Applied Mathematics E-Notes, 21(2021), 718-733 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Co-Ordinated σ-Convex Function And Related Hermite Hadamard Type Inequalities*

Hira Ashraf Baig†, Naveed Ahmad‡, Qurat-ul-Ain Ahmad§, Muhammad Atif Idrees¶

Received 14 December 2020

Abstract

An advanced class of convexity has been introduced in this article, named as co-ordinated σ-convexity. This variant holds some other types of co-ordinated convex functions as its special cases. We also constituted integral inequalities enmeshed with the Hermite-Hadamard type for co-ordinated σ-convex functions, as application.

1 Introduction

A function $\vartheta : \mathcal{B} \to \mathbb{R}$ is known as convex in classical sense when $\mathcal{B} \subset \mathbb{R}$ is a convex set, and the following inequality is true $\forall\ x, y \in \mathcal{B}$ and $t \in [0,1]$

$$\vartheta(\theta x + (1-\theta)y) \leq \theta\vartheta(x) + (1-\theta)\vartheta(y).$$

The ideology of convexity plays a significant role in various fields of applied and pure sciences. That is the reason why the classical notion of convex sets as well as convex functions have been generalized in numerous ways. For further details, reader may see [2]–[4]. One more perspective for which the theory of convex functions has captivated a large number of researchers is its compact relationship with theory of inequalities. There are so many renowned inequalities which have been obtained using the concept of convexity. For more information, see [5]–[15]. Hermite-Hadamard's inequality is the most eminent name among these inequalities, which actually yields a necessary and sufficient condition for a function to be convex. This famous result of Hadamard and Hermite is as follows by

Theorem 1 *Assume a convex function $\vartheta : [e_a, e_b] \subset \mathbb{R} \to \mathbb{R}$ which is integrable on its domain. Then*

$$\vartheta\left(\frac{e_a + e_b}{2}\right) \leq \frac{1}{e_b - e_a}\int_{e_a}^{e_b} \vartheta(x)dx \leq \frac{\vartheta(e_a) + \vartheta(e_b)}{2}.$$

S. S. Dragomir presented the concept of **co-ordinated convex functions** in 1999 in [1]. He defined a function from a bi-dimensional interval $\Delta := [e_a, e_b] \times [e_c, e_d] \in \mathbb{R}^2$ with $e_a < e_b$ and $e_c < e_d$ to $\mathbb{R}$, i.e $\vartheta : \Delta \to \mathbb{R}$ is called convex on co-ordinates if the partial mappings $\vartheta_y : [e_a, e_b] \to \mathbb{R}$, $\vartheta_y(u) := \vartheta(u, y)$, and $\vartheta_x := [e_c, e_d] \to \mathbb{R}$, $\vartheta_x(v) = \vartheta(x, v)$, are convex which are defined for all $y \in [e_c, e_d]$ and $x \in [e_a, e_b]$. He also proved that, every convex mapping $\vartheta : \Delta \to \mathbb{R}$ is convex on the co-ordinates, but the converse is not generally true. He also furnished the Hermite-Hadamard inequality for co-ordinated convex functions.

*Mathematics Subject Classifications: 26D10, 26B25, 26D15.

†School of Mathematics and Computer Science, Institute of Business Administration (IBA), Karachi, Pakistan
‡School of Mathematics and Computer Science, Institute of Business Administration (IBA), Karachi, Pakistan
§School of Mathematics and Computer Science, Institute of Business Administration (IBA), Karachi, Pakistan
¶Department of Electrical Engineering, Muhammad Ali Jinnah University (MAJU), Karachi, Pakistan

Theorem 2 *For co-ordinated convex function $\vartheta : \Delta \to \mathbb{R}$ on Δ, the following inequalities are true*

$$\begin{aligned}
\vartheta\left(\frac{e_a+e_b}{2}, \frac{e_c+e_d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{e_b-e_a}\int_{e_a}^{e_b}\vartheta\left(x, \frac{e_c+e_d}{2}\right)dx + \frac{1}{e_d-e_c}\int_{e_c}^{e_d}\vartheta\left(\frac{e_a+e_b}{2}, y\right)dy\right] \\
&\leq \frac{1}{(e_b-e_a)(e_d-e_c)}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\ dydx \\
&\leq \frac{1}{4}\left[\frac{1}{e_b-e_a}\int_{e_a}^{e_b}\vartheta(x,e_c)dx + \frac{1}{e_b-e_a}\int_{e_a}^{e_b}\vartheta(x,e_d)dx\right. \\
&\quad \left. + \frac{1}{e_d-e_c}\int_{e_c}^{e_d}\vartheta(e_a,y)dy + \frac{1}{e_d-e_c}\int_{e_c}^{e_d}\vartheta(e_b,y)dy\right] \\
&\leq \frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4},
\end{aligned}$$

these inequalities are sharp.

Furthermore an analytical definition of two variable convex functions on co-ordinates has been presented in [16].

Definition 1 *A function $\vartheta : \Delta \to \mathbb{R}$ will be called co-ordinated convex on Δ, for all $\theta, \phi \in [0,1]$ and $(x,y),(u,v) \in \Delta$, if the following inequality holds*

$$\vartheta(\theta x + (1-\theta)y\ ,\ \phi u + (1-\phi)v) \leq \theta\phi\vartheta(x,u) + \phi(1-\theta)\vartheta(y,u) + \theta(1-\phi)\vartheta(x,v) + (1-\theta)(1-\phi)\vartheta(y,v).$$

Moreover few class of convexity has been introduced in [17] named as **σ-convexity** which comprises various other classes of convexities.

The aim of the present article is to combine σ-convexity with convex functions of two variables on co-ordinates, which emerged the notions of coordinated σ-convex sets and coordinated σ-convex functions. Thus we define the coordinated σ-convex functions through the formula

$$\mathcal{M}_{[\sigma_2]}((x_1,y_1),(x_2,y_2)) := (\sigma^{-1}(\theta\sigma(x_1) + (1-\theta)\sigma(y_1)), \sigma^{-1}(\phi\sigma(x_2) + (1-\phi)\sigma(y_2))$$

which has a relationship with the strictly monotonic continuous function σ, where $\mathcal{M}_p$ is the quasi-airthmatic mean for $p \in \mathbb{R}$, which binds all the power means together.

Additionally, as applications of the coordinated σ-convex functions, we acquire some new Hermite-Hadamard type inequalities. Simultaneously we also discuss some important special cases in detail.

2 Co-ordinated σ-Convex Functions

This section is devoted to formulate co-ordinated σ-convexity.

