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Fractional Integral Inequalities For Preinvex Functions*

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Abstract

This work is concerned with the Hermite-Hadamard type integral inequalities for various classes of preinvex functions via Katugampola fractional integral. For preinvex functions we obtain new approximations for the right side of the Hermite-Hadamard integral inequality. We show that the obtained results have relationship with Hermite-Hadamard type inequalities obtained via Riemann-Liouville integral.

1 Introduction

The convex functions and the relevant theories have been playing important roles in many branches of science including engineering, probability theory, mathematical programming and optimization theory. In recent times, the notion of convexity has been generalized and extended in many disciplines of mathematical and engineering science. A marked generalization of the convex function is the invex functions [6]. Hanson's early results provided a broader spectrum to study the contribution of invexity in optimization and many branches of applied and pure sciences. Ben-Israel and Mond [2] established the concept of preinvex functions, which is the generalization of invex functions. It have been shown [14] that preinvex functions have same properties as convex functions.

It is an interesting fact that the convex functions are becoming of striking importance due to their nature. Along with generalizations of convex functions, several noteworthy inequalities have been established associated with them. Among these inequalities is the Hermite-Hadamard type inequality [5]. The Hermite-Hadamard integral inequality on the interval $[a, b]$ for a convex function φ is,

$$\varphi\Big(\frac{a+b}{2}\Big) \leq \frac{1}{(b-a)}\int_a^b \varphi(x)dx \leq \frac{\varphi(a)+\varphi(b)}{2}. \tag{1}$$

If function φ is concave, then the inequality (1) reverses in opposite direction. Recently many researchers have generalized and extended the Hermite-Hadamard integral inequality. The Hermite-Hadamard integral inequality for preinvex functions has been established by Noor [15].

Let φ be a preinvex function on the interval $[a, a + \zeta(b, a)]$, where $\zeta(b, a) > 0$. Noor [15] proved that a function φ is preinvex function, if and only if, the function φ satisfies the inequality

$$\varphi\Big(\frac{2a+\zeta(b,a)}{2}\Big) \leq \frac{1}{\zeta(b,a)}\int_a^{a+\zeta(b,a)} \varphi(x)dx \leq \frac{\varphi(a)+\varphi(b)}{2}, \tag{2}$$

which is called the Hermite-Hadamard inequality. It is note worthy that for $\zeta(b, a) = b - a$, the preinvex function is the same as the convex function and (2) becomes Hermite-Hadamard Integral (1). Hence the inequality (2) can be viewed as a novel generalization of the classical Hermite-Hadamard inequality.

In this work, we derive some new Hermite-Hadamard type integral inequalities for some preinvex functions through Katugampola fractional integrals using new techniques. These inequalities generalize the Hermite-Hadamard integral inequalities for convex functions.

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2 Preliminaries

Definition 1 *The function φ prescribed on the interval $[a,b]$ is called to be a convex function if for $u,\upsilon \in [a,b]$ and $\tau \in [0,1]$, it satisfies*

$$\varphi(\tau u + (1-\tau)\upsilon) \leq (1-\tau)\varphi(u) + \tau\varphi(\upsilon).$$

The function φ will become a concave function if $-\varphi$ is a convex function.

Definition 2 *[2, 13] A set K_ζ is said to be an invex set with respect to the arbitrary bifunction $\zeta(.,.)$, if, for every $u,\upsilon \in K_\zeta$ and $\tau \in [0,1]$, we have*

$$u + \tau\zeta(\upsilon,u) \in K_\zeta.$$

Definition 3 *A function $\varphi \in K_\zeta$ is called a preinvex function with respect to the arbitrary bifunction $\zeta(.,.)$, if, for $\tau \in [0,1]$ and for every $u,\upsilon \in K_\zeta$,*

$$\varphi(u + \tau\zeta(\upsilon,u)) \leq (1-\tau)\varphi(u) + \tau\varphi(\upsilon). \tag{3}$$

Definition 4 *[7] A function prescribed on an invex set K_ζ is termed an s-convex function if for any $u,\upsilon \in K_\zeta$ and $\tau, s \in [0,1]$*

$$\varphi(\tau u + (1-\tau)\upsilon) \leq \tau^s\varphi(u) + (1-\tau)^s\varphi(\upsilon).$$

The function φ on the set K_ζ is an s-preinvex function, if, for every $u,\upsilon \in K_\zeta$ and $s \in [0,1]$,

$$\varphi(u + \tau\zeta(\upsilon,u)) \leq \tau^s\varphi(\upsilon) + (1-\tau)^s\varphi(u).$$

Definition 5 *[4] A set K_ζ with respect to $\zeta(\upsilon,u)$ is called an m-invex set, if, for each $u, m\upsilon \in K_\zeta$, $u + \tau\zeta(m\upsilon,u) \in K_\zeta$ holds, where $m \in (0,1]$.*

This set reduces to an invex set on K_ζ for $m = 1$.

Definition 6 *For $m \in (0,1]$ and $u,\upsilon \in K_\zeta$, a function φ on the invex set K_ζ is said to be m-preinvex, if*

$$\varphi(u + \tau\zeta(m\upsilon,u)) \leq (1-\tau)\varphi(u) + m\tau\varphi(\upsilon). \tag{4}$$

The function φ is an m-preconcave, if $-\varphi$ is m-preinvex function.

Condition C. [13] Let K_ζ be an invex defined as $\zeta\colon K_\zeta \times K_\zeta \to R^n$. Then, for any $u,\upsilon \in K_\zeta$ and $\tau \in [0,1]$, the bifunction $\zeta(.,.)$ meets the condition, if

$$\zeta(u, u + \tau\zeta(\upsilon,u)) = -\tau\zeta(\upsilon,u),$$
$$\zeta(\upsilon, u + \tau\zeta(\upsilon,u)) = (1-\tau)\zeta(\upsilon,u).$$

Its also clear from above conditions that for every $x,y \in K_\zeta$ and $t_1,t_2 \in [0,1]$ we have

$$\zeta(x + t_1\zeta(y,x), x + t_2\zeta(y,x)) = (t_1 - t_2)\zeta(y,x). \tag{5}$$

Beta Function. The classical Beta function is described as

$$\mathbf{B}(u,\nu) = \frac{\Gamma(u)\Gamma(\nu)}{\Gamma(u+\nu)} = \int_0^1 (1-\chi)^{\nu-1}\chi^{u-1}d\chi,$$

whereas Γ is known as the Gamma function described as

$$\Gamma(\beta) = \int_0^\infty e^{-\chi}\chi^{\beta-1}d\chi.$$

The generalization of Beta function is given [12] as:

$${}^\rho\gamma(a,b) = \int_0^1 x^{\rho a-1}(1-x^\rho)^b dx. \tag{6}$$

Clearly, for $\rho \to 1$ the generalized Beta function turns into the classical Beta function.

