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The Effect Of Time Delay On The Transmission Of Impulses In A Biological Neural Network*

Evgeny Georgievich Fedorov†, Anton Igorevich Popov‡, Igor Yurievich Popov§

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Abstract

Transmission of signal through simple biological neural network is considered. The investigation is performed in the framework of FitzHugh-Nagumo model. Two models are suggested: a discrete model based on discrete graph and a continuous model based on metric graph. The effect of time delay on the impulse transmission is studied.

1 Introduction

The Hodgkin-Huxley model is widely used to describe a biological neuron [1, 2, 3]. From a computational point of view, this model is complex. To reduce the computational complexity, FitzHugh and Nagumo suggested a more simple model [4, 5] presenting, in fact, a modification of the van der Pol model [6], called the FitzHugh-Nagumo model. It is intensively investigated last decade (see, e.g., [7, 8, 9, 10, 11, 12]). The model possesses the main features of the Hodgkin-Huxley model and quite accurately describes the dynamics of a biological neuron and at the same time has a relatively small computational complexity. In this article we suggest two graph type models of a simple neural network including two or three neurons. The first model is discrete. The neurons are described using the FitzHugh-Nagumo model. Communication between them occurs with a delay. The second model describes the actual impulse transmission through axons from one neuron to another and back. The time delay has an important role in the dynamics of the system. For the second model the time delay, really, corresponds to the length of the axons. It is shown that types of system behavior fundamentally depend on these parameters. One can observe oscillation or relaxation modes. We found numerically the corresponding critical values of the parameters.

2 Discrete Model

Consider two identical neurons, each of which is described by a FitzHugh-Nagumo system

$$\begin{cases} \dot{u}_i = -au_i + (a+1)u_i^2 - u_i^3 - v_i + I, \\ \dot{v}_i = bu_i - \gamma v_i, \end{cases} \tag{1}$$

where $i = 1, 2$, the functions $u_i(t)$ and $v_i(t)$ describe the state of the corresponding neuron at time t (they are related to some physical potentials in the neuron but we don't describe here physical processes ensuring signal transmission in biological neuron), a, b, γ are constants, and I is the external current to the neuron.

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†ITMO University, Kronverkskii, 49, St. Petersburg, 197101, Russia
‡ITMO University, Kronverkskii, 49, St. Petersburg, 197101, Russia
§ITMO University, Kronverkskii, 49, St. Petersburg, 197101, Russia

Next, we write the system for two neurons connected by a sigmoidal connection (that is independent of the postsynaptic neuron) [13, 14]:

$$\begin{cases} \dot{u}_1 = -au_1 + (a+1)u_1^2 - u_1^3 - v_1 + c\tanh(u_2^\tau), \\ \dot{v}_1 = bu_1 - \gamma v_1, \\ \dot{u}_2 = -au_2 + (a+1)u_2^2 - u_2^3 - v_2 + c\tanh(u_1^\tau), \\ \dot{v}_2 = bu_2 - \gamma v_2, \end{cases} \tag{2}$$

where $i = 1, 2$, $u_i = u_i(t), u_i^\tau = u_i(t-\tau)$, c is a constant corresponding to the strength of the coupling between neurons, and τ is the time delay associated with the impulse transmission between neurons. A schematic illustration of this system is shown in Fig. 1.