Definition 2 *A bi-dimensional set $\Delta_{\sigma_2} \subset \mathbb{R}^2$ is known as bi-dimensional σ-convex set related to a strictly monotonic continuous function σ if*

$$\mathcal{M}_{[\sigma_2]}((x_1,y_1),(x_2,y_2)) := (\sigma^{-1}(\theta\sigma(x_1) + (1-\theta)\sigma(y_1)), \sigma^{-1}(\phi\sigma(x_2) + (1-\phi)\sigma(y_2)) \in \Delta_{\sigma_2},$$

for all $(x_1,y_1),(x_2,y_2) \in \Delta_{\sigma 2}$ and $\theta, \phi \in [0,1]$.

Definition 3 *A function $\vartheta : \Delta_{\sigma_2} \to \mathbb{R}$ is said to be co-ordinated σ-convex on Δ_{σ_2} if*

$$\begin{aligned}
&\vartheta(\mathcal{M}_{[\sigma_2]}((x_1,y_1),(x_2,y_2)) \\
\leq\ & \theta\phi\vartheta(x_1,x_2) + \phi(1-\theta)\vartheta(y_1,x_2) + \theta(1-\phi)\vartheta(x_1,y_2) + (1-\theta)(1-\phi)\vartheta(y_1,y_2), \qquad (1)
\end{aligned}$$

for all $(x_1,y_1),(x_2,y_2) \in \Delta_{\sigma 2} := [e_a,e_b] \times [e_c,e_d]$ and $\theta, \phi \in [0,1]$.

Now we can extract various other types of co-ordinated convex function as special cases of co-ordinated σ-convex function by assuming different mappings in place of σ in Definition 3.

Case I: If we take $\sigma(x_1) = ln(x_1)$, then (1) becomes

$$\begin{aligned} &\vartheta\big(x_1^{\theta}\ y_1^{(1-\theta)},\ x_2^{\phi}\ y_2^{(1-\phi)}\big) \\ \leq\ &\theta\phi\ \vartheta(x_1,x_2) + \phi(1-\theta)\ \vartheta(y_1,x_2) + \theta(1-\phi)\ \vartheta(x_1,y_2) + (1-\theta)(1-\phi)\ \vartheta(y_1,y_2), \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in \Delta_{\sigma 2} := [e_a, e_b] \times [e_c, e_d] \subset (0,\infty) \times (0,\infty)$ and $\theta, \phi \in [0, 1]$, this is the concept of co-ordinated geometric convexity.

Case II: If we take $\sigma(x_1) = \frac{1}{x_1}$, then (1) becomes

$$\begin{aligned} &\vartheta\left(\frac{x_1 y_1}{(1-\theta)x_1+\theta y_1}, \frac{x_2 y_2}{(1-\phi)x_2+\phi y_2}\right) \\ \leq\ &\theta\phi\vartheta(x_1,x_2) + (1-\theta)\phi\vartheta(y_1,x_2) + \theta(1-\phi)\vartheta(x_1,y_2) + (1-\theta)(1-\phi)\vartheta(y_1,y_2), \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in \Delta_{\sigma 2} := [e_a, e_b] \times [e_c, e_d] \subset (0,\infty) \times (0,\infty)$ and $\theta, \phi \in [0, 1]$, this is the concept of co-ordinated harmonic convexity [18].

Case III: If we take $\sigma(x_1) = x_1^p$, then (1) becomes

$$\begin{aligned} &\vartheta\left(\big(\theta x_1^p + (1-\theta)y_1^p\big)^{\frac{1}{p}}, \big(\phi x_2^p + (1-\phi)y_2^p\big)^{\frac{1}{p}}\right) \\ \leq\ &\theta\phi\vartheta(x_1,x_2) + \phi(1-\theta)\vartheta(y_1,x_2) + \theta(1-\phi)\vartheta(x_1,y_2) + (1-\theta)(1-\phi)\vartheta(y_1,y_2), \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in \Delta_{\sigma 2} := [e_a, e_b] \times [e_c, e_d] \subset (0,\infty) \times (0,\infty)$ and $\theta, \phi \in [0, 1]$, this is the concept of co-ordinated p-convexity.

Case IV: If we take $\sigma(x_1) = \exp(x_1)$, then (1) becomes

$$\begin{aligned} &\vartheta\Big(ln\big(\theta\exp(x_1) + (1-\theta)\exp(y_1)\big), ln\big(\phi\exp(x_2) + (1-\phi)\exp(y_2)\big)\Big) \\ \leq\ &\theta\phi\vartheta(x_1,x_2)\ +\ (1-\theta)\phi\vartheta(y_1,x_2) + \theta(1-\phi)\vartheta(x_1,y_2) + (1-\theta)(1-\phi)\vartheta(y_1,y_2), \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in \Delta_{\sigma 2} := [e_a, e_b] \times [e_c, e_d]$ and $\theta, \phi \in [0, 1]$, this is the concept of co-ordinated log-exponential convexity.

3 Applications of Co-ordinated σ-Convex Functions to Integral Inequalities

Lemma 1 *Every σ-convex mapping $\vartheta : \Delta_{\sigma_2} := [e_a, e_b] \times [e_c, e_d] \to \mathbb{R}$ is co-ordinated σ-convex, although the converse is not true in general.*

Proof. Suppose that $\vartheta : \Delta_{\sigma_2} \to \mathbb{R}$ is σ-convex in Δ_{σ_2}. Let $\vartheta_x : [e_c, e_d] \to \mathbb{R}$ be defined by

$$\vartheta_x(y) := \vartheta\Big(\sigma^{-1}\big(\theta\sigma(x) + (1-\theta)\sigma(x)\big), y\Big).$$

Then for all $\theta \in [0,1]$ and $y_1, y_2 \in [e_c, e_d]$ one has

$$\begin{aligned}
&\vartheta_x\Big(\sigma^{-1}\big(\theta\sigma(y_1) + (1-\theta)\sigma(y_2)\big)\Big) \\
=\;& \vartheta\Big(\sigma^{-1}\big(\theta\sigma(x) + (1-\theta)\sigma(x)\big), \sigma^{-1}\big(\theta\sigma(y_1) + (1-\theta)\sigma(y_2)\big)\Big) \\
\leq\;& \theta\vartheta(x, y_1) + (1-\theta)\vartheta(x, y_2) \\
=\;& \theta\vartheta\Big(\sigma^{-1}\big(\sigma(x)\big), y_1\Big) + (1-\theta)\vartheta\Big(\sigma^{-1}\big(\sigma(x)\big), y_2\Big) \\
=\;& \theta\vartheta_x(y_1) + (1-\theta)\vartheta_x(y_2),
\end{aligned}$$

which shows the σ-convexity of ϑ_x. The fact that $\vartheta_y : [e_a, e_b] \to \mathbb{R}$ defined by $\vartheta_y(x) := \vartheta\Big(x, \sigma^{-1}\big(\theta\sigma(y) + (1-\theta)\sigma(y)\big)\Big)$, is also convex on $[e_a, e_b]$ goes likewise.