Definition 7 *Let $\alpha > 0$ such that $n-1 < \alpha < n$, where n is a natural number and $x \in (a,b)$. The Riemann-Liouville integrals of the order α for the function φ are defined as [18]*

$$J^{\alpha}_{a^+}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\varphi(\tau)d\tau,$$

$$J^{\alpha}_{b^-}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\tau-x)^{\alpha-1}\varphi(\tau)d\tau.$$

The above integrals are called left and right Riemann-Liouville Integrals respectively with the property $J^0_{a^+}\varphi(x) = J^0_{b^-}\varphi(x) = \varphi(x)$. For $\alpha = 1$, these integrals reduces to classical integral.

Definition 8 *Let $\alpha > 0$ such that $n-1 < \alpha < n$ where n is a natural number and $x \in (a,b)$. The Katugampola integrals of order α for the function φ are given by [9]*

$${}^{\rho}I^{\alpha}_{a^+}\varphi(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{\rho}-\tau^{\rho})^{\alpha-1}\tau^{\rho-1}\varphi(\tau)d\tau,$$

$${}^{\rho}I^{\alpha}_{b^-}\varphi(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (\tau^{\rho}-x^{\rho})^{\alpha-1}\tau^{\rho-1}\varphi(\tau)d\tau.$$

For $\rho = 1$, these integrals reduces to the Riemann-Liouville fractional integrals. Hence the Katugampola fractional integral operator generalizes the Riemann-Liouville fractional integral.

3 Integral Inequalities for s-Preinvex Functions

In this section, we derive the Hermite-Hadamard type inequalities for s-preinvex functions.

Theorem 1 *For $\alpha > 0, \rho > 0$, consider a function $\varphi \colon I_{\zeta} = [a^{\rho}, a^{\rho} + \zeta(a^{\rho}, b^{\rho})] \to R$ with $\zeta(b^{\rho}, a^{\rho}) > 0$. If φ is an s-preinvex function and satisfies the condition C, then*

$$\begin{aligned} 2^{s-1}\varphi(\frac{2a^{\rho}+\zeta(b^{\rho},a^{\rho})}{2}) &\leq \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2\zeta^{\alpha}(b^{\rho},a^{\rho})}[{}^{\rho}I^{\alpha}_{a^+}\varphi(a^{\rho}+\zeta(b^{\rho},a^{\rho}) + {}^{\rho}I^{\alpha}_{(a^{\rho}+\zeta(b^{\rho},a^{\rho}))^-}\varphi(a^{\rho}))] \\ &\leq \Big(\frac{\alpha}{\alpha+s} + \alpha B(\alpha, s+1)\Big)\Big(\frac{\varphi(a^{\rho})+\varphi(b^{\rho})}{2}\Big). \end{aligned} \tag{7}$$

Proof. By the s-preinvexity of φ on $[a^{\rho}, a^{\rho} + \zeta(b^{\rho}, a^{\rho})]$, where $t \in [0,1]$ with $t = \frac{1}{2}$,

$$\varphi\Big(\frac{2x^{\rho}+\zeta(y^{\rho},x^{\rho})}{2}\Big) \leq \frac{\varphi(x^{\rho})+\varphi(y^{\rho})}{2^s}. \tag{8}$$

Taking $x^{\rho} = a^{\rho} + (1-t^{\rho})\zeta(b^{\rho}, a^{\rho})$, $y^{\rho} = a^{\rho} + t^{\rho}\zeta(b^{\rho}, a^{\rho})$, in (8), and using condition C, we have

$$\begin{aligned} &\varphi\Big(\frac{2a^{\rho}+2(1-t^{\rho})\zeta(b^{\rho},a^{\rho})+\zeta(a^{\rho}+t^{\rho}\zeta(b^{\rho},a^{\rho}), a^{\rho}+t^{\rho}\zeta(b^{\rho},a^{\rho})+(1-2t^{\rho})\zeta(b^{\rho},a^{\rho})}{2}\Big) \\ &= 2^s\varphi(\frac{2a^{\rho}+\zeta(b^{\rho},a^{\rho})}{2}) \leq \varphi(a^{\rho}+(1-t^{\rho})\zeta(b^{\rho},a^{\rho})) + \varphi(a^{\rho}+t^{\rho}\zeta(b^{\rho},a^{\rho})). \end{aligned} \tag{9}$$

Multiplying (9) by $t^{\alpha\rho-1}$ and integrating both sides over t, we obtain

$$\begin{aligned} &\int_0^1 t^{\alpha\rho-1}2^s\varphi(\frac{2a^{\rho}+\zeta(b^{\rho},a^{\rho})}{2})dt = \frac{2^s}{\alpha\rho}\varphi(\frac{2a^{\rho}+\zeta(b^{\rho},a^{\rho})}{2}) \\ &\leq \int_0^1 t^{\alpha\rho-1}\varphi(a^{\rho}+(1-t^{\rho})\zeta(b^{\rho},a^{\rho}))dt + \int_0^1 t^{\alpha\rho-1}\varphi(a^{\rho}+t^{\rho}\zeta(b^{\rho},a^{\rho}))dt \\ &= I_1 + I_2. \end{aligned} \tag{10}$$

Now by using change of variable technique $a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho) = z^\rho$,

$$\begin{aligned} I_1 &= \int_0^1 t^{\alpha\rho-1}\varphi(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))dt \\ &= \int_{(a^\rho+\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}}}^{a} \Big(\frac{a^\rho + \zeta(b^\rho, a^\rho) - z^\rho}{\zeta(b^\rho, a^\rho)}\Big)^{\alpha-1}\varphi(z^\rho)\frac{-z^{\rho-1}}{\zeta(b^\rho, a^\rho)}dz \\ &= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\zeta^\alpha(b^\rho, a^\rho)}{}^\rho I^\alpha_{a+}\varphi(a^\rho + \zeta(b^\rho, a^\rho)),. \end{aligned} \tag{11}$$

Analogously

$$I_2 = \int_0^1 t^{\alpha\rho-1}\varphi(a^\rho + t^\rho\zeta(b^\rho, a^\rho))dt = \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\zeta^\alpha(b^\rho, a^\rho)}{}^\rho I^\alpha_{(a^\rho+\zeta(b^\rho,a^\rho))-}\varphi(a^\rho).$$