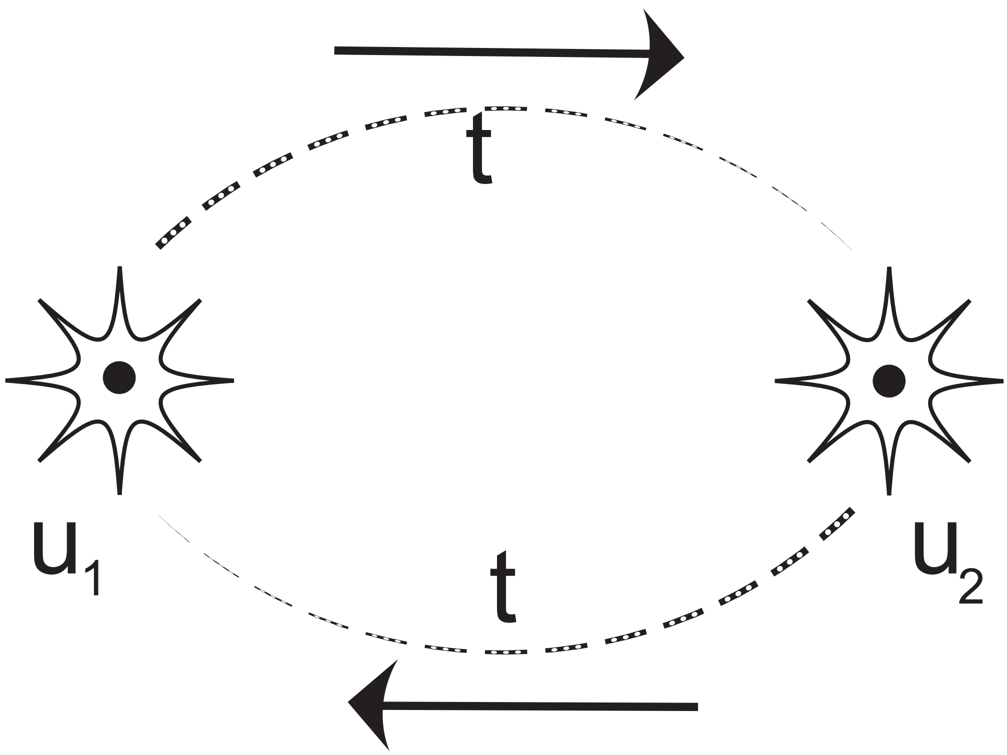

Figure 1: Schematic illustration of the considered system of two neurons

Let us take the values of the system (2) parameters from [13, 18]: $a = 0.25$, $b = 0.02$, $\gamma = 0.02$, $c = 0.2$. Consider the case of the delay absence. Let there be an initial above-threshold impulse in one of the neurons. Then a single excitation of both neurons will occur, as shown in the Fig. 2. For calculations we used the Wolfram Mathematica environment (NDSolve method) and the following initial conditions:

$$\begin{cases} u_1(t) = 0, & t < 0, \\ u_1(t) = 0.5, & t = 0, \\ u_2(t) = 0, & t \leq 0. \end{cases}$$

Next, we consider the following time delays: 10, 14.9, 14.94973, 14.94974 (Fig. 3, 4, 5, 6). One can see that at first the number of "cycles" increases with increasing delay, but their amplitude decays quickly. However, if the delay exceeds a certain value (for these values of the parameters the critical value of the time delay is 14.94974), then the impulse transmission between neurons becomes periodic. This effect can be explained by the presence of relaxation in biological neurons and in this system. That is, at low time delay values the impulse returns to the original neuron earlier than it has completely passed the relaxation period, so it has not completely recovered. It results in the attenuation of the impulse. But if the delay is large enough, then both neurons have enough time to fully recover by the moment the impulse returns. This means that it can transmit the impulse further without an attenuation.