Now, consider the mapping $\vartheta : [0,1]^2 \to [0,\infty)$ defined as $\vartheta(x,y) = xy$. Let $\sigma : \mathbb{R} \to \mathbb{R}$ defined as $\sigma(x) = x^p$, $p \in (0,1]$. If $(u,0), (0,w) \in [0,1]^2$ and $t \in [0,1]$, we get

$$\begin{aligned}
&\vartheta\Bigg(\sigma^{-1}\Big(\theta\sigma(u) + (1-\theta)\sigma(0)\Big)\ ,\ \sigma^{-1}\Big(\theta\sigma(0) + (1-\theta)\sigma(w)\Big)\Bigg) \\
=\;& \vartheta\Bigg(\sigma^{-1}\Big(\theta(u)^p + (1-\theta)(0)^p\Big)\ ,\ \sigma^{-1}\Big(\theta(0)^p + (1-\theta)(w)^p\Big)\Bigg) \\
=\;& \vartheta\Big((\theta)^{\frac{1}{p}}u\ ,\ (1-\theta)^{\frac{1}{p}}w\Big) = \Big(\theta(1-\theta)\Big)^{\frac{1}{p}}uw
\end{aligned}$$

and

$$\theta\vartheta(u,0) + \theta(1-\theta)\vartheta(0,0) + \theta(1-\theta)\vartheta(u,w) + (1-\theta)^2\vartheta(0,w) = \theta(1-\theta)uw.$$

This shows that ϑ is co-ordinated σ-convex on $[0,1]^2$.

Now,

$$\begin{aligned}
&\vartheta\Bigg(\sigma^{-1}\Big((1-\theta)\sigma(u) + \theta\sigma(0)\Big)\ ,\ \sigma^{-1}\Big((1-\theta)\sigma(0) + \theta\sigma(w)\Big)\Bigg) \\
=\;& \vartheta\Bigg(\sigma^{-1}\Big((1-\theta)u^p\Big)\ ,\ \sigma^{-1}\Big(\theta w^p\Big)\Bigg) \\
=\;& \vartheta\Big((1-\theta)^{\frac{1}{p}}u\ ,\ \theta^{\frac{1}{p}}w\Big) \\
=\;& \Big(\theta(1-\theta)\Big)^{\frac{1}{p}}uw
\end{aligned}$$

and

$$(1-\theta)\vartheta(u,0) + \theta\vartheta(0,w) = 0.$$

Thus, $\forall\ \theta \in (0,1)$, $u, w \in (0,1)$ and $p \in (0,1]$, we get

$$\vartheta\Bigg(\sigma^{-1}\Big((1-\theta)\sigma(u) + \theta\sigma(0)\Big)\ ,\ \sigma^{-1}\Big((1-\theta)\sigma(0) + \theta\sigma(w)\Big)\Bigg) > (1-\theta)\vartheta(u,0) + \theta\vartheta(0,w),$$

hence, it proves that ϑ is not σ-convex on $[0,1]^2$. ■

Theorem 3 *Suppose that $\vartheta : \Delta_{\sigma_2} := [e_a, e_b] \times [e_c, e_d] \to \mathbb{R}$ is co-ordinated σ-convex on Δ_{σ_2}. Then the following inequalities are true*

$$\begin{aligned}
&\vartheta\left(\sigma^{-1}\left(\frac{\sigma(e_a)+\sigma(e_b)}{2}\right), \sigma^{-1}\left(\frac{\sigma(e_c)+\sigma(e_d)}{2}\right)\right)\\
\leq\ & \frac{1}{2}\left[\frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}\vartheta\left(x, \sigma^{-1}\left(\frac{\sigma(e_c)+\sigma(e_d)}{2}\right)\right)\sigma'(x)dx\right]\\
&+\frac{1}{2}\left[\frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}\vartheta\left(\sigma^{-1}\left(\frac{\sigma(e_a)+\sigma(e_b)}{2}\right), y\right)\sigma'(y)dy\right]\\
\leq\ & \frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx\\
\leq\ & \frac{1}{4}\left[\frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}\vartheta(x,e_c)\sigma'(x)dx + \frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}\vartheta(x,e_d)\sigma'(x)dx\right]\\
&+\frac{1}{4}\left[\frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}\vartheta(e_a,y)\sigma'(y)dy + \frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}\vartheta(e_b,y)\sigma'(y)dy\right]\\
\leq\ & \frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}. \qquad (2)
\end{aligned}$$

Proof. Since $\vartheta : \Delta_{\sigma_2} := [e_a, e_b] \times [e_c, e_d] \to \mathbb{R}$ is co-ordinated σ-convex on Δ_{σ_2}, it follows that the mapping $g_\theta : [e_c, e_d] \to \mathbb{R}$ defined by $g_\theta(y) := \vartheta(\sigma^{-1}(\theta\sigma(e_a) + (1-\theta)\sigma(e_b)), y)$ is σ-convex on $[e_c, e_d]$ for all $\theta \in [0, 1]$. Then by Hadamard's inequality we have

$$g_\theta\left(\sigma^{-1}\left(\frac{\sigma(e_c)+\sigma(e_d)}{2}\right)\right) \leq \frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d} g_\theta(y)\sigma'(y)dy \leq \frac{g_\theta(e_c)+g_\theta(e_d)}{2}, \quad \theta \in [0,1].$$

That is,

$$\begin{aligned}
&\vartheta\left(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)), \sigma^{-1}\left(\frac{\sigma(e_c)+\sigma(e_d)}{2}\right)\right)\\
\leq\ & \frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}\vartheta(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)), y)\sigma'(y)dy\\
\leq\ & \frac{\vartheta(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b), e_c)+\vartheta(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b), e_d)}{2}, \quad \theta \in [0,1].
\end{aligned}$$

Integarting the above inequality on $[0, 1]$ by substituting $\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)) = x$ and $d\theta = \frac{\sigma'(x)}{\sigma(e_b)-\sigma(e_a)}dx$, we have

$$\begin{aligned}
&\frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}\vartheta\left(x, \sigma^{-1}\left(\frac{\sigma(e_c)+\sigma(e_d)}{2}\right)\right)\sigma'(x)dx\\
\leq\ & \frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dxdy\\
\leq\ & \frac{1}{2[\sigma(e_b)-\sigma(e_a)]}\int_{e_a}^{e_b}[\vartheta(x,e_c)+\vartheta(x,e_d)]\sigma'(x)dx. \qquad (3)
\end{aligned}$$