From (10), we have

$$\begin{aligned} 2^{s-1}\varphi\Big(\frac{2a^\rho + \zeta(b^\rho, a^\rho)}{2}\Big) &\leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho, a^\rho)}[{}^\rho I^\alpha_{a+}\varphi(a^\rho + \zeta(b^\rho, a^\rho)) + {}^\rho I^\alpha_{(a^\rho+\zeta(b^\rho,a^\rho))-}\varphi(a^\rho)]. \\ &\leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho, a^\rho)}[{}^\rho I^\alpha_{a+}\varphi(b^\rho) + {}^\rho I^\alpha_{(a^\rho+\zeta(b^\rho,a^\rho))-}\varphi(a^\rho)]. \end{aligned} \tag{12}$$

In order to establish the second inequality in (7), using the s-preinvexity of function φ on $[a^\rho, a^\rho + \zeta(b^\rho, a^\rho)]$,

$$\varphi(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho)) \leq (1-t^\rho)^s\varphi(a^\rho + \zeta(b^\rho, a^\rho)) + (t^\rho)^s\varphi(a^\rho) \tag{13}$$

$$\varphi(a^\rho + t^\rho\zeta(b^\rho, a^\rho)) \leq (t^\rho)^s\varphi(a^\rho + \zeta(b^\rho, a^\rho)) + (1-t^\rho)^s\varphi(a^\rho), \tag{14}$$

so by adding (13) and (14)

$$\begin{aligned} \varphi(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho)) + \varphi(a^\rho + t^\rho\zeta(b^\rho, a^\rho)) &\leq [(1-t^\rho)^s + (t^\rho)^s]\varphi(a^\rho + \zeta(b^\rho, a^\rho)) \\ &\leq [(1-t^\rho)^s + (t^\rho)^s](\varphi(a^\rho) + \varphi(b^\rho)). \end{aligned} \tag{15}$$

Now multiplying both sides of (15) by $t^{\alpha\rho-1}$ and integrating on t and by using (11) and (**??**)

$$\begin{aligned} &\int_0^1 t^{\alpha\rho-1}[\varphi(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho)) + \varphi(a^\rho + t^\rho\zeta(b^\rho, a^\rho))]dt \\ &= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\zeta^\alpha(b^\rho, a^\rho)}[{}^\rho I^\alpha_{a+}\varphi(a^\rho + \zeta(b^\rho, a^\rho)) + {}^\rho I^\alpha_{(a^\rho+\zeta(b^\rho,a^\rho))+}\varphi(a^\rho)] \\ &\leq \int_0^1 t^{\alpha\rho-1}[(1-t^\rho)^s + (t^\rho)^s](\varphi(a^\rho) + \varphi(b^\rho))dt. \end{aligned} \tag{16}$$

Since $\int_0^1 t^{\alpha\rho-1+\rho s}dt = \frac{1}{\rho(\alpha+s)}$, and $\int_0^1 t^{\alpha\rho-1}(1-t^\rho)^s dt = \dfrac{B(\alpha, s+1)}{\rho}$, therefore (16) becomes

$$\begin{aligned} &\frac{\rho^\alpha\Gamma(\alpha)}{2\zeta^\alpha(b^\rho, a^\rho)}[{}^\rho I^\alpha_{a+}\varphi(a^\rho + \zeta(b^\rho, a^\rho)) + {}^\rho I^\alpha_{(a^\rho+\zeta(b^\rho,a^\rho))+}\varphi(a^\rho)] \\ \leq\ &\Big(\frac{1}{\alpha+s} + B(\alpha, s+1)\Big)\Big(\frac{\varphi(a^\rho) + \varphi(b^\rho)}{2}\Big), \end{aligned} \tag{17}$$

this gives us second inequality. Hence by combining (12) and (17) we obtain

$$\begin{aligned} 2^{s-1}\varphi(\frac{2a^\rho + \zeta(b^\rho, a^\rho)}{2}) &\leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho, a^\rho)}[{}^\rho I^\alpha_{a+}\varphi(a^\rho + \zeta(b^\rho, a^\rho) +{}^\rho I^\alpha_{(a^\rho+\zeta(b^\rho,a^\rho))-}\varphi(a^\rho))] \\ &\leq \Big(\frac{\alpha}{\alpha+s} + \alpha B(\alpha, s+1)\Big)\Big(\frac{\varphi(a^\rho) + \varphi(b^\rho)}{2}\Big), \end{aligned}$$

which is the required result. ■

Some special cases:

i. If $\zeta(b^\rho, a^\rho) = b^\rho - a^\rho$, then, we get inequality (5) of [12].

ii. If $\zeta(b^\rho, a^\rho) = b^\rho - a^\rho$ and $\rho = 1$, then the theorem 7 reduces to a result of [20].

iii. If $\rho = 1, s = 1$, then, we obtain

$$\varphi(\frac{2a+\zeta(b,a)}{2}) \leq \frac{\Gamma(\alpha+1)}{2\zeta^\alpha(b,a)}[J^\alpha_{a^+}\varphi(a+\zeta(b,a)) + J^\alpha_{(a+\zeta(b,a))^-}\varphi(a)] \leq \frac{\varphi(a)+\varphi(b)}{2},$$

which is the same as in [8].

iv. For $\alpha = 1$ in inequality (18), we obtain the generalized Hermite-Hadamard inequality.

Moreover, for $\zeta(b,a) = b - a$, the result is the same as the classical Hermite-Hadamard inequality.

Now we prove an auxiliary lemma, which is needed for proving our next results.