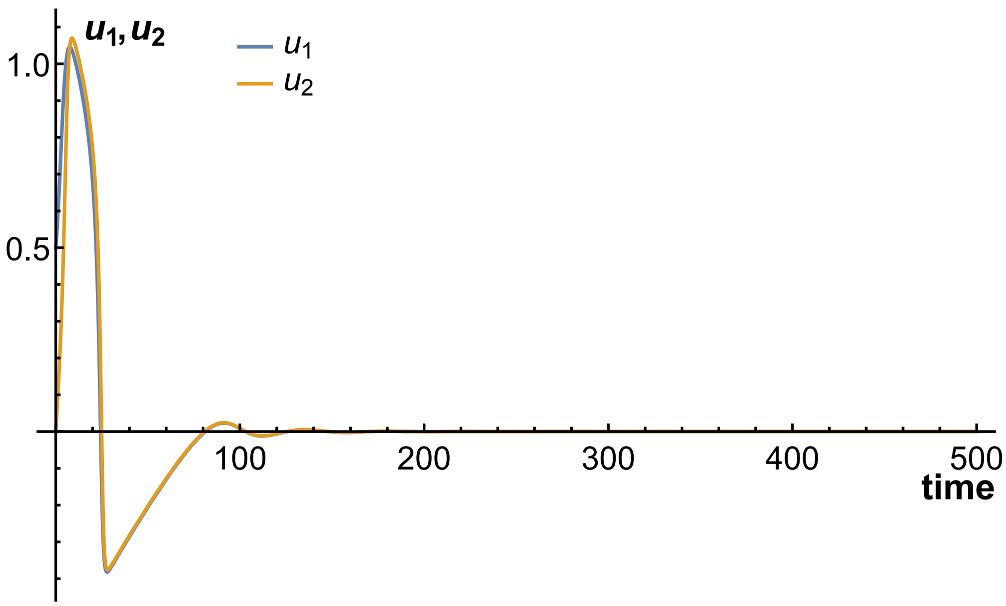

Figure 2: Dependence of two neurons potentials on time at $\tau = 0$

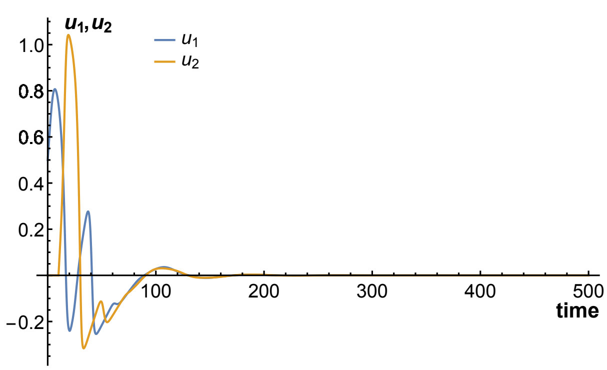

Figure 3: Dependence of two neurons potentials on time at $\tau = 10$

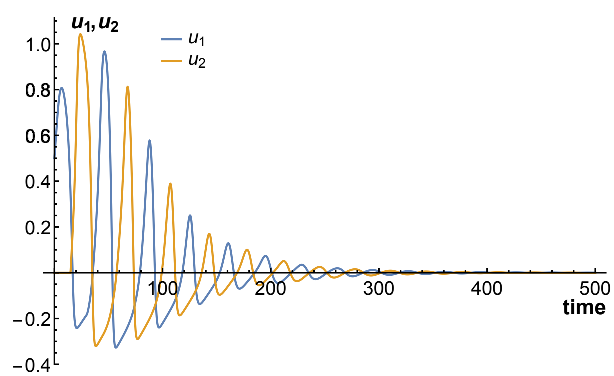

Figure 4: Dependence of two neurons potentials on time at $\tau = 14.9$

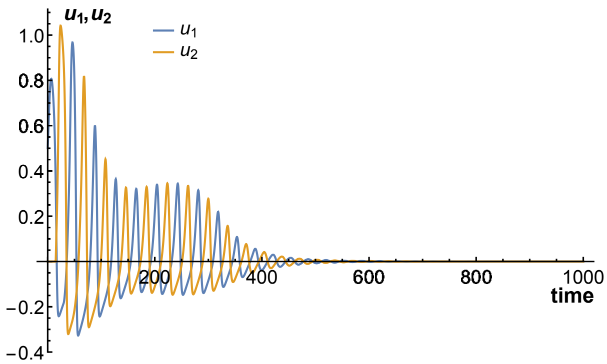

Figure 5: Dependence of two neurons potentials on time at $\tau = 14.94973$

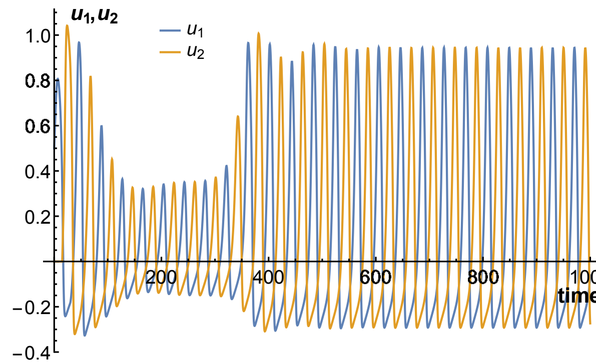

Figure 6: Dependence of two neurons potentials on time at $\tau = 14.94974$

3 Continuous Model

Now we consider the model of three neurons. The system is modeled as a metric graph with a loop shown in Fig. 7. At each edge of the graph the FitzHugh-Nagumo partial differential equations are treated:

$$\begin{cases} \frac{\partial u_1}{\partial t} = D\frac{\partial^2 u_1}{\partial x^2} - au_1 + (a+1)u_1^2 - u_1^3 - v_1, \\ \frac{\partial v_1}{\partial t} = bu_1 - \gamma v_1, \\ \frac{\partial u_2}{\partial t} = D\frac{\partial^2 u_2}{\partial x^2} - au_2 + (a+1)u_2^2 - u_2^3 - v_2, \\ \frac{\partial v_2}{\partial t} = bu_2 - \gamma v_2, \\ \frac{\partial u_3}{\partial t} = D\frac{\partial^2 u_3}{\partial x^2} - au_3 + (a+1)u_3^2 - u_3^3 - v_3, \\ \frac{\partial v_3}{\partial t} = bu_3 - \gamma v_3, \end{cases} \tag{3}$$

