}\left(C_n\left|a_{n,1}\right|+D_n\left|b_{n,1}\right|\right)+\frac{(1+j)}{2}\sum_{n=2}^{\infty}\left(C_n\left|a_{n,2}\right|+D_n\left|b_{n,2}\right|\right)\\
\leq & \frac{(1-j)}{2}(Q-P)+\frac{(1+j)}{2}(Q-P)=(Q-P).
\end{aligned}$$

Hence by Theorem 3, $w_j(z) \in \mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$. ■

Theorem 7 *We have*

$$E\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)=\{h_n : n\in\mathbb{N}\}\cup\{g_n : n\in\{2,3,\cdots\}\},$$

where

$$h_1(z)=z,\; h_n(z)=z\;-\frac{Q-P}{C_n}z^n,\; g_n(z)=(-1)^k\,\frac{Q-P}{D_n}\overline{z}^n\;\;(n=2,3,\cdots;\;z\in\mathfrak{A}). \tag{20}$$

Proof. Assume that $0<u<1$ and

$$g_n(z)=uf_1(z)+(1-u)\,f_2(z),$$

where the functions f_1,$f_2 \in \mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ of the form (16). Then, by (7) we have

$$\left|b_{1,n}\right|=\left|b_{2,n}\right|=\frac{Q-P}{D_n},$$

and in consequence, $a_{1,k}=a_{2,k}=0$ for $k\in\{2,3,\cdots\}$ and $b_{1,k}=b_{2,k}=0$ for $k\in\{2,3,\cdots\}\setminus\{n\}$. It follows that $g_n=f_1=f_2$, and consequently $g_n\in E\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$. Similarly, we can verify that the functions h_n of the form (20) are extreme points of the class $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$. Now, suppose that f of the form (16) is in the class of extreme point of the class $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ and f is not of the form (20). Then there exists $k\in\{2,3,\cdots\}$ such that

$$0<\left|a_k\right|<\frac{Q-P}{C_k}\quad\text{or}\quad 0<\left|b_k\right|<\frac{Q-P}{D_k}.$$

If $0<|a_k|<\frac{Q-P}{C_k}$, then putting

$$u=\frac{\left|a_k\right|C_k}{Q-P},\;\;\phi=\frac{1}{1-u}\left(f-uh_k\right),$$

we obtain $0<u<1$, h_k, $\phi\in\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$, $h_k\neq\phi$ and

$$f=uh_k+(1-u)\,\phi.$$

Thus,the function f is not in the class of extreme point of the class $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$. Similarly, if $0<|b_k|<\frac{Q-P}{D_k}$, then putting

$$u=\frac{\left|b_k\right|D_k}{Q-P},\quad\phi=\frac{1}{1-u}\left(f-ug_k\right),$$

we obtain $0<u<1$, g_k, $\phi\in\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$, $g_k\neq\phi$ and

$$f=ug_k+(1-u)\,\phi.$$

It follows that $f\notin\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$. This proves the theorem. ■

4 Radii of Starlikeness and Convexity

A function $f \in \mathcal{H}_0$ is said to be starlike of order α in $\mathfrak{A}(r)$ if

$$\frac{\partial}{\partial t}\left(\arg f\left(\rho e^{it}\right)\right) > \alpha \ (0 \leq t \leq 2\pi; \ (0 < \rho < r < 1). \tag{21}$$

Also a function $f \in \mathcal{H}_0$ is said to be convex of order α in $\mathfrak{A}(r)$ if

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(\arg f\left(\rho e^{it}\right)\right)\right) > \alpha \ (0 \leq t \leq 2\pi; \ 0 < \rho < r < 1).$$

It is very simple to verify that the condition (21) is identical to the following

$$\mathrm{Re}\frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} > \alpha \quad (z \in \mathfrak{A}(r)),$$

or equivalently,

$$\left|\frac{\mathcal{D}_{\mathcal{H}} f(z) - (1+\alpha) f(z)}{\mathcal{D}_{\mathcal{H}} f(z) + (1+\alpha) f(z)}\right| < 1 \ \ (z \in \mathfrak{A}(r)). \tag{22}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{H}_0$. Now we define the radii of starlikeness and convexity, respectively, as follows:

$$R_{\alpha}^{*}(\mathcal{B}) = \inf_{f\in\mathcal{B}}\left(\sup\left\{0 < r \leq 1 : f \text{ is starlike of order } \alpha \text{ in } \mathfrak{A}(r)\right\}\right)$$

and

$$R_{\alpha}^{c}(\mathcal{B}) = \inf_{f\in\mathcal{B}}\left(\sup\left\{0 < r \leq 1 : f \text{ is convex of order } \alpha \text{ in } \mathfrak{A}(r)\right\}\right).$$