By the similar argument applied for the mapping $g_\phi : [e_a, e_b] \to \mathbb{R}$ defined by $g_\phi(x) := \vartheta(x, \sigma^{-1}(\phi\sigma(e_c) + (1-\phi)\sigma(e_d)))$, we get

$$\begin{aligned}
&\frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}\vartheta\left(\sigma^{-1}\left(\frac{\sigma(e_a)+\sigma(e_b)}{2}\right), y\right)\sigma'(y)dy\\
\leq\ & \frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dxdy\\
\leq\ & \frac{1}{2[\sigma(e_d)-\sigma(e_c)]}\int_{e_c}^{e_d}[\vartheta(e_a,y)+\vartheta(e_b,y)]\sigma'(y)dy. \qquad (4)
\end{aligned}$$

By adding the above inequalities (3) and (4), we get the second and the third inequalities in (2). From the Hadamard's inequality, we also have

$$\vartheta\left(\sigma^{-1}\left(\frac{\sigma(e_a)+\sigma(e_b)}{2}\right),\sigma^{-1}\left(\frac{\sigma(e_c)+\sigma(e_d)}{2}\right)\right)\leq\frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}\vartheta\left(x,\sigma^{-1}\left(\frac{\sigma(e_c)+\sigma(e_d)}{2}\right)\right)\sigma'(x)dx,$$

and

$$\vartheta\left(\sigma^{-1}\left(\frac{\sigma(e_a)+\sigma(e_b)}{2}\right),\sigma^{-1}\left(\frac{\sigma(e_c)+\sigma(e_d)}{2}\right)\right)\leq\frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}\vartheta\left(\sigma^{-1}\left(\frac{\sigma(e_a)+\sigma(e_b)}{2}\right),y\right)\sigma'(y)dy,$$

by addition, it gives the first inequality in (2). Finally, we can also write by the same inequality

$$\frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}\vartheta(x,e_c)\sigma'(x)dx\leq\frac{\vartheta(e_a,e_c)+\vartheta(e_b,e_c)}{2},$$

$$\frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}\vartheta(x,e_d)\sigma'(x)dx\leq\frac{\vartheta(e_a,e_d)+\vartheta(e_b,e_d)}{2},$$

$$\frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}\vartheta(e_a,y)\sigma'(y)dy\leq\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)}{2},$$

$$\frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}\vartheta(e_b,y)\sigma'(y)dy\leq\frac{\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{2},$$

which give, by addition, the last inequality in (2). ■

Theorem 4 *Let $\vartheta:\Delta_{\sigma_2}\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable mapping on $\Delta:=[e_a,e_b]\times[e_c,e_d]$ in $\mathbb{R}^2$ with $e_a<e_b$ and $e_c<e_d$. If $\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\in L(\Delta)$, then*

$$\begin{aligned}
&\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}\\
&+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx\\
&-\frac{1}{2}\left[\frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}(\vartheta(x,e_c)+\vartheta(x,e_d))\sigma'(x)dx\right.\\
&\left.+\frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}(\vartheta(e_a,y)+\vartheta(e_b,y))\sigma'(y)dy\right]\\
=\;&\frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{4}\int_0^1\int_0^1\Big\{\,(1-2\phi)(1-2\theta)\\
&\times\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)),\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))\,\Big\}d\theta d\phi. \qquad (5)
\end{aligned}$$

Proof. Using integration by parts, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left\{(1-2\phi)(1-2\theta)\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b))\ ,\ \sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))\right\}d\theta d\phi\\
=&\ \frac{1}{\sigma(e_a)-\sigma(e_b)}\int_0^1(1-2\phi)\Big\{(1-2\theta)\frac{\partial\vartheta}{\partial\phi}(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b))\ ,\ \sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))\Big|_0^1\Big\}d\phi\\
&+\frac{2}{\sigma(e_a)-\sigma(e_b)}\int_0^1(1-2\phi)\left\{\int_0^1\frac{\partial\vartheta}{\partial\phi}(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)),\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))d\theta\right\}d\phi,\\
=&\ \frac{1}{\sigma(e_b)-\sigma(e_a)}\int_0^1(1-2\phi)\Big\{\frac{\partial\vartheta}{\partial\phi}(e_a,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))\\
&+\frac{\partial\vartheta}{\partial\phi}(e_b,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))\Big\}d\phi\\
&-\frac{2}{\sigma(e_b)-\sigma(e_a)}\int_0^1\int_0^1(1-2\phi)\frac{\partial\vartheta}{\partial\phi}(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)),\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))d\theta d\phi. \quad (6)
\end{aligned}$$

Again using integration by parts on the right hand side of (6), we get

$$\begin{aligned}
&\int_0^1(1-2\phi)\left(\frac{\partial\vartheta}{\partial\phi}(e_a,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))+\frac{\partial\vartheta}{\partial\phi}(e_b,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))\right)d\phi\\
&-2\int_0^1\int_0^1(1-2\phi)\frac{\partial\vartheta}{\partial\phi}(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)),\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))d\theta d\phi\\
=&\ (1-2\phi)\frac{\vartheta(e_a,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))+\vartheta(e_b,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))}{\sigma(e_c)-\sigma(e_d)}\Big|_0^1\\
&+\frac{2}{\sigma(e_c)-\sigma(e_d)}\int_0^1\left\{\vartheta(a,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))+(\vartheta(b,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))\right\}d\phi\\
&-2\int_0^1(1-2\phi)\frac{\vartheta(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)),\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))}{\sigma(e_c)-\sigma(e_d)))}\Big|_0^1 d\theta\\
&-\frac{4}{\sigma(e_c)-\sigma(e_d)}\int_0^1\int_0^1\vartheta(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)),\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))d\phi d\theta\\
=&\ \frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{\sigma(e_d)-\sigma(e_c)}\\
&+\frac{4}{\sigma(e_d)-\sigma(e_c)}\int_0^1\int_0^1\vartheta(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)),\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))d\phi d\theta\\
&-\frac{2}{\sigma(e_d)-\sigma(e_c)}\Bigg\{\int_0^1\Big[\vartheta(a,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))+\vartheta(b,\sigma^{-1}(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)))\Big]d\phi\\
&+\int_0^1\Big[\vartheta(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b),c)+\vartheta(\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b),e_d)\Big]d\theta\Bigg\}. \quad (7)
\end{aligned}$$