where $i = 1, 2, 3$, functions $u_i(t, x)$ and $v_i(t, x)$ describe the states of the corresponding neurons at time t at the axon point x, a, b, γ and D are constants, a, b, γ are the same as in the discrete model and D is the diffusion constant. Choosing of coupling condition at the vertices is the crucial point. As for quantum graphs, there are works investigating unusual coupling conditions (reflectionless conditions, conditions with preferred orientation, etc. [15, 16]) At the graph vertices we pose the following conditions ensuring a proper coupling (the chosen directions at the graph edges is shown in Fig. 7):

$$\begin{cases} u_2(t,0) = u_1(t,L) + u_3(t,L), \\ u_3(t,0) = u_2(t,L), \\ \frac{\partial u_1}{\partial x}(t,0) = 0, \quad \frac{\partial v_1}{\partial x}(t,0) = 0, \\ \frac{\partial u_1}{\partial x}(t,L) = 0, \quad \frac{\partial v_1}{\partial x}(t,L) = 0, \\ \frac{\partial u_2}{\partial x}(t,L) = 0, \quad \frac{\partial v_2}{\partial x}(t,L) = 0, \\ \frac{\partial u_3}{\partial x}(t,L) = 0, \quad \frac{\partial v_3}{\partial x}(t,L) = 0, \end{cases} \tag{4}$$

where L is the axon length (we assume that all axons have the same length, this assumption is not essential).

Thus, the system describes three neurons: the first is the start, the second and the third are tied to each other (see. Fig. 7).