Lemma 2 *For $\alpha, \rho > 0$, consider an open invex subset $K_\zeta \subseteq R$ and we define $\zeta\colon K_\zeta \times K_\zeta \to R$ and $a^\rho, b^\rho \in K_\zeta$ with $\zeta(b^\rho, a^\rho) > 0$. Let $\varphi\colon K_\zeta \to R$ is differentiable and $\varphi^{'} \in L_1[a^\rho, a^\rho + \zeta(b^\rho, a^\rho)]$. If φ' is a preinvex function, then*

$$\begin{aligned}&\frac{\varphi(a^\rho) + \varphi(a^\rho + \zeta(b^\rho, a^\rho))}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho, a^\rho)}[{}_\rho I^\alpha_{a^+}\varphi(a^\rho + \zeta^\alpha(b^\rho, a^\rho)) + {}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)] \\ &= \frac{\rho\zeta(b^\rho, a^\rho)}{2}\int_0^1 [(1-t^\rho)^\alpha - (t^\rho)^\alpha]t^{\rho-1}\varphi'(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))dt. \end{aligned} \tag{18}$$

Proof. Consider

$$\begin{aligned} I &= \int_0^1 [(1-t^\rho)^\alpha - (t^\rho)^\alpha]t^{\rho-1}\varphi\prime(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))dt \\ &= \int_0^1 (1-t^\rho)^\alpha t^{\rho-1}\varphi\prime(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))dt - \int_0^1 (t^\rho)^\alpha t^{\rho-1}\varphi\prime(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))dt \\ &= I_1 - I_2. \end{aligned} \tag{19}$$

Now

$$\begin{aligned} I_1 =& \int_0^1 (1-t^\rho)^\alpha t^{\rho-1}\varphi\prime(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))dt \\ =&(1-t^\rho)^\alpha\left(\frac{-1}{\rho\zeta(b^\rho, a^\rho)}\right)\varphi(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))\Bigg|_0^1 \\ &+ \int_0^1 \left(\frac{-1}{\rho\zeta(b^\rho, a^\rho)}\right)\varphi(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))\alpha\rho t^{\rho-1}(1-t^\rho)^{\alpha-1}dt \\ =&\frac{\alpha}{\zeta(b^\rho, a^\rho)}\int_{((a^\rho+\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}})}^a \left(1 - \frac{a^\rho + \zeta(b^\rho, a^\rho) - u^\rho}{\zeta(b^\rho, a^\rho)}\right)^{\alpha-1}\varphi(u^\rho)\frac{u^{\rho-1}}{\zeta(b^\rho, a^\rho)}du + \frac{\varphi(a^\rho + \zeta(b^\rho, a^\rho))}{\rho\zeta(b^\rho, a^\rho)} \\ =&\frac{\varphi(a^\rho + \zeta(b^\rho, a^\rho))}{\rho\zeta(b^\rho, a^\rho)} - \frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{\zeta^{\alpha+1}(b^\rho, a^\rho)}{}^\rho I^\alpha_{((a^\rho+\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}})^-}\varphi(a^\rho), \end{aligned} \tag{20}$$

where we have used the change of variable technique as $a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho) = u^\rho$. Similarly

$$I_2 = -\frac{\varphi(a^\rho)}{\rho\zeta(b^\rho, a^\rho)} + \frac{\Gamma(\alpha+1)\rho^{\alpha-1}}{\zeta^{\alpha+1}(b^\rho, a^\rho)}{}^\rho I^\alpha_{(a^\rho)^+}\varphi(a^\rho + \zeta(b^\rho, a^\rho)). \tag{21}$$

By using (20) and (21) in (19), we get

$$
\begin{aligned}
I =&\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{\rho\zeta(b^\rho,a^\rho)} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\zeta^{\alpha+1}(b^\rho,a^\rho)}[{}^\rho I^\alpha_{((a^\rho+\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}})^-}\varphi(a^\rho) +^\rho I^\alpha_{(a^\rho)^+}\varphi(a^\rho+\zeta(b^\rho,a^\rho))]\\
=&\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{\rho\zeta(b^\rho,a^\rho)} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{2\zeta^{\alpha}(b^\rho,a^\rho)}[{}_\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho)) + {}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)]\\
\leq&\frac{\varphi(a^\rho)+\varphi(b^\rho)}{\rho\zeta(b^\rho,a^\rho)} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{2\zeta^{\alpha}(b^\rho,a^\rho)}[{}_\rho I^\alpha_{a^+}\varphi(b^\rho) + {}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)]. \qquad (22)
\end{aligned}
$$

Multiplying (22) by $\zeta(b^\rho,a^\rho)/2$, we obtain

$$
\begin{aligned}
&\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2\zeta^{\alpha}(b^\rho,a^\rho)}[{}_\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho)) + {}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)]\\
=&\frac{\rho\zeta(b^\rho,a^\rho)}{2}\int_0^1[(1-t^\rho)^\alpha-(t^\rho)^\alpha]t^{\rho-1}\varphi'(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))dt, \qquad (23)
\end{aligned}
$$

which is the required result. ■

Theorem 3 *Consider an open invex subset $K_\zeta \subseteq R$ and define $\zeta\colon K_\zeta\times K_\zeta \to R$ and $a^\rho, b^\rho \in K_\zeta$ with $\zeta(b^\rho,a^\rho)>0$. Let $\varphi\colon K_\zeta \to R$ be differentiable and $\varphi^{'} \in L_1[a^\rho+\zeta(b^\rho,a^\rho)]$. If $|\varphi'|$ is an s-preinvex function, then*

$$
\begin{aligned}
&\left|\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2\zeta^{\alpha}(b^\rho,a^\rho)}\left[{}_\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho)) + {}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)\right]\right|\\
&\qquad\leq\left(\frac{1}{\alpha+s+1}+B(\alpha+1,s+1)\right)\frac{\zeta(b^\rho,a^\rho)}{2\rho}[|\varphi'(a^\rho)|+|\varphi'(b^\rho)|]. \qquad (24)
\end{aligned}
$$

Proof. Consider

$$
\frac{\rho^{\alpha-1}\Gamma(\alpha)}{\zeta^{\alpha}(b^\rho,a^\rho)}[{}_\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho)) + {}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)],
$$

which we have already deduced using I_1 and I_2 as right hand side of (12).

$$
=\int_0^1 t^{\alpha\rho-1}\varphi(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))dt+\int_0^1 t^{\alpha\rho-1}\varphi(a^\rho+t^\rho\zeta(b^\rho,a^\rho))dt,
$$

$$
\begin{aligned}
&\frac{\rho^{\alpha-1}\Gamma(\alpha)}{\zeta^{\alpha}(b^\rho,a^\rho)}[{}_\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho)) + {}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)]\\
=&\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{\alpha\rho} - \frac{\zeta(b^\rho,a^\rho)}{\alpha}\int_0^1 t^{\alpha\rho+\rho-1}[\varphi'(a^\rho+t^\rho\zeta(b^\rho,a^\rho))\\
&-\varphi'(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))]dt.
\end{aligned}
$$

By suitable arrangement and applying triangular inequality, we have

$$
\begin{aligned}
&\left|\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2\zeta^{\alpha}(b^\rho,a^\rho)}[{}_\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho)) + {}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)]\right|\\
\leq&\frac{\zeta(b^\rho,a^\rho)}{2}\int_0^1\left|t^{\alpha\rho+\rho-1}[\varphi'(a^\rho+t^\rho\zeta(b^\rho,a^\rho))-\varphi'(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))]\right|dt\\
\leq&\frac{\zeta(b^\rho,a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}[|\varphi'(a^\rho+t^\rho\zeta(b^\rho,a^\rho))|+|\varphi'(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))|]dt.
\end{aligned}
$$

