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Hybrid Pair Of Suzuki-Type Contractions And Related Coincidence Point Results*

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Abstract

In this paper, we introduce the concept of hybrid pair of Suzuki-type (α, φ, ζ)-contractions via Bianchini-Grandolfi gauge functions and utilize this to obtain points of coincidence for a hybrid pair of mappings on metric spaces. As some consequences of this study we obtain several important results of the existing literature.

1 Introduction

Let (X, d) be a metric space, $CL(X)$ be the family of all nonempty closed subsets of X and $CB(X)$ be the family of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, define $\mathcal{H}(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$. Then $\mathcal{H}$ is called Pompeiu-Hausdorff metric on $CB(X)$. Let $T : X \to CB(X)$ be a multivalued mapping. If there exists $\lambda \in (0, 1)$ such that $\mathcal{H}(Tx, Ty) \leq \lambda\, d(x, y)$ for all $x, y \in X$, then T is called a multivalued contraction. In 1969, Nadler [9] proved that every multivalued contraction on a complete metric space has a fixed point. Thereafter, several authors successfully established some interesting fixed point results for multivalued mappings with application in control theory, differential equations and convex optimization(see [3, 4, 5]). The study of fixed point theory combining simulation functions and gauge functions is a new development in the domain of contractive type multivalued theory. Following Nadler, many researchers have developed fixed point theory for multivalued mappings in different spaces; see for examples [7, 11, 12, 13, 14]. In this study, we introduce the concept of hybrid pair of Suzuki-type (α, φ, ζ)-contractions via Bianchini-Grandolfi gauge functions and obtain a sufficient condition for existence of points of coincidence for a hybrid pair of mappings on metric spaces. Finally, an example is given to justify the validity of our main result.

2 Basic Definitions and Results

We begin with some basic notations, definitions, and necessary results that will be needed in the sequel.

Definition 1 ([6]) *A mapping $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a simulation function if:*

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$ *for all* $t, s > 0$;

(ζ_3) *if* (t_n), (s_n) *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ *then* $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We note that $\zeta(t, t) < 0$ for all $t > 0$.

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Example 1 ([6]) *Let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by*

(i) $\zeta(t,s) = \varphi_1(s) - \varphi_2(t)$ *for all* $t,\ s \in [0,+\infty)$*, where* $\varphi_1, \varphi_2 : [0,\infty) \to [0,\infty)$ *are continuous functions such that* $\varphi_1(t) = \varphi_2(t) = 0$ *if and only if* $t = 0$ *and* $\varphi_1(t) < t \leq \varphi_2(t)$ *for all* $t > 0$*;*

(ii) $\zeta(t,s) = s - \frac{f(t,s)}{g(t,s)}t$ *for all* $t,\ s \in [0,\infty)$*, where* $f,\ g : [0,\infty) \times [0,\infty) \to (0,\infty)$ *are continuous functions with respect to each variable such that* $f(t,s) > g(t,s)$ *for all* $t, s > 0$*;*

(iii) $\zeta(t,s) = s - \varphi(s) - t$ *for all* $t,\ s \in [0,\infty)$*, where* $\varphi : [0,\infty) \to [0,\infty)$ *is a lower semicontinuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$*;*

(iv) $\zeta(t,s) = \varphi(s) - t$ *for all* $t,\ s \in [0,+\infty)$*, where* $\varphi : [0,\infty) \to [0,\infty)$ *is an upper semicontinuous function such that* $\varphi(t) < t$ *for all* $t > 0$ *and* $\varphi(0) = 0$*;*

(v) $\zeta(t,s) = s - \int_0^t \nu(u)du$ *for all* $t,\ s \in [0,\infty)$*, where* $\nu : [0,\infty) \to [0,\infty)$ *is a function such that* $\int_0^\epsilon \nu(u)du$ *exists and* $\int_0^\epsilon \nu(u)du > \epsilon$ *for all* $\epsilon > 0$*.*

Each of the function considered in (i)–(v) is a simulation function.

In 2017, Alolaiyan et al.[1] considered the following family of mappings:

$$\Phi = \left\{ \varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \mid \varphi \text{ satisfies } \varphi(r_1, r_2) \leq \frac{1}{2}r_1 - r_2 \right\}.$$

The function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $\varphi(r_1, r_2) = \frac{1}{2}r_1 - r_2$ is an element of Φ.

Throughout the paper J denotes an interval on $\mathbb{R}^+$ containing 0, that is, an interval of the form $[0,a],\ [0,a)$ or $[0,\infty)$.