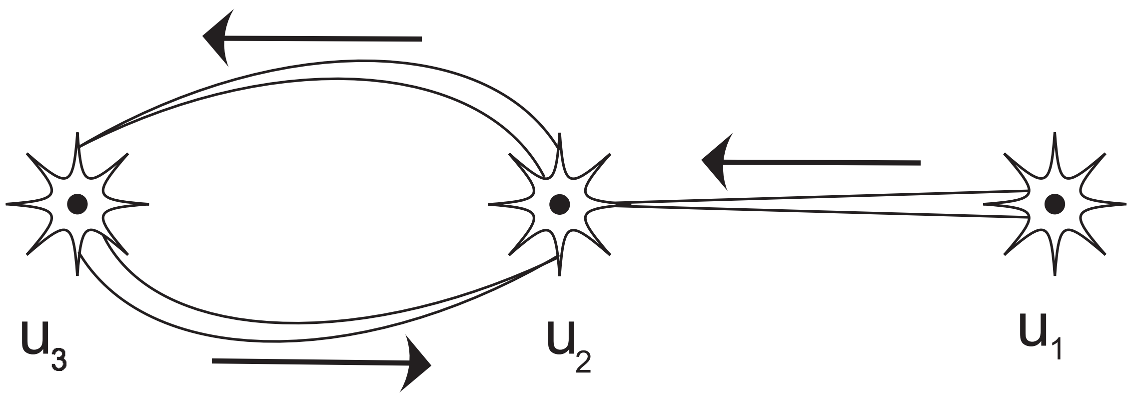

Figure 7: Neuron connections

To numerically solve the system (3), the method described below was used. Consider the first neuron:

$$\begin{cases} \frac{\partial u_1}{\partial t} - D\frac{\partial^2 u_1}{\partial x^2} = f(u_1) - v_1, \\ \frac{\partial v_1}{\partial t} = bu_1 - \gamma v_1, \\ \frac{\partial u_1}{\partial x}(t,0) = 0, \;\; \frac{\partial v_1}{\partial x}(t,0) = 0, \\ \frac{\partial u_1}{\partial x}(t,L) = 0, \;\; \frac{\partial v_1}{\partial x}(t,L) = 0, \end{cases} \tag{5}$$

where $f(x) = -ax+(a+1)x^2-x^3$. We divide the segment $[0,L]$ into n equal segments (with lengths $h = \frac{L}{n}$) by points $x_0 = 0$, $x_1 = \frac{L}{n}, \ldots, x_{n-1} = \frac{L(n-1)}{n}$, $x_n = L$. We introduce a similar grid along the time axis with the step τ and the points $t_k = k\tau$. Then we introduce the notation: $u_1^{(i,k)} = u_1(x_i, t_k)$. Now system (5) can be rewritten in the form of the following difference equations:

$$\begin{cases} \frac{u_1^{(i,k)}-u_1^{(i,k-1)}}{\tau} - \frac{D}{2}\left(\frac{u_1^{(i+1,k)}-2u_1^{(i,k)}+u_1^{(i-1,k)}}{h^2} + \frac{u_1^{(i+1,k-1)}-2u_1^{(i,k-1)}+u_1^{(i-1,k-1)}}{h^2}\right) \\ \quad = f(u_1^{(i,k-1)}) - v_1^{(i,k-1)}, \qquad i = 1\ldots n-1, \\ \frac{v_1^{(i,k)}-v_1^{(i,k-1)}}{\tau} = bu_1^{(i,k-1)} - \gamma v_1^{(i,k-1)}, \qquad i = 1\ldots n-1, \\ -3u_1^{(0,k)} + 4u_1^{(1,k)} - u_1^{(2,k)} = 0, \\ u_1^{(n-2,k)} - 4u_1^{(n-1,k)} + 3u_1^{(n,k)} = 0, \\ -3v_1^{(0,k)} + 4v_1^{(1,k)} - v_1^{(2,k)} = 0, \\ v_1^{(n-2,k)} - 4v_1^{(n-1,k)} + 3v_1^{(n,k)} = 0. \end{cases}$$