Writing (7) in (6), using the change of the variable

$$x=\sigma^{-1}(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)) \qquad \text{and} \qquad y=\sigma^{-1}(\phi(\sigma(e_c)+(1-\phi)\sigma(e_d)),$$

for θ, $\phi\in[0,1]^2$, and multiplying the both sides by

$$\frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{4},$$

we obtain (5),which completes the proof. ■

Theorem 5 *Let $\vartheta : \Delta_{\sigma_2} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta_{\sigma_2} := [e_a, e_b] \times [e_c, e_d]$ in $\mathbb{R}^2$ with $e_a < e_b$ and $e_c < e_d$. If $\left|\frac{\partial^2 \vartheta}{\partial\theta\partial\phi}\right|$ is coordinated σ-convex function on Δ_{σ_2}, then one has the inequality*

$$\begin{aligned} &\Bigg|\frac{\vartheta(e_a, e_c) + \vartheta(e_a, e_d) + \vartheta(e_b, e_c) + \vartheta(e_b, e_d)}{4} \\ &+ \frac{1}{\big(\sigma(e_b) - \sigma(e_a)\big)\big(\sigma(e_d) - \sigma(e_c)\big)} \int_{e_a}^{e_b}\int_{e_c}^{e_d} \vartheta(x, y)\sigma'(x)\sigma'(y)dydx - A\Bigg| \\ \leq\ & \frac{\big(\sigma(e_b) - \sigma(e_a)\big)\big(\sigma(e_d) - \sigma(e_c)\big)}{16} \\ &\times\left(\frac{\left|\frac{\partial^2\vartheta}{\partial\phi\partial\theta}(e_a, e_c)\right| + \left|\frac{\partial^2\vartheta}{\partial\phi\partial\theta}(e_a, e_d)\right| + \left|\frac{\partial^2\vartheta}{\partial\phi\partial\theta}(e_b, e_c)\right| + \left|\frac{\partial^2\vartheta}{\partial\phi\partial\theta}(e_b, e_d)\right|}{4}\right), \end{aligned} \tag{8}$$

where

$$A = \frac{1}{2}\Big\{\frac{1}{\sigma(e_b) - \sigma(e_a)} \int_{e_a}^{e_b} [\vartheta(x, e_c) + \vartheta(x, e_d)]\sigma'(x)dx + \frac{1}{\sigma(e_d) - \sigma(e_c)} \int_{e_c}^{e_d} [\vartheta(e_a, y) + \vartheta(e_b, y)]\sigma'(y)dy\Big\}.$$

Proof. From Theorem 4, we have

$$\begin{aligned} &\Bigg|\frac{\vartheta(e_a, e_c) + \vartheta(e_a, e_d) + \vartheta(e_b, e_c) + \vartheta(e_b, e_d)}{4} \\ &+ \frac{1}{\big(\sigma(e_b) - \sigma(e_a)\big)\big(\sigma(e_d) - \sigma(e_c)\big)} \int_{e_a}^{e_b}\int_{e_c}^{e_d} \vartheta(x, y)\sigma'(x)\sigma'(y)dydx - A\Bigg| \\ \leq\ & \frac{\big(\sigma(e_b) - \sigma(e_a)\big)\big(\sigma(e_d) - \sigma(e_c)\big)}{4} \int_0^1\int_0^1 \Big|(1 - 2\theta)(1 - 2\phi)\Big| \\ &\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\left(\sigma^{-1}\Big(\theta\sigma(e_a) + (1 - \theta)\sigma(e_b)\Big), \sigma^{-1}\Big(\phi\sigma(e_c) + (1 - \phi)\sigma(e_d)\Big)\right)\right| d\theta d\phi. \end{aligned}$$

Since $\vartheta : \Delta_{\sigma_2} \to \mathbb{R}$ is coordinated σ-convex function on $\Delta_{\sigma 2}$, one has

$$\begin{aligned} &\Bigg|\frac{\vartheta(e_a, e_c) + \vartheta(e_a, e_d) + \vartheta(e_b, e_c) + \vartheta(e_b, e_d)}{4} \\ &+ \frac{1}{\big(\sigma(e_b) - \sigma(e_a)\big)\big(\sigma(e_d) - \sigma(e_c)\big)} \int_{e_a}^{e_b}\int_{e_c}^{e_d} \vartheta(x, y)\sigma'(x)\sigma'(y)dydx - A\Bigg| \\ \leq\ & \frac{\big(\sigma(e_b) - \sigma(e_a)\big)\big(\sigma(e_d) - \sigma(e_c)\big)}{4} \int_0^1 \left[\int_0^1 \Big|(1 - 2\theta)(1 - 2\phi)\Big|\right. \\ &\times\left\{\theta\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_a, \sigma^{-1}\big(\phi\sigma(e_c) + (1 - \phi)\sigma(e_d)\big)\Big)\right|\right. \\ &\left.\left.+(1 - \theta)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_b, \sigma^{-1}\big(\phi\sigma(e_c) + (1 - \phi)\sigma(e_d)\big)\Big)\right|\right\} d\theta\right] d\phi. \end{aligned}$$

Firstly, by calculating the integral in above inequality, that is

$$\int_0^1 |1 - 2\theta| \left\{\theta\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_a, \sigma^{-1}\big(\phi\sigma(e_c) + (1 - \phi)\sigma(e_d)\big)\Big)\right| + (1 - \theta)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_b, \sigma^{-1}\big(\phi\sigma(e_c) + (1 - \phi)\sigma(e_d)\big)\Big)\right|\right\} d\theta,$$

by breaking the limits of integration, $0 \leq \theta \leq \frac{1}{2}$ and $\frac{1}{2} \leq \theta \leq 1$ then integrating we get

$$\frac{1}{4}\left(\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_a, \sigma^{-1}\big(\phi\sigma(e_c) + (1-\phi)\sigma(e_d)\big)\Big)\right| + \left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_b, \sigma^{-1}\big(\phi\sigma(e_c) + (1-\phi)\sigma(e_d)\big)\Big)\right|\right).$$

Thus, we obtain

$$\begin{aligned} &\Bigg|\frac{\vartheta(e_a, e_c) + \vartheta(e_a, e_d) + \vartheta(e_b, e_c) + \vartheta(e_b, e_d)}{4} \\ &+ \frac{1}{[\sigma(e_b) - \sigma(e_a)][\sigma(e_d) - \sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x, y)\sigma'(x)\sigma'(y)dydx - A\Bigg| \\ \leq\ & \frac{[\sigma(e_b) - \sigma(e_a)][\sigma(e_d) - \sigma(e_c)]}{16} \\ &\times\int_0^1|1-2\phi|\left\{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_a, \sigma^{-1}\big(\phi\sigma(e_c) + (1-\phi)\sigma(e_d)\big)\Big)\right|\right. \\ &\left.+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_b, \sigma^{-1}\big(\phi\sigma(e_c) + (1-\phi)\sigma(e_d)\big)\Big)\right|\right\}d\phi. \end{aligned} \tag{9}$$