Definition 2 ([10]) *Let $r \geq 1$. A function $\eta : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:*

(η_1) $\eta(\lambda t) \leq \lambda^r \eta(t)$ *for all* $\lambda \in (0,1)$ *and* $t \in J$*;*

(η_2) $\eta(t) < t$ *for all* $t \in J \setminus \{0\}$*.*

It is easy to verify that condition (η_1) is equivalent to the following one:

$$\eta(0) = 0 \text{ and } \frac{\eta(t)}{t^r} \text{ is nondecreasing on } J \backslash \{0\}.$$

Definition 3 ([10]) *A nondecreasing gauge function $\eta : J \to J$ is said to be a Bianchini-Grandolfi gauge function on J if*

$$\sigma(t) = \sum_{i=0}^{\infty} \eta^i(t) < \infty \text{ for all } t \in J. \tag{1}$$

A function $\eta : J \to J$ satisfying condition (1) is called a rate of convergence on J and noticed that η satisfies the following functional equation

$$\sigma(t) = \sigma(\eta(t)) + t.$$

Lemma 1 ([2]) *Let (X,d) be a metric space, $B \in CL(X)$ and $u \in X$. Then, for each $\epsilon > 0$, there exists $v \in B$ such that*

$$d(u,v) \leq d(u,B) + \epsilon.$$

Definition 4 *Let (X, d) be a metric space and C be a nonempty subset of X. Let $T : C \to CL(X)$ be a multivalued mapping and $g : C \to X$ be a single valued mapping. Then T is called α-admissible w.r.t. g if there exists a mapping $\alpha : g(C) \times g(C) \to [0, \infty)$ such that*

$$a,\, b \in C,\ \alpha(ga, gb) \geq 1 \Rightarrow \alpha(u, v) \geq 1$$

for all $u \in Ta \cap g(C)$ and $v \in Tb \cap g(C)$.

Definition 5 *Let (X, d) be a metric space and C be a nonempty subset of X. Let $T : C \to CL(X)$ be a multivalued mapping and $g : C \to X$ be a single valued mapping. If for $x_0 \in C$, there exists a sequence (gx_n) in $g(C)$ such that $gx_n \in Tx_{n-1},\ n \in \mathbb{N}$, then $O(T, x_0) = \{gx_0, gx_1, \cdots\}$ is called an orbit of T at x_0 in $g(C)$.*

Definition 6 *Let (X, d) be a metric space and C be a nonempty subset of X. A function $h : C \to \mathbb{R}$ is said to be T-orbitally lower semicontinuous w.r.t. g at $t \in C$ if (gx_n) is a sequence in $O(T, x_0)$ and $gx_n \to gt$ implies $h(t) \leq \liminf_n h(x_n)$.*

For $A,\, B \in CL(X)$, define

$$H(A, B) = \begin{cases} \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\} & \text{if the maximum exists,} \\ \infty & \text{otherwise,} \end{cases}$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. Such a map H is called Pompeiu-Hausdorff metric on $CL(X)$ induced by d.

Theorem 1 ([8]) *Let (X, d) be a metric space and let $T : X \to CL(X)$ and $g : X \to X$ be a hybrid pair of mappings such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that there exists $r \in (0, 1)$ such that*

$$H(Tx, Ty) \leq r d(gx, gy) \tag{2}$$

for all $x,\, y \in X$. Then g and T have a point of coincidence in $g(X)$.

3 Coincidence Point Results

In this section, we prove some coincidence point results for a hybrid pair of mappings in metric spaces.

Definition 7 *Let (X, d) be a metric space, C a closed subset of X, and let η be a Bianchini-Grandolfi gauge function on J. Let $T : C \to CL(X)$ be a multivalued mapping and $g : C \to X$ be a single valued mapping with $T(C) \subseteq g(C)$. Then (T, g) is called a hybrid pair of Suzuki-type (α, φ, ζ)-contraction, if there exist $\alpha : g(C) \times g(C) \to [0, \infty)$, $\varphi \in \Phi$, and a simulation function ζ such that T is α-admissible w.r.t. g and*

$$\varphi\left(d(gx, Tx \cap C), d(gx, gy)\right) < 0$$

implies that

$$\zeta\left(\alpha(gx, gy)\, H(Tx \cap C, Ty \cap C),\, \eta(d(gx, gy))\right) \geq 0 \tag{3}$$

for all $x \in C$, $gy \in Tx \cap C$ with $gx \neq gy$ and $d(gx, gy) \in J$.