By s-preinvexity of $|\varphi'|$, we have

$$\leq \frac{\zeta(b^\rho,a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}[(1-t^\rho)^s|\varphi'(a^\rho)|+(t^\rho)^s|\varphi'(a^\rho)|+(t^\rho)^s|\varphi'(a^\rho)|+(1-t^\rho)^s|\varphi'(b^\rho)|]dt$$
$$\leq \frac{\zeta(b^\rho,a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}[((1-t^\rho)^s+(t^\rho)^s)(|\varphi'(a^\rho)|+|\varphi'(b^\rho)|)]dt$$
$$= \frac{\zeta(b^\rho,a^\rho)}{2}(|\varphi'(a^\rho)|+|\varphi'(b^\rho)|)\int_0^1 t^{\alpha\rho+\rho-1}((1-t^\rho)^s+(t^\rho)^s)dt$$
$$= \frac{\zeta(b^\rho,a^\rho)}{2\rho}(|\varphi'(a^\rho)|+|\varphi'(b^\rho)|)\Big(\frac{1}{\alpha+s+1}+B(\alpha+1,s+1)\Big),$$

which is the required result. ■

Remark If $\rho=s=1$ in (24), then we have

$$\left|\frac{\varphi(a)+\varphi(b)}{2}-\frac{\Gamma(\alpha+1)}{2\zeta^\alpha(b,a)}[J^\alpha_{a^+}\varphi(a+\zeta^\alpha(b,a))+J^\alpha_{(a+\zeta^\alpha(b,a))^-}\varphi(a)]\right|$$
$$\leq\Big(\frac{1}{\alpha+2}+B(\alpha+1,2)\Big)\frac{\zeta(b,a)}{2}[|\varphi'(a)|+|\varphi'(b)|]=\frac{\zeta(b,a)}{2(\alpha+1)}[|\varphi'(a)|+|\varphi'(b)|].$$

Theorem 4 *Consider an open invex set K_ζ. Define $\zeta\colon K_\zeta\times K_\zeta\to R$, and $a^\rho,b^\rho\in K_\zeta$ with $\zeta(b^\rho,a^\rho)>0$. Let φ be differentiable, where $\varphi'\in L_1(a^\rho,a^\rho+\zeta(b^\rho,a^\rho))$. If for some $q>1$, $|\varphi'|^q$ is an s-preinvex function, then*

$$\Big|\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho,a^\rho)}\big[{}_\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho))+{}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)]\Big|$$
$$\leq\frac{\rho\zeta(b^\rho,a^\rho)}{2}[{}^\rho\gamma(\frac{1}{2};1,\alpha+1)-{}^\rho\gamma(\frac{1}{2};\alpha+1,1)+\frac{1}{\alpha\rho+\rho}(1-\frac{2}{2^{\alpha\rho+\rho}})]^{1-\frac{1}{q}}$$
$$\times\Bigg[\Big({}^\rho\gamma(\frac{1}{2};s+1,\alpha+1)-{}^\rho\gamma(\frac{1}{2};\alpha+1,s+1)+\frac{1}{\rho(\alpha+s+1)(1-\frac{2}{2^{\rho(\alpha+s+1)}})}\Big)|\varphi'(a^\rho)|^q$$
$$+\Big({}^\rho\gamma(\frac{1}{2};1,\alpha+s+1)-{}^\rho\gamma(\frac{1}{2};\alpha+s+1,1)+{}^\rho\gamma(\frac{1}{2};s+1,\alpha+1)-{}^\rho\gamma(\frac{1}{2};\alpha+1,s+1)\Big)|\varphi'(b^\rho)|^q\Bigg]^{\frac{1}{q}}.$$

Proof. Using Holder's inequality on Lemma (2), we have

$$\left|\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho,a^\rho)}[{}^\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho))+{}^\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))^-}\varphi(a^\rho)]\right|$$
$$=\left|\frac{\rho\zeta(b^\rho,a^\rho)}{2}\int_0^1[(1-t^\rho)^\alpha-(t^\rho)^\alpha]t^{\rho-1}\varphi'(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))dt\right|$$
$$\leq\frac{\rho\zeta(b^\rho,a^\rho)}{2}\int_0^1|[(1-t^\rho)^\alpha-(t^\rho)^\alpha]t^{\rho-1}||\varphi'(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))|dt$$
$$\leq\frac{\rho\zeta(b^\rho,a^\rho)}{2}\left(\int_0^1 t^{\rho-1}|[(1-t^\rho)^\alpha-(t^\rho)^\alpha]|dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 t^{\rho-1}|[(1-t^\rho)^\alpha-(t^\rho)^\alpha]||\varphi\prime(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))|^q dt\right)^{\frac{1}{q}}. \tag{25}$$

Consider

$$\begin{aligned}
&\int_0^1 |(1-t^\rho)^\alpha - (t^\rho)^\alpha| t^{\rho-1} dt \\
&= \int_0^{\frac{1}{2}} [(1-t^\rho)^\alpha - (t^\rho)^\alpha] t^{\rho-1} dt + \int_{\frac{1}{2}}^1 [(t^\rho)^\alpha - (1-t^\rho)^\alpha] t^{\rho-1} dt \\
&= {}^\rho\gamma(\frac{1}{2}; 1, \alpha+1) - {}^\rho\gamma(\frac{1}{2}; \alpha+1, 1) + \frac{1}{\alpha\rho+\rho}\Big(1 - \frac{2}{2^{\alpha\rho+\rho}}\Big),
\end{aligned} \tag{26}$$

where

$$\int_0^{\frac{1}{2}} t^s (1-t^\rho)^\alpha t^{\rho-1} dt = \int_{\frac{1}{2}}^1 (1-t^\rho)^s t^{\rho-1} (t^\rho)^\alpha dt = {}^\rho\gamma(\frac{1}{2}; s+1, \alpha+1)$$

and

$$\int_0^{\frac{1}{2}} (1-t^\rho)^s t^\alpha t^\rho - 1 dt = \int_{\frac{1}{2}}^1 (1-t^\rho)^s (t^\rho)^\alpha t^{\rho-1} dt = {}^\rho\gamma(\frac{1}{2}; \alpha+1, s+1).$$