Similarly for other integral, since $\vartheta : \Delta_{\sigma 2} \to \mathbb{R}$ is coordinated σ-convex on $\Delta_{\sigma 2}$, we have

$$\int_0^1|1-2\phi|\left\{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_a, \sigma^{-1}\big(\phi\sigma(e_c) + (1-\phi)\sigma(e_d)\big)\Big)\right| + \left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_b, \sigma^{-1}\big(\phi\sigma(e_c) + (1-\phi)\sigma(e_d)\big)\Big)\right|\right\}d\phi,$$

by breaking the limits of integration, $0 \leq \phi \leq \frac{1}{2}$ and $\frac{1}{2} \leq \phi \leq 1$ then integrating we get

$$\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a, e_c)\right| + \left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a, e_d)\right| + \left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b, e_c)\right| + \left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b, e_d)\right|}{4}. \tag{10}$$

By (9) and (10), we get (8). ■

Theorem 6 *Let $\vartheta : \Delta_{\sigma_2} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta_{\sigma_2} := [e_a, e_b] \times [e_c, e_d]$ in $\mathbb{R}^2$ with $e_a < e_b$ and $e_c < e_d$. If $\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\right|^q$, $q > 1$, is a coordinated σ-convex function on Δ_{σ_2}, then one has the inequaliy*

$$\begin{aligned} &\Bigg|\frac{\vartheta(e_a, e_c) + \vartheta(e_a, e_d) + \vartheta(e_b, e_c) + \vartheta(e_b, e_d)}{4} \\ &+ \frac{1}{[\sigma(e_b) - \sigma(e_a)][\sigma(e_d) - \sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x, y)\sigma'(x)\sigma'(y)dydx - A\Bigg| \\ \leq\ & \frac{[\sigma(e_b) - \sigma(e_a)][\sigma(e_d) - \sigma(e_c)]}{4(p+1)^{\frac{2}{p}}} \\ &\times\left(\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a, e_c)\right|^q + \left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a, e_d)\right|^q + \left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b, e_c)\right|^q + \left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b, e_d)\right|^q}{4}\right)^{\frac{1}{q}}, \end{aligned}$$

where

$$A = \frac{1}{2}\Big\{\frac{1}{\sigma(e_b) - \sigma(e_a)}\int_{e_a}^{e_b}[\vartheta(x, e_c) + \vartheta(x, e_d)]\sigma'(x)dx + \frac{1}{\sigma(e_d) - \sigma(e_c)}\int_{e_c}^{e_d}[\vartheta(e_a, y) + \vartheta(e_b, y)]\sigma'(y)dy\Big\},$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Theorem 4, we have

$$\begin{aligned}
&\left|\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}\right.\\
&\left.+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx-A\right|\\
\leq\ &\frac{\big(\sigma(e_b)-\sigma(e_a)\big)\big(\sigma(e_d)-\sigma(e_c)\big)}{4}\int_0^1\int_0^1\Big|(1-2\theta)(1-2\phi)\Big|\\
&\times\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\left(\sigma^{-1}\Big(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)\Big),\sigma^{-1}\Big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\Big)\right)\right|d\theta d\phi.
\end{aligned}$$

As $\vartheta:\Delta\sigma_2\to\mathbb{R}$ is coordinated σ-convex on Δ_{σ_2}, now by using Holder's inequality for double integrals, we can get

$$\begin{aligned}
&\left|\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}\right.\\
&\left.+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx-A\right|\\
\leq\ &\frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{4}\left(\int_0^1\int_0^1\Big|(1-2\theta)(1-2\phi)\Big|^p d\theta d\phi\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\big(\sigma^{-1}\big(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)\big),\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\big)\right|^q d\theta d\phi\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\right|^q$ is coordinated σ-convex function on Δ_{σ_2}, we know that for $\theta\in[0,1]$,

$$\begin{aligned}
&\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\big(\sigma^{-1}\big(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)\big),\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\big)\right|^q\\
\leq\ &\theta\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\big(e_a,\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\big)\right|^q+(1-\theta)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\big(e_b,\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\big)\right|^q,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\big(\sigma^{-1}\big(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)\big),\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\big)\right|^q\\
\leq\ &\theta\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+\theta(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q+(1-\theta)\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+(1-\theta)(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q,
\end{aligned}$$

hence, it follows that

$$\begin{aligned}
&\Bigg|\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4} \\
&+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx-A\Bigg| \\
\leq\;& \frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{4(p+1)^{\frac{2}{p}}}\Bigg(\int_0^1\int_0^1\Bigg\{\theta\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+\theta(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q \\
&+(1-\theta)\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+(1-\theta)(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q\Bigg\}d\theta d\phi\Bigg)^{\frac{1}{q}} \\
=\;& \frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{4(p+1)^{\frac{2}{p}}} \\
&\times\left(\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

■

Theorem 7 *Let $\vartheta:\Delta_{\sigma_2}\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable mapping on $\Delta_{\sigma_2}:=[e_a,e_b]\times[e_c,e_d]$ in $\mathbb{R}^2$ with $e_a<e_b$ and $e_c<e_d$. If $\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\right|^q$, $q\geq 1$, is coordinated σ-convex function on Δ_{σ_2}, then one has the inequality*

$$\begin{aligned}
&\Bigg|\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4} \\
&+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx-A\Bigg| \\
\leq\;& \frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{16} \\
&\times\left(\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q}{4}\right)^{\frac{1}{q}}, \qquad (11)
\end{aligned}$$

where

$$A=\frac{1}{2}\Big\{\frac{1}{\sigma(e_b)-\sigma(e_a)}\int_{e_a}^{e_b}[\vartheta(x,e_c)+\vartheta(x,e_d)]\sigma'(x)dx+\frac{1}{\sigma(e_d)-\sigma(e_c)}\int_{e_c}^{e_d}[\vartheta(e_a,y)+\vartheta(e_b,y)]\sigma'(y)dy\Big\}.$$

Proof. From Theorem 4, we have

$$\begin{aligned}
&\Bigg|\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4} \\
&+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx-A\Bigg| \\
\leq\;& \frac{\big(\sigma(e_b)-\sigma(e_a)\big)\big(\sigma(e_d)-\sigma(e_c)\big)}{4}\int_0^1\int_0^1\Big|(1-2\theta)(1-2\phi)\Big| \\
&\times\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(\sigma^{-1}\Big(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)\Big),\sigma^{-1}\Big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\Big)\Big)\right|d\theta d\phi.
\end{aligned}$$

By using power mean inequality for double integrals, $\vartheta : \Delta\sigma_2 \to \mathbb{R}$ is coordinated σ-convex on Δ_{σ_2}, we can get

$$\begin{aligned} &\Bigg|\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4} \\ &+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx - A\Bigg| \\ \leq\ &\frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{4}\left(\int_0^1\int_0^1\Big|(1-2\theta)(1-2\phi)\Big|d\theta d\phi\right)^{1-\frac{1}{q}} \\ &\times\Bigg(\int_0^1\int_0^1\Big|(1-2\theta)(1-2\phi)\Big| \\ &\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(\sigma^{-1}\big(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)\big),\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\Big)\right|^q d\theta d\phi\Bigg)^{\frac{1}{q}}. \end{aligned}$$