Theorem 8 *The radius of starlikeness of order α for the class $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ is given by*

$$R_{\alpha}^{*}\left(\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)\right) = \inf_{n\geq 2}\left(\frac{1-\alpha}{Q-P}\min\left\{\frac{C_n}{n-\alpha}, \frac{D_n}{n+\alpha}\right\}\right), \tag{23}$$

where C_n and D_n are define by (8) and (9), respectively.

Proof. Let $f \in \mathcal{S}_{\tau}^{d,\eta,k}(A,B)$ be expressed in the series expansion (12). Then, for $|z| = r < 1$ we have

$$\begin{aligned}\left|\frac{\mathcal{D}_{\mathcal{H}} f(z) - (1+\alpha) f(z)}{\mathcal{D}_{\mathcal{H}} f(z) + (1+\alpha) f(z)}\right| &= \left|\frac{-\alpha z + \sum_{n=2}^{\infty}\left((n-1-\alpha)|a_n| z^n - (n+1+\alpha)|b_n| \overline{z}^n\right)}{(2-\alpha) z + \sum_{n=2}^{\infty}\left((n+1-\alpha)|a_n| z^n - (n-1+\alpha)|b_n| \overline{z}^n\right)}\right| \\ &\leq \frac{\alpha + \sum_{n=2}^{\infty}\left((n-1-\alpha)|a_n| - (n+1+\alpha)|b_n|\right) r^{n-1}}{(2-\alpha) - \sum_{n=2}^{\infty}\left((n+1-\alpha)|a_n| - (n-1+\alpha)|b_n|\right) r^{n-1}}.\end{aligned}$$

So, the condition (22) is true if and only if

$$\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) r^{n-1} \leq 1. \tag{24}$$

By Theorem 2, we have

$$\sum_{n=2}^{\infty}\left(\frac{C_n}{Q-P}|a_n| + \frac{D_n}{Q-P}|b_n|\right) \leq 1z, \tag{25}$$

where C_n and D_n are defined by (8) and (9) respectively. Thus the conditions (24) is true if

$$\frac{n-\alpha}{1-\alpha} r^{n-1} \leq \frac{C_n}{Q-P}, \frac{n+\alpha}{1-\alpha} r^{n-1} \leq \frac{D_n}{Q-P} \ (n = 2, 3, \cdots),$$

i.e.,

$$r \leq \left(\frac{1-\alpha}{Q-P} \min\left\{\frac{C_n}{n-\alpha}, \frac{D_n}{n+\alpha}\right\}\right)^{n-1} \quad (n = 2, 3, \cdots).$$

It follows that the function f is starlike of order α in the disc $\mathfrak{A}(r^*)$, where r^*

$$r^* := \inf \left(\frac{1-\alpha}{Q-P} \min\left\{\frac{C_n}{n-\alpha}, \frac{D_n}{n+\alpha}\right\}\right)^{n-1}.$$

The functions h_n and g_n defined by (20) realize equality in (25), and the radius r^* cannot be larger. Thus we get (23). ■

Theorem 9 *The radius of convexity of order α for the class $\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)$ is given by*

$$R_{\alpha}^{c}\left(\mathcal{S}_{\tau}^{d,\eta,k}(P,Q)\right) = \inf_{n\geq 2} \left(\frac{1-\alpha}{Q-P} \min\left\{\frac{C_n}{n-\alpha}, \frac{D_n}{n+\alpha}\right\}\right),$$

where C_n and D_n are define in (8) and (9) respectively.

Proof. The proof of this theorem is identical as Theorem 8. So we omitted for details. ■

5 Conclusion

In this article, we defined an integral operator using polylogrithm functions. On the basis of this operator we introduced a new subclass of harmonic functions. For better understanding of this class we investigated its various geometric and topological properties. With the numerous applications of harmonic functions in pure and applied sciences, the theory developed here will serve as a potential ingredient for research. The means and methods used here can also be utilized for various new directions in the area of geometric function theory.

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