Now applying the s-preinvexity of $|\varphi'|^q$, it follows that

$$\begin{aligned}
&\int_0^1 |[(1-t^\rho)^\alpha - (t^\rho)^\alpha]| t^{\rho-1} |\varphi'(a^\rho + (1-t^\rho)\zeta(b^\rho, a^\rho))|^q dt \\
&\leq \int_0^1 |[(1-t^\rho)^\alpha - (t^\rho)^\alpha]| t^{\rho-1} [(t^\rho)^s |\varphi'(a^\rho)|^q + (1-t^\rho)^s |\varphi'(b^\rho)|^q] dt \\
&\leq \int_0^{\frac{1}{2}} [(1-t^\rho)^\alpha - (t^\rho)^\alpha] t^{\rho-1} [(t^\rho)^s |\varphi'(a^\rho)|^q + (1-t^\rho)^s |\varphi'(b^\rho)|^q] dt \\
&+ \int_{\frac{1}{2}}^1 [(t^\rho)^\alpha - (1-t^\rho)^\alpha] t^{\rho-1} [(t^\rho)^s |\varphi'(a^\rho)|^q + (1-t^\rho)^s |\varphi'(b^\rho)|^q] dt \\
&= \Big({}^\rho\gamma(\frac{1}{2}; s+1, \alpha+1) - {}^\rho\gamma(\frac{1}{2}; \alpha+1, s+1) + \frac{1}{\rho(\alpha+s+1)} + \big(1 - \frac{2}{2^{\rho(\alpha+s+1)}}\big)\Big) |\varphi'(a^\rho)|^q
\end{aligned}$$

$$+\Big({}^\rho\gamma(\frac{1}{2}; 1, \alpha+s+1) - {}^\rho\gamma(\frac{1}{2}; \alpha+s+1, 1) + {}^\rho\gamma(\frac{1}{2}; s+1, \alpha+1) - {}^\rho\gamma(\frac{1}{2}; \alpha+1, s+1)\Big) |\varphi'(b^\rho)|^q.$$

By using (26) and (**??**) in (25), we obtain the required result. ■

Special Cases

i. For $\rho = 1, s = 1$, we obtain theorem (10) of [8].

ii. For $\zeta(b^\rho, a^\rho) = b - a, \rho = 1, \alpha = 1$, we obtain theorem(1) of [10].

iii. For $\zeta(b^\rho, a^\rho) = b - a, s = 1, \alpha = 1$, we obtain theorem(1) of [17].

iv. For $\zeta(b^\rho, a^\rho) = b - a, \rho = 1, \alpha = 1$, we obtain the result same as theorem (4) of [20].

Theorem 5 *Consider an open invex set K_ζ. Define $\zeta\colon K_\zeta \times K_\zeta \to R$, and $a^\rho, b^\rho \in K_\zeta$ with $\zeta(b^\rho, a^\rho) > 0$. Let φ is differentiable such that $\varphi' \in L_1(a^\rho, a^\rho + \zeta(b^\rho, a^\rho))$. If, for some $q > 1$, $|\varphi'|^q$ is an s-preinvex, where $\frac{1}{p} = \frac{q-1}{q}$, then*

$$\begin{aligned}
&\left| \frac{\varphi(a^\rho) + \varphi(a^\rho + \zeta(b^\rho, a^\rho))}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho, a^\rho)} [{}_\rho I^\alpha_{a^+} \varphi(a^\rho + \zeta^\alpha(b^\rho, a^\rho)) + {}_\rho I^\alpha_{(a^\rho + \zeta^\alpha(b^\rho, a^\rho))^-} \varphi(a^\rho)] \right| \\
&\leq \frac{\rho\zeta(b^\rho, a^\rho)}{2} \Big(\frac{1}{2(\alpha\rho+1)}\Big)^{\frac{1}{p}} \Big[\frac{1}{\rho s+1} |\varphi'(a^\rho)|^q + \frac{B(s+1, \frac{1}{\rho})}{\rho} |\varphi'(b^\rho)|^q\Big]^{\frac{1}{q}}.
\end{aligned}$$

Proof. Using lemma (2) and applying the Holder's inequality, we obtain

$$\Big|\frac{\varphi(a^\rho)+\varphi(a^\rho+\zeta(b^\rho,a^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho,a^\rho)}\big[{}_\rho I^\alpha_{a+}\varphi(a^\rho+\zeta^\alpha(b^\rho,a^\rho))+{}_\rho I^\alpha_{(a^\rho+\zeta^\alpha(b^\rho,a^\rho))-}\varphi(a^\rho)\big]\Big|$$
$$=\Big|\frac{\rho\zeta(b^\rho,a^\rho)}{2}\int_0^1[(1-t^\rho)^\alpha-(t^\rho)^\alpha]t^{\rho-1}\varphi'(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))dt\Big|$$
$$\le\frac{\rho\zeta(b^\rho,a^\rho)}{2}\Big(\int_0^1|[(1-t^\rho)^\alpha-(t^\rho)^\alpha]t^{\rho-1}|^p dt\Big)^{\frac{1}{p}}\times\Big(\int_0^1|\varphi'(a^\rho+(1-t^\rho)\zeta(b^\rho,a^\rho))|^q dt\Big)^{\frac{1}{q}}$$
$$\le\frac{\rho\zeta(b^\rho,a^\rho)}{2}\Big(\int_0^1[(2t^\rho-1)^\alpha t^{\rho-1}]^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1[(t^\rho)^s|\varphi'(a^\rho)|^q+(1-t^\rho)^s|\varphi'(b^\rho)|^q]dt\Big)^{\frac{1}{q}}$$
$$\le\frac{\rho\zeta(b^\rho,a^\rho)}{2}\Big(\frac{1}{2(\alpha\rho+1)}\Big)^{\frac{1}{p}}\Big[\frac{1}{\rho s+1}|\varphi'(a^\rho)|^q+\frac{B(s+1,\frac{1}{\rho})}{\rho}|\varphi'(b^\rho)|^q\Big]^{\frac{1}{q}},\tag{27}$$

which completes the proof. ■

4 Integral Inequalities for m-Preinvex Functions

In following section, we will discuss Hermite-Hadamard type integral inequalities for m-preinvex functions.