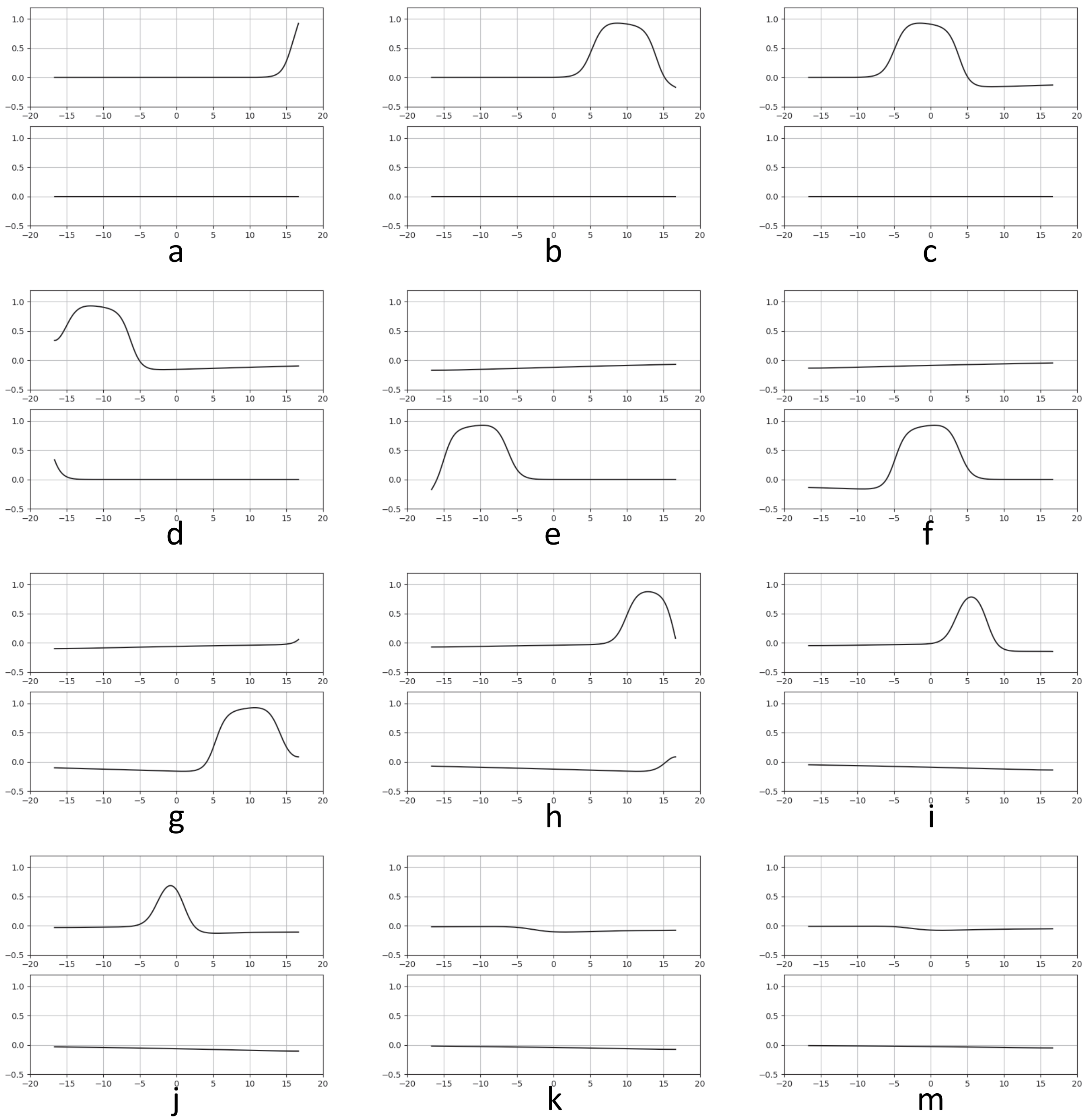

Figure 8: Signal movement through neural system shown in Fig. 7. At each fragment, upper curve shows the signal in upper neuron from Fig. 7, lower curve corresponds to lower neuron. Different fragments correspond to different time moments (arbitrary units): a) $t = 5$, b) $t = 65$, c) $t = 125$, d) $t = 185$, e) $t = 245$, f) $t = 305$, g) $t = 365$, h) $t = 425$, i) $t = 485$, j) $t = 545$, k) $t = 605$, m) $t = 665$.