In particular, if

$$\zeta(t, s) = s - \int_0^t \nu(r)\, dr\ \forall\, t,\, s \geq 0,$$

where $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\int_0^\epsilon \nu(r)\, dr$ exists and $\int_0^\epsilon \nu(r)\, dr > \epsilon$ for all $\epsilon > 0$, then (3) reduces to

$$\int_0^{\alpha(gx,gy)\, H(Tx\cap C, Ty\cap C)} \nu(r)\, dr \leq \eta(d(gx, gy))$$

for all $x \in C$, $gy \in Tx \cap C$ with $gx \neq gy$ and $d(gx, gy) \in J$. In this case, (T, g) is called a hybrid pair of Suzuki-integral type (α, φ, ζ)-contraction.

Remark 1 *In case $g = I$, the identity map on C, we call T is a multivalued Suzuki-type (α, φ, ζ)-contraction, instead of saying that (T, I) is a hybrid pair of Suzuki-type (α, φ, ζ)-contraction.*

Definition 8 *Let (X, d) be a metric space, C a nonempty subset of X and let $T : C \to CL(X)$, $g : C \to X$ be two mappings. If $y = gx \in Tx$ for some x in C, then x is called a coincidence point of T and g, and y is called a point of coincidence of T and g.*

Theorem 2 *Let (X, d) be a metric space, C a closed subset of X and let η be a Bianchini-Grandolfi gauge function on an interval J. Suppose that (T, g) is a hybrid pair of Suzuki-type (α, φ, ζ)-contraction and $g(C)$ is a complete subspace of (X, d). Also assume that there exists $x_0 \in C$ with $d(gx_0, z) \in J$ for some $z \in Tx_0 \cap C$ and $\alpha(gx_0, z) \geq 1$. Then,*

(a) there exist an orbit $\{gx_0, gx_1, \cdots\}$ of T at x_0 in $g(C)$ and $u \in g(C)$ such that $\lim_{n\to\infty} gx_n = u = gt$, for some $t \in C$;

(b) u is a point of coincidence of g and T in $g(C)$ if the function $h(x) = d(gx, Tx \cap C)$ is T-orbitally lower semicontinuous w.r.t. g at t.

Proof. Suppose there exists $x_0 \in C$ such that $d(gx_0, z) \in J$ for some $z \in Tx_0 \cap C$ and $\alpha(gx_0, z) \geq 1$. Since $T(C) \subseteq g(C)$, there exists $x_1 \in C$ such that $gx_1 = z$. If $gx_0 = gx_1$, then gx_0 is a point of coincidence of g and T. So, we assume that $gx_0 \neq gx_1$. Now,

$$\begin{aligned}
\varphi\left(d(gx_0, Tx_0 \cap C), d(gx_0, gx_1)\right) &\leq \frac{1}{2} d(gx_0, Tx_0 \cap C) - d(gx_0, gx_1) \\
&\leq \frac{1}{2} d(gx_0, gx_1) - d(gx_0, gx_1) \\
&< d(gx_0, gx_1) - d(gx_0, gx_1) \\
&= 0.
\end{aligned}$$

Since $d(gx_0, gx_1) \in J$, we obtain from (3) that

$$\begin{aligned}
0 &\leq \zeta\left(\alpha(gx_0, gx_1)\, H(Tx_0 \cap C, Tx_1 \cap C), \eta(d(gx_0, gx_1))\right) \\
&< \eta(d(gx_0, gx_1)) - \alpha(gx_0, gx_1)\, H(Tx_0 \cap C, Tx_1 \cap C).
\end{aligned}$$

This gives that

$$\alpha(gx_0, gx_1)\, H(Tx_0 \cap C, Tx_1 \cap C) < \eta(d(gx_0, gx_1)).$$

We can choose an $\epsilon_1 > 0$ such that

$$\alpha(gx_0, gx_1)\, H(Tx_0 \cap C, Tx_1 \cap C) + \epsilon_1 \leq \eta(d(gx_0, gx_1)).$$

Therefore,

$$\begin{aligned}
d(gx_1, Tx_1 \cap C) + \epsilon_1 &\leq H(Tx_0 \cap C, Tx_1 \cap C) + \epsilon_1 \\
&\leq \alpha(gx_0, gx_1)\, H(Tx_0 \cap C, Tx_1 \cap C) + \epsilon_1 \\
&\leq \eta(d(gx_0, gx_1)). \qquad (4)
\end{aligned}$$

By using Lemma 1, there exists $gx_2 \in Tx_1 \cap C$ for some $x_2 \in C$ such that

$$d(gx_1, gx_2) \leq d(gx_1, Tx_1 \cap C) + \epsilon_1. \qquad (5)$$

Using conditions (4) and (5), we get

$$d(gx_1, gx_2) \leq \eta(d(gx_0, gx_1)). \qquad (6)$$

Suppose that $d(gx_1, gx_2) \neq 0$, otherwise gx_1 is a point of coincidence of g and T in $g(C)$. By (η_2) and (6), we get

$$d(gx_1, gx_2) \leq \eta(d(gx_0, gx_1)) < d(gx_0, gx_1),$$

which implies that $d(gx_1, gx_2) \in J$. Since T is α-admissible w.r.t. g and $\alpha(gx_0, gx_1) \geq 1$, we have $\alpha(gx_1, gx_2) \geq 1$.