Theorem 6 *Consider a positive valued function $\varphi:[a^\rho,a^\rho+\zeta(b^\rho,a^\rho)]\to R$ with $\rho>0$ and $\zeta(b^\rho,a^\rho)>0$. If for $m\in(0,1]$ and $\alpha>0$, φ is an m-preinvex function, then*

$$\varphi\Big(\frac{m^\rho(2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\le\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho I^\alpha_{ma+}\varphi(m^\rho a^\rho+a^\rho\zeta(b^\rho,a^\rho))}{(m^\rho\zeta(b^\rho,a^\rho))^\alpha}+\frac{{}^\rho I^\alpha_{(\zeta(b^\rho,a^\rho))-}\varphi(a^\rho)}{\zeta^\alpha(b^\rho,a^\rho)}\Big]$$
$$\le\frac{\varphi(m^\rho a^\rho)+(m^\rho)^2\varphi(\frac{b^\rho}{m^\rho})}{\rho(\alpha+1)}+\frac{m^\rho(\varphi(a^\rho)+\varphi(b^\rho))}{\rho\alpha(\alpha+1)}.\tag{28}$$

Proof. Since φ is an m-preinvex function, so using (4), we have

$$\varphi(x^\rho+t^\rho\zeta(m^\rho y^\rho,x^\rho))\le(1-t^\rho)\varphi(x^\rho)+m^\rho t^\rho\varphi(y^\rho).$$

Taking $t^\rho=1/2$ in the above inequality, we obtain

$$\varphi\big(x^\rho+\frac{1}{2}\zeta(m^\rho y^\rho,x^\rho)\big)\le\frac{\varphi(x^\rho)+m^\rho\varphi(y^\rho)}{2}.\tag{29}$$

Taking $x^\rho=m^\rho a^\rho+m^\rho(1-t^\rho)\zeta(b^\rho,a^\rho)$, $y^\rho=a^\rho+t^\rho\zeta(b^\rho,a^\rho)$, and using the condition C, we obtain

$$\varphi\Big(\frac{m^\rho(2a^\rho+\zeta(b^\rho,a^\rho))}{2}\Big)\le\frac{1}{2}\Big(\varphi(m^\rho a^\rho+m^\rho(1-t^\rho)\zeta(b^\rho,a^\rho))+m^\rho\varphi(a^\rho+t^\rho\zeta(b^\rho,a^\rho))\Big).$$

Multiplying by $t^{\alpha\rho-1}$ and integrating with respect to t, we have

$$2\int_0^1 t^{\alpha\rho-1}\varphi\Big(\frac{m^\rho(2a^\rho+\zeta(b^\rho,a^\rho))}{2}\Big)dt=\frac{2}{\alpha\rho}\varphi\Big(\frac{m^\rho(2a^\rho+\zeta(b^\rho,a^\rho))}{2}\Big)$$
$$\le\int_0^1 t^{\alpha\rho-1}\varphi(m^\rho a^\rho+m^\rho(1-t^\rho)\zeta(b^\rho,a^\rho))d+m^\rho\int_0^1 t^{\alpha\rho-1}\varphi(a^\rho+t^\rho\zeta(b^\rho,a^\rho))dt.$$

Setting $m^\rho a^\rho + m^\rho(1-t^\rho)\zeta(b^\rho,a^\rho) = u^\rho$ and $a^\rho + t^\rho\zeta(b^\rho,a^\rho) = z^\rho$, we obtain

$$\leq \int_{m^\rho(a^\rho+m^\rho\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}}}^{ma} \Big[\frac{u^\rho - (m^\rho(a^\rho + m^\rho\zeta(b^\rho,a^\rho)))}{-m^\rho\zeta(b^\rho,a^\rho)}\Big]^{\alpha-1} \varphi(u^\rho)\frac{u^{\rho-1}}{-m^\rho\zeta(b^\rho,a^\rho)}du$$
$$+ \int_a^{(a^\rho+m^\rho\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}}} \Big[\frac{z^\rho - a^\rho}{\zeta(b^\rho,a^\rho)}\Big]^{\alpha-1}\varphi(z^\rho)\frac{z^{\rho-1}}{\zeta(b^\rho,a^\rho)}dz$$
$$\leq \frac{1}{(m^\rho\zeta(b^\rho,a^\rho))^\alpha}\rho^{\alpha-1}\Gamma(\alpha)^\rho I^\alpha_{ma^+}\varphi(m^\rho a^\rho + a^\rho\zeta(b^\rho,a^\rho)) + \frac{m^\rho\rho^{\alpha-1}}{\zeta^\alpha(b^\rho,a^\rho)}{}^\rho I^\alpha_{(\zeta(b^\rho,a^\rho))^-}\varphi(a^\rho).$$

Consequently, we obtain

$$\varphi\Big(\frac{m^\rho(2a^\rho+\zeta(b^\rho,a^\rho))}{2}\Big) \leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2(m^\rho\zeta(b^\rho,a^\rho))^\alpha}{}^\rho I^\alpha_{ma^+}\varphi(m^\rho a^\rho + a^\rho\zeta(b^\rho,a^\rho)) + \frac{m^\rho\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho,a^\rho)}{}^\rho I^\alpha_{(\zeta(b^\rho,a^\rho))^-}\varphi(a^\rho),$$

which is the ist part.
Using the m-preinvexity of φ, we have

$$\varphi(m^\rho a^\rho + (1-t^\rho)m^\rho\zeta(b^\rho,a^\rho)) + m^\rho\varphi(a^\rho + t^\rho\zeta(b^\rho,a^\rho))$$
$$\leq t^\rho\varphi(m^\rho a^\rho) + (1-t^\rho)m^\rho\varphi(b^\rho) + m^\rho\Big((1-t^\rho)\varphi(a^\rho) + t^\rho(m^\rho)^2\varphi(\frac{b^\rho}{m^\rho})\Big).$$

Multiplying by $t^{\alpha\rho-1}$ and integrating with respect to t, we obtain

$$\frac{\rho^\alpha\Gamma(\alpha+1)}{2(m^\rho\zeta(b^\rho,a^\rho))^\alpha}{}^\rho I^\alpha_{ma^+}\varphi(m^\rho a^\rho + a^\rho\zeta(b^\rho,a^\rho)) + \frac{m^\rho\rho^\alpha\Gamma(\alpha+1)}{2\zeta^\alpha(b^\rho,a^\rho)}{}^\rho I^\alpha_{(\zeta(b^\rho,a^\rho))^-}\varphi(a^\rho)$$
$$\leq \int_0^1 t^{\alpha\rho-1}[t^\rho\varphi(m^\rho a^\rho) + m^\rho(1-t^\rho)(\varphi(a^\rho)+\varphi(b^\rho)) + t^\rho(m^\rho)^2\varphi(\frac{b^\rho}{m^\rho})]$$
$$= \frac{\varphi(m^\rho a^\rho) + (m^\rho)^2\varphi(\frac{b^\rho}{m^\rho})}{\rho(\alpha+1)} + \frac{m^\rho,(\varphi(a^\rho)+\varphi(b^\rho))}{\rho\alpha(\alpha+1)}, \quad (30)$$

which is the required inequality. ■

For suitable and proper choice of ρ, α, m and $\zeta(b,a)$, one can obtain various inequalities for the m-preinvex functions.