After transformations the following system of linear equations is obtained:

$$\begin{cases} -ru_1^{(i-1,k)} + (1+2r)u_1^{(i,k)} - ru_1^{(i+1,k)} = w_i, \qquad i = 1 \dots n-1, \\ -3u_1^{(0,k)} + 4u_1^{(1,k)} - u_1^{(2,k)} = 0, \\ u_1^{(n-2,k)} - 4u_1^{(n-1,k)} + 3u_1^{(n,k)} = 0, \\ v_1^{(i,k)} = v_1^{(i,k-1)} + \tau\left(bu_1^{(i,k-1)} - \gamma v_1^{(i,k-1)}\right), \qquad i = 1 \dots n-1, \\ -3v_1^{(0,k)} + 4v_1^{(1,k)} - v_1^{(2,k)} = 0, \\ v_1^{(n-2,k)} - 4v_1^{(n-1,k)} + 3v_1^{(n,k)} = 0, \end{cases}$$

where $r = \frac{D\tau}{2h^2}$,

$$w_i = u_1^{(i,k-1)} + \tau\left(f(u_1^{(i,k-1)}) - v_1^{(i,k-1)}\right) + r\left(u_1^{(i+1,k-1)} - 2u_1^{(i,k-1)} + u_1^{(i-1,k-1)}\right).$$

We transform the system to the following form

$$\begin{cases} -2ru_1^{(0,k)} + (2r-1)u_1^{(1,k)} = -w_1, \\ -ru_1^{(i-1,k)} + (1+2r)u_1^{(i,k)} - ru_1^{(i+1,k)} = w_i, \qquad i = 1 \dots n-1, \\ (2r-1)u_1^{(n-1,k)} - 2ru_1^{(n,k)} = -w_{n-1}, \\ v_1^{(i,k)} = v_1^{(i,k-1)} + \tau\left(bu_1^{(i,k-1)} - \gamma v_1^{(i,k-1)}\right), \qquad i = 1 \dots n-1, \\ v_1^{(0,k)} = \frac{1}{3}\left(4v_1^{(1,k)} - v_1^{(2,k)}\right), \\ v_1^{(n,k)} = \frac{1}{3}\left(v_1^{(n-2,k)} - 4v_1^{(n-1,k)}\right). \end{cases}$$

Initial values $u_1^{i,0}, v_1^{i,0}, i = 0 \dots n$ being given, the system can be solved by the three-diagonal matrix algorithm:

$$\begin{cases} \alpha_1 = \frac{2r-1}{2r}, \quad \beta_1 = \frac{w_1}{2r}, \\ \alpha_{i+1} = \frac{r}{1+2r-r\alpha_i}, \qquad i = 1 \dots n-1, \\ \beta_{i+1} = \frac{r\beta_i + w_i}{1+2r-r\alpha_i}, \qquad i = 1 \dots n-1, \\ u_1^{(n,k)} = \frac{w_{n-1} + (2r-1)\beta_n}{2r-(2r-1)\alpha_n}, \\ u_1^{(i,k)} = \alpha_{i+1}u_1^{(i+1,k)} + \beta_{i+1}, \qquad i = n-1 \dots 0, \\ v_1^{(i,k)} = v_1^{(i,k-1)} + \tau\left(bu_1^{(i,k-1)} - \gamma v_1^{(i,k-1)}\right), \qquad i = 1 \dots n-1, \\ v_1^{(0,k)} = \frac{1}{3}\left(4v_1^{(1,k)} - v_1^{(2,k)}\right), \\ v_1^{(n,k)} = \frac{1}{3}\left(v_1^{(n-2,k)} - 4v_1^{(n-1,k)}\right). \end{cases}$$