Since $\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\right|^q$ is coordinated σ-convex function on Δ_{σ_2}, we know that for $\theta \in [0,1]$

$$\begin{aligned} &\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(\sigma^{-1}\big(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)\big),\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\Big)\right|^q \\ \leq\ &\theta\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_a,\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\Big)\right|^q+(1-\theta)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(e_b,\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\Big)\right|^q, \end{aligned}$$

and

$$\begin{aligned} &\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}\Big(\sigma^{-1}\big(\theta\sigma(e_a)+(1-\theta)\sigma(e_b)\big),\sigma^{-1}\big(\phi\sigma(e_c)+(1-\phi)\sigma(e_d)\big)\Big)\right|^q \\ \leq\ &\theta\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+\theta(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q+(1-\theta)\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+(1-\theta)(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q, \end{aligned}$$

hence, it follows that

$$\begin{aligned} &\Bigg|\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4} \\ &+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx - A\Bigg| \\ \leq\ &\frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{4}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \\ &\times\Bigg(\int_0^1\int_0^1\Big|(1-2\theta)(1-2\phi)\Big|\Bigg\{\theta\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+\theta(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q \\ &+(1-\theta)\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+(1-\theta)(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q\Bigg\}d\theta d\phi\Bigg)^{\frac{1}{q}}. \end{aligned}$$

By calculating the integral in above inequality, we obtain

$$\int_0^1\Big|1-2\theta\Big|\left(\theta\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q\theta(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q+(1-\theta)\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+(1-\theta)(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q\right)d\theta,$$

now by breaking the limits of integration, $0 \leq \theta \leq \frac{1}{2}$ and $\frac{1}{2} \leq \theta \leq 1$, then after integrating we get

$$\frac{\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q}{24}+\frac{(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q}{24}+\frac{5\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q}{24}+\frac{5(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q}{24}$$
$$+\frac{5\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q}{24}+\frac{5(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q}{24}+\frac{\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q}{24}+\frac{(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q}{24}$$
$$=\frac{\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q}{4}+\frac{\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q}{4}.$$

Thus, we obtain

$$\left|\frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}\right.$$
$$\left.+\frac{1}{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\sigma'(x)\sigma'(y)dydx-A\right|$$
$$\leq \frac{[\sigma(e_b)-\sigma(e_a)][\sigma(e_d)-\sigma(e_c)]}{16}\left[\int_0^1|1-2\phi|\left(\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q\right.\right.$$
$$\left.\left.+(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q+\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q\right)d\phi\right]^{\frac{1}{q}}. \tag{12}$$

Similarly for other integral

$$\int_0^1|1-2\phi|\left(\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q+\phi\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+(1-\phi)\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q\right)d\phi,$$

since $\vartheta:\Delta_{\sigma_2}\to\mathbb{R}$ is coordinated σ-convex on Δ_{σ_2}, now by breaking the limits of integration, $0 \leq \phi \leq \frac{1}{2}$ and $\frac{1}{2} \leq \phi \leq 1$, then after integrating we get

$$\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q}{24}+\frac{5\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q}{24}+\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q}{24}+\frac{5\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q}{24}$$
$$+\frac{5\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q}{24}+\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q}{24}+\frac{5\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q}{24}+\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q}{24}$$
$$=\frac{\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_c)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_a,e_d)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_c)\right|^q+\left|\frac{\partial^2\vartheta}{\partial\theta\partial\phi}(e_b,e_d)\right|^q}{4}. \tag{13}$$

By (12) and (13), we get the inequality (11). ■

Conclusion

1. In this paper we introduced co-ordinated σ-convexity and we have shown that co-ordinated σ-convexity implies co-ordinated convexity when we consider σ as an identity function.

   If we take $\sigma(x)=x$ and $\sigma^{-1}(x)=x$, then Theorems 3, 4, 5, 6 and 7 given in this paper coincides with Theorem 1, Lemma 1, Theorems 2, 3 and 4 respectively given in [16].

2. **Hermite-Hadamard inequality for co-ordinated geometric convexity:** If we take $\sigma(x) = \ln(x)$ and $\sigma^{-1}(x) = \exp(x)$, then (2) becomes

$$\begin{aligned}
\vartheta\left(\frac{e_a e_b}{2}, \frac{e_c e_d}{2}\right) &\leq \frac{1}{2}\left[\ln\frac{e_a}{e_b}\int_{e_a}^{e_b}\vartheta\left(x, \frac{e_c e_d}{2}\right)\frac{1}{x}dx + \ln\frac{e_c}{e_d}\int_{e_c}^{e_d}\vartheta\left(\frac{e_a e_b}{2}, y\right)\frac{1}{y}dy\right] \\
&\leq \ln\frac{e_a}{e_b}\ln\frac{e_c}{e_d}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\frac{1}{xy}\,dydx \\
&\leq \frac{1}{4}\ln\frac{e_a}{e_b}\left[\int_{e_a}^{e_b}\Big(\vartheta(x,e_c)+\vartheta(x,e_d)\Big)\frac{1}{x}dx\right] \\
&\quad + \frac{1}{4}\ln\frac{e_c}{e_d}\left[\int_{e_c}^{e_d}\Big(\vartheta(e_a,y)+\vartheta(e_b,y)\Big)\frac{1}{y}dy\right] \\
&\leq \frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}.
\end{aligned}$$

3. **Hermite-Hadamard inequality for co-ordinated harmonic convexity:** If we take $\sigma(x) = \frac{1}{x}$ and $\sigma^{-1}(x) = \frac{1}{x}$, then (2) becomes

$$\begin{aligned}
\vartheta\left(\frac{2e_a e_b}{e_a+e_b}, \frac{2e_c e_d}{e_c+e_d}\right) &\leq \frac{1}{2}\left[\frac{e_a e_b}{e_b-e_a}\int_{e_a}^{e_b}\vartheta\left(x, \frac{2e_c e_d}{e_c+e_d}\right)\frac{1}{x^2}dx + \frac{e_c e_d}{e_d-e_c}\int_{e_c}^{e_d}\vartheta\left(\frac{2e_a e_b}{e_a+e_b}, y\right)\frac{1}{y^2}dy\right] \\
&\leq \frac{e_a e_b e_c e_d}{(e_a-e_b)(e_c-e_d)}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\frac{1}{x^2y^2}dydx \\
&\leq \frac{1}{4}\frac{e_a e_b}{e_b-e_a}\left[\int_{e_a}^{e_b}\Big(\vartheta(x,e_c)+\vartheta(x,e_d)\Big)\frac{1}{x^2}dx\right] \\
&\quad + \frac{1}{4}\frac{e_c e_d}{e_d-e_c}\left[\int_{e_c}^{e_d}\Big(\vartheta(e_a,y)+\vartheta(e_b,y)\Big)\frac{1}{y^2}dy\right] \\
&\leq \frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}.
\end{aligned}$$