Theorem 7 *Let $m \in (0,1]$ and φ be a m-preinvex function on $L_1[a^\rho, a^\rho + \zeta(b^\rho,a^\rho)]$, such that $\zeta(b^\rho,a^\rho) > 0$. If*

$$F(x^\rho, x^\rho + \zeta(y^\rho,x^\rho))|_{t^\rho} = \frac{1}{2}\big[\varphi(x^\rho + (1-t^\rho)\zeta(m^\rho y^\rho,x^\rho)) + \varphi(x^\rho + t^\rho\zeta(m^\rho y^\rho,x^\rho))\big],$$

then

$$\frac{1}{\zeta^\alpha(b^\rho,a^\rho)}\int_a^{((a^\rho+\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}})} \zeta(b^\rho,a^\rho)^{\alpha-1}F\Big(u^\rho, \frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big|_{\frac{u^\rho-(a^\rho+\zeta(b^\rho,a^\rho))}{-\zeta(b^\rho,a^\rho)}}du$$
$$\leq \frac{\rho^{\alpha-1}\Gamma\alpha}{2\zeta^\alpha(b^\rho,a^\rho)}{}^\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta(b^\rho,a^\rho)) + \frac{m^\rho}{2\alpha\rho}\varphi\Big(\frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big).$$

Proof. Since φ is an m-preinvex function, we see that

$$F\Big(x^\rho, x^\rho+\zeta(y^\rho,x^\rho)\Big)\Big|_{t^\rho} = \frac{1}{2}\big[\varphi(x^\rho + (1-t^\rho)\zeta(m^\rho y^\rho,x^\rho)) + \varphi(x^\rho + t^\rho\zeta(m^\rho y^\rho,x^\rho))\big]$$
$$\leq \frac{1}{2}[t^\rho\varphi(x^\rho) + (1-t^\rho)m^\rho\varphi(y^\rho) + (1-t^\rho)m^\rho\varphi(x^\rho) + t^\rho m^\rho\varphi(y^\rho)]$$
$$= \frac{1}{2}[\varphi(x^\rho + m^\rho\varphi(y^\rho))].$$

Also
$$F\Big(x^\rho, \frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big|_{t^\rho} \le \frac{1}{2}\Big[\varphi(x^\rho) + m^\rho\varphi\Big(\frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big].$$
Taking $x^\rho = a^\rho + (1-t^\rho)\zeta(b^\rho,a^\rho)$, we obtain
$$F\Big(a^\rho + (1-t^\rho)\zeta(b^\rho,a^\rho), \frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big|_{t^\rho} \le \frac{1}{2}\Big[\varphi(a^\rho + (1-t^\rho)\zeta(b^\rho,a^\rho)) + m^\rho\varphi\Big(\frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big].$$
Now multiplying by $t^{\alpha\rho-1}$ and then integrating $t\in[0,1]$, we have
$$\begin{aligned}
&\int_0^1 t^{\alpha\rho-1}F\Big(a^\rho + (1-t^\rho)\zeta(b^\rho,a^\rho), \frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big|_{t^\rho}dt\\
&\le \frac{1}{2}\int_0^1 t^{\alpha\rho-1}\Big[\varphi(a^\rho + (1-t^\rho)\zeta(b^\rho,a^\rho)) + m^\rho\varphi\Big(\frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big]dt\\
&= \frac{1}{2}\int_0^1 t^{\alpha\rho-1}\varphi(a^\rho + (1-t^\rho)\zeta(b^\rho,a^\rho))dt + \frac{1}{2}m^\rho\varphi\Big(\frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\int_0^1 t^{\alpha\rho-1}dt\\
&= \frac{1}{2}\int_{(a^\rho+\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}}}^{a}\Big(\frac{u^\rho-(a^\rho+\zeta(b^\rho,a^\rho))}{-\zeta(b^\rho,a^\rho)}\Big)^{\alpha-1}\varphi(u^\rho)\frac{u^{\rho-1}}{-\zeta(b^\rho,a^\rho)}du + \frac{m^\rho}{2\alpha\rho}\varphi\Big(\frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big).\\
&= \frac{1}{2\zeta^\alpha(b^\rho,a^\rho)}\Gamma(\alpha)^\rho I^\alpha_{a^+}\varphi(a^\rho+\zeta(b^\rho,a^\rho)) + \frac{m^\rho}{2\alpha\rho}\varphi\Big(\frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big).
\end{aligned}\tag{31}$$
Also
$$\begin{aligned}
&\int_0^1 t^{\alpha\rho-1}F\Big(a^\rho + (1-t^\rho)\zeta(b^\rho,a^\rho), \frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big|_{t^\rho}dt\\
&= \int_{((a^\rho+\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}})}^{a}\Big(\frac{u^\rho-(a^\rho+\zeta(b^\rho,a^\rho))}{-\zeta(b^\rho,a^\rho)}\Big)^{\alpha-1}F\Big(u^\rho, \frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big|_{t^\rho}\frac{u^{\rho-1}}{-\zeta(b^\rho,a^\rho)}du\\
&= \frac{1}{\zeta^\alpha(b^\rho,a^\rho)}\int_a^{((a^\rho+\zeta(b^\rho,a^\rho))^{\frac{1}{\rho}})}\zeta(b^\rho,a^\rho)^{\alpha-1}F\Big(u^\rho, \frac{2a^\rho+\zeta(b^\rho,a^\rho)}{2}\Big)\Big|_{\frac{u^\rho-(a^\rho+\zeta(b^\rho,a^\rho))}{-\zeta(b^\rho,a^\rho)}}du.
\end{aligned}\tag{32}$$
By combining (31) and (32), we obtain the required result. ■

5 Conclusion

In this work, we have obtained the Hermite-Hadamard type fractional integral inequalities for various classes of preinvex functions in their generalized form using generalized fractional integral. For different values of α, ρ and bifunctional $\zeta(b^\rho,a^\rho)$, we obtained various new results associated with m-preinvex, s-preinvex, preinvex and convex functions. The ideas and techniques of this paper may inspire further research in this dynamics field.

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