Similarly, one can obtain solutions for the second and the third neurons:

$$\begin{cases} \alpha_1 = 0, \beta_1 = u_1^{(n,k-1)} + u_3^{(n,k-1)}, \\ \alpha_{i+1} = \frac{r}{1+2r-r\alpha_i}, \qquad i = 1\dots n-1, \\ \beta_{i+1} = \frac{r\beta_i + w_i}{1+2r-r\alpha_i}, \qquad i = 1\dots n-1, \\ u_2^{(n,k)} = \frac{w_{n-1}+(2r-1)\beta_n}{2r-(2r-1)\alpha_n}, \\ u_2^{(i,k)} = \alpha_{i+1} u_2^{(i+1,k)} + \beta_{i+1}, \qquad i = n-1\dots 0, \\ v_2^{(i,k)} = v_2^{(i,k-1)} + \tau\left(bu_2^{(i,k-1)} - \gamma v_2^{(i,k-1)}\right), \qquad i = 1\dots n-1, \\ v_2^{(0,k)} = v_1^{(n,k-1)} + v_3^{(n,k-1)}, \\ v_2^{(n,k)} = \frac{1}{3}\left(v_2^{(n-2,k)} - 4v_2^{(n-1,k)}\right), \end{cases}$$

$$\begin{cases} \alpha_1 = 0, \beta_1 = u_2^{(n,k-1)}, \\ \alpha_{i+1} = \frac{r}{1+2r-r\alpha_i}, \qquad i = 1\dots n-1, \\ \beta_{i+1} = \frac{r\beta_i + w_i}{1+2r-r\alpha_i}, \qquad i = 1\dots n-1, \\ u_3^{(n,k)} = \frac{w_{n-1}+(2r-1)\beta_n}{2r-(2r-1)\alpha_n}, \\ u_3^{(i,k)} = \alpha_{i+1} u_1^{(i+1,k)} + \beta_{i+1}, \qquad i = n-1\dots 0, \\ v_3^{(i,k)} = v_2^{(i,k-1)} + \tau\left(bu_3^{(i,k-1)} - \gamma v_3^{(i,k-1)}\right), \qquad i = 1\dots n-1, \\ v_3^{(0,k)} = v_2^{(n,k-1)}, \\ v_3^{(n,k)} = \frac{1}{3}\left(v_3^{(n-2,k)} - 4v_3^{(n-1,k)}\right). \end{cases}$$

This scheme was implemented in the Python 3 programming language. Zero initial values were taken at all points except for the vicinity of the beginning of the first axon, where the above-threshold disturbance was considered as the starting impulse of the system. Taking the following values of the parameters: $a = 0.25$, $b = 0.002$, $\gamma = 0.002$, $D = 0.3$ (as in [17]), we constructed several solutions for various axon lengths L. Fig. 8 shows several successive states of the system at different times for $L = 16.7$ (the upper part corresponds to the potential of the second axon and the lower to the third). The figures show that when returning to the second neuron, the signal decays. Numerical simulation shows that at $L = 16.8$, the attenuation does not occur. That is, the same situation is observed as in the discrete system, and the critical value lies between $L = 16.7$ and $L = 16.8$.

4 Conclusion

We suggest two mathematical models of graph types for a small neural network based on the FitzHugh-Nagumo equations. Both models showed a significant effect of the time delay on impulse transmission between the nuclei of neurons and on the dynamics of the network as a whole. With similar values of this parameter the impulse in the system can quickly decay or infinitely transmit between neurons. From the physical point of view, it turns out that small delays in the transmission of an impulse do not allow the impulse to pass through recursive systems. It is not essential why the time delay is small: because of short axons or fast transmission speed along them. The main reason is simple. If the impulse comes to neuron during the relaxation period, it can not pass through it without an attenuation. It leads to a limitation in number of signals travelling inside a neural network.

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