4. **Hermite-Hadamard inequality for co-ordinated p-convexity:** If we take $\sigma(x) = x^p$ and $\sigma^{-1}(x) = x^{\frac{1}{p}}$, then (2) becomes

$$\begin{aligned}
\vartheta\left(\left(\frac{{e_a}^p+{e_b}^p}{2}\right)^{\frac{1}{p}}, \left(\frac{{e_c}^p+{e_d}^p}{2}\right)^{\frac{1}{p}}\right) &\leq \frac{1}{2}\left[\frac{p}{{e_b}^p-{e_a}^p}\int_{e_a}^{e_b}\vartheta\left(x, \left(\frac{{e_c}^p+{e_d}^P}{2}\right)^{\frac{1}{p}}\right)x^{p-1}dx\right] \\
&\quad + \frac{1}{2}\left[\frac{p}{{e_d}^p-{e_c}^p}\int_{e_c}^{e_d}\vartheta\left(\left(\frac{{e_a}^p+{e_b}^P}{2}\right)^{\frac{1}{p}}, y\right)y^{p-1}dy\right] \\
&\leq \frac{p^2}{({e_b}^p-{e_a}^p)\,({e_d}^p-{e_c}^p)}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)x^{p-1}y^{p-1}dydx \\
&\leq \frac{1}{4}\frac{p}{{e_b}^p-{e_a}^p}\left[\int_{e_a}^{e_b}\Big(\vartheta(x,e_c)+\vartheta(x,e_d)\Big)x^{p-1}dx\right] \\
&\quad + \frac{1}{4}\frac{p}{{e_d}^p-{e_c}^p}\left[\int_{e_c}^{e_d}\Big(\vartheta(e_a,y)+\vartheta(e_b,y)\Big)y^{p-1}dy\right] \\
&\leq \frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}.
\end{aligned}$$

5. **Hermite-Hadamard inequality for co-ordinated log-exponential convexity:** If we take $\sigma(x) =$

$\exp(x)$ and $\sigma^{-1}(x) = \ln x$, then (2) becomes

$$\begin{aligned}
&\vartheta\left(\ln\left(\frac{\exp(e_a)+\exp(e_b)}{2}\right), \ln\left(\frac{\exp(e_c)+\exp(e_d)}{2}\right)\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{\exp(e_b)-\exp(e_a)}\int_{e_a}^{e_b}\vartheta\left(x, \ln\left(\frac{\exp(e_c)+\exp(e_d)}{2}\right)\right)\exp(x)dx\right] \\
&+\frac{1}{2}\left[\frac{1}{\exp(e_d)-\exp(e_c)}\int_{e_c}^{e_d}\vartheta\left(\ln\left(\frac{\exp(e_a)+\exp(e_b)}{2}\right), y\right)\exp(y)dy\right] \\
\leq\ & \frac{1}{(\exp(e_b)-\exp(e_a))(\exp(e_d)-\exp(e_c))}\int_{e_a}^{e_b}\int_{e_c}^{e_d}\vartheta(x,y)\exp(x+y)dydx \\
\leq\ & \frac{1}{4}\frac{1}{\exp(e_b)-\exp(e_a)}\left[\int_{e_a}^{e_b}\Big(\vartheta(x,e_c)+\vartheta(x,e_d)\Big)\exp(x)dx\right] \\
&+\frac{1}{4}\frac{1}{\exp(e_d)-\exp(e_c)}\left[\int_{e_c}^{e_d}\Big(\vartheta(e_a,y)+\vartheta(e_b,y)\Big)\exp(y)dy\right] \\
\leq\ & \frac{\vartheta(e_a,e_c)+\vartheta(e_a,e_d)+\vartheta(e_b,e_c)+\vartheta(e_b,e_d)}{4}.
\end{aligned}$$

6. Furthermore, in the same manner we can obtain all other inequalities established in section 3 for the above four types of co-ordinated convexity.

References

[1] S. S. Dragomir, On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese J. Math., 5(2001), 775–788.

[2] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, J. Math. Anal. Appl., 335(2007), 1294–1308.

[3] C. Gabriela, and L. Lupsa, Non-connected Convexities and Applications, Vol. 68. Springer Science and Business Media, 2002.

[4] D. E. Varberg and A. W. Roberts, Convex Functions, Academic Press, New-York. MR 56 :1201 (1973)

[5] E. A. Youness, E-convex sets, E-convex functions, and E-convex programming, J. Optim. Theory Appl., 102(1999), 439–450

[6] T. Du, Y. Li and Z. Yang, A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions, Appl. Math. Comput., 293(2017), 358–369.

[7] S. Wu, I. A. Baloch and I. Iscan, On harmonically (p, h, m)-preinvex functions, J. Funct. Spaces, 2017, 9 pp.

[8] J. E. Pecaric, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, 1992.

[9] S. Wu, Generalization and sharpness of the power means inequality and their applications, J. Math. Anal. Appl., 312(2005), 637–652.

[10] M. U. Awan, K. I. Noor and M. A. Noor, Simpson-type inequalities for geometrically relative convex functions, Ukraïn. Mat. Zh., 70(2018), 992–1000.

[11] Y. M. Bai, S. Wu and Y. Wu, Hermite-Hadamard type integral inequalities for functions whose second-order mixed derivatives are coordinated (s, m)-P-convex, J. Funct. Spaces 2018, 7 pp.

[12] C. Y. Luo and Y. Zhang, Certain new bounds considering the weighted Simpson-like type inequality and applications, J. Inequal. Appl., 2018, Paper No. 332, 20 pp.

[13] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications. Springer, New York (2006)

[14] Y. Zhang, Different types of quantum integral inequalities via (α, m)-convexity, J. Inequal. Appl., 2018, Paper No. 264, 24 pp.

[15] E. Set and B. Celik, On generalizations related to the left side of Fejer's inequality via fractional integral operator, Miskolc Math. Notes, 18(2017), 1043–1057.

[16] M. Z. Sarikaya, E. Set, M. E. Ozdemir and S. S. Dragomir, New Hadamard's type inequalities for co-ordinated convex functions, Tamsui Oxf. J. Inf. Math. Sci., 28(2012), 137–152.

[17] S. Wu, M. U. Awan, M. A. Noor, K. I. Noor and S. Iftikhar, On a new class of convex functions and integral inequalities, J. Inequal. Appl., 2019, Paper No. 131, 14 pp.

[18] M. A. Noor, K. I. Noor, S. Iftikhar and C. Ionescu, Hermite-Hadamard inequalities for co-ordinated harmonic convex functions, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 79(2017), 25–34.