Again, we have

$$\begin{aligned} \varphi\left(d(gx_1, Tx_1 \cap C), d(gx_1, gx_2)\right) &\leq \frac{1}{2} d(gx_1, Tx_1 \cap C) - d(gx_1, gx_2) \\ &\leq \frac{1}{2} d(gx_1, gx_2) - d(gx_1, gx_2) \\ &< d(gx_1, gx_2) - d(gx_1, gx_2) \\ &= 0. \end{aligned}$$

Since $d(gx_1, gx_2) \in J$, we obtain from (3) that

$$\begin{aligned} 0 &\leq \zeta\left(\alpha(gx_1, gx_2)\, H(Tx_1 \cap C, Tx_2 \cap C), \eta(d(gx_1, gx_2))\right) \\ &< \eta(d(gx_1, gx_2)) - \alpha(gx_1, gx_2)\, H(Tx_1 \cap C, Tx_2 \cap C). \end{aligned}$$

This gives that

$$\alpha(gx_1, gx_2)\, H(Tx_1 \cap C, Tx_2 \cap C) < \eta(d(gx_1, gx_2)).$$

We choose an $\epsilon_2 > 0$ such that

$$\alpha(gx_1, gx_2)\, H(Tx_1 \cap C, Tx_2 \cap C) + \epsilon_2 \leq \eta(d(gx_1, gx_2)).$$

Thus,

$$\begin{aligned} d(gx_2, Tx_2 \cap C) + \epsilon_2 &\leq H(Tx_1 \cap C, Tx_2 \cap C) + \epsilon_2 \\ &\leq \alpha(gx_1, gx_2)\, H(Tx_1 \cap C, Tx_2 \cap C) + \epsilon_2 \\ &\leq \eta(d(gx_1, gx_2)). \end{aligned} \tag{7}$$

By Lemma 1, there exists $gx_3 \in Tx_2 \cap C$ for some $x_3 \in C$ such that

$$d(gx_2, gx_3) \leq d(gx_2, Tx_2 \cap C) + \epsilon_2. \tag{8}$$

From conditions (7) and (8), we get

$$d(gx_2, gx_3) \leq \eta(d(gx_1, gx_2)) \leq \eta^2(d(gx_0, gx_1)). \tag{9}$$

We assume that $d(gx_2, gx_3) \neq 0$, otherwise gx_2 is a point of coincidence of g and T in $g(C)$. From (9), it follows that $d(gx_2, gx_3) < d(gx_1, gx_2)$ and so $d(gx_2, gx_3) \in J$. Continuing in this way, we can construct a sequence (gx_n) in $g(C)$ such that $gx_n \in Tx_{n-1} \cap C \subseteq g(C)$, $gx_{n-1} \neq gx_n$ with $\alpha(gx_{n-1}, gx_n) \geq 1$, $d(gx_{n-1}, gx_n) \in J$ and

$$d(gx_n, gx_{n+1}) \leq \eta^n(d(gx_0, gx_1)), \ \forall n \in \mathbb{N}. \tag{10}$$

For m, $n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ &\leq \eta^n(d(gx_0, gx_1)) + \eta^{n+1}(d(gx_0, gx_1)) + \cdots + \eta^{m-1}(d(gx_0, gx_1)) \\ &= \sum_{i=n}^{m-1} \eta^i(d(gx_0, gx_1)). \end{aligned}$$

Since $\sum_{i=0}^{\infty} \eta^i(t) < \infty$ for each $t \in J$, it follows that

$$\lim_{n,m\to\infty} d(gx_n, gx_m) = 0.$$

This proves that (gx_n) is a Cauchy sequence in $g(C)$. Since $g(C)$ is complete, there exists $u \in g(C)$ such that $gx_n \to u = gt$ for some $t \in C$. This proves part (a) of the theorem.

As $gx_{n+1} \in Tx_n \cap C$, we have

$$\begin{aligned} \varphi\left(d(gx_n, Tx_n \cap C), d(gx_n, gx_{n+1})\right) &\leq \frac{1}{2} d(gx_n, Tx_n \cap C) - d(gx_n, gx_{n+1}) \\ &\leq \frac{1}{2} d(gx_n, gx_{n+1}) - d(gx_n, gx_{n+1}) \\ &< 0. \end{aligned}$$

Therefore, from (3) we get

$$\begin{aligned} 0 &\leq \zeta\left(\alpha(gx_n, gx_{n+1})\, H(Tx_n \cap C, Tx_{n+1} \cap C), \eta(d(gx_n, gx_{n+1}))\right) \\ &< \eta(d(gx_n, gx_{n+1})) - \alpha(gx_n, gx_{n+1})\, H(Tx_n \cap C, Tx_{n+1} \cap C). \end{aligned}$$

This gives that

$$\alpha(gx_n, gx_{n+1})\, H(Tx_n \cap C, Tx_{n+1} \cap C) < \eta(d(gx_n, gx_{n+1})). \tag{11}$$

Since $gx_{n+1} \in Tx_n \cap C$, using (10) and (11), we get

$$\begin{aligned} d(gx_{n+1}, Tx_{n+1} \cap C) &\leq \alpha(gx_n, gx_{n+1})\, H(Tx_n \cap C, Tx_{n+1} \cap C) \\ &< \eta(d(gx_n, gx_{n+1})) \\ &\leq \eta^{n+1}(d(gx_0, gx_1)). \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(gx_{n+1}, Tx_{n+1} \cap C) = 0.$$

Since $h(x) = d(gx, Tx \cap C)$ is T-orbitally lower semicontinuous w.r.t. g at t, we have

$$d(gt, Tt \cap C) = h(t) \leq \liminf_n h(x_{n+1}) = \liminf_n d(gx_{n+1}, Tx_{n+1} \cap C) = 0.$$

This gives that $d(gt, Tt \cap C) = 0$. As $Tt \cap C$ is closed, it follows that $u = gt \in Tt$ and hence u is a point of coincidence of g and T in $g(C)$. ■

Corollary 1 *Let (X, d) be a metric space, C a closed subset of X and let η be a Bianchini-Grandolfi gauge function on an interval J. Suppose that (T, g) is a hybrid pair of Suzuki-integral type (α, φ, ζ)-contraction and $g(C)$ is a complete subspace of (X, d). Also assume that there exists $x_0 \in C$ with $d(gx_0, z) \in J$ for some $z \in Tx_0 \cap C$ and $\alpha(gx_0, z) \geq 1$. Then,*

(a) there exist an orbit $\{gx_0, gx_1, \cdots\}$ of T at x_0 in $g(C)$ and $u \in g(C)$ such that $\lim_{n\to\infty} gx_n = u = gt$, for some $t \in C$;

(b) u is a point of coincidence of g and T in $g(C)$ if the function $h(x) = d(gx, Tx \cap C)$ is T-orbitally lower semicontinuous w.r.t. g at t.

Proof. The proof can be obtained from Theorem 2 by taking

$$\zeta(t, s) = s - \int_0^t \nu(r)\, dr \;\forall\, t,\, s \geq 0,$$

where $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\int_0^\epsilon \nu(r)\, dr$ exists and $\int_0^\epsilon \nu(r)\, dr > \epsilon$ for all $\epsilon > 0$. ■

Corollary 2 *Let (X, d) be a metric space, C a closed subset of X, and let η be a Bianchini-Grandolfi gauge function on J. Let $T : C \to CL(X)$ be a multivalued mapping and $g : C \to X$ be a single valued mapping with $T(C) \subseteq g(C)$ and $g(C)$ a complete subspace of (X, d). Suppose there exists $\alpha : g(C) \times g(C) \to [0, \infty)$ such that T is α-admissible w.r.t. g and*

$$\alpha(gx, gy)\, H(Tx \cap C, Ty \cap C) \le \psi(\eta(d(gx, gy))) \tag{12}$$

for all $x \in C$, $gy \in Tx \cap C$ with $gx \neq gy$ and $d(gx, gy) \in J$, where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. Also assume that there exists $x_0 \in C$ with $d(gx_0, z) \in J$ for some $z \in Tx_0 \cap C$ and $\alpha(gx_0, z) \ge 1$. Then,

(a) *there exist an orbit $\{gx_0, gx_1, \cdots\}$ of T at x_0 in $g(C)$ and $u \in g(C)$ such that $\lim_{n\to\infty} gx_n = u = gt$, for some $t \in C$;*

(b) *u is a point of coincidence of g and T in $g(C)$ if the function $h(x) = d(gx, Tx \cap C)$ is T-orbitally lower semicontinuous w.r.t. g at t.*

Proof. Taking $\varphi(r_1, r_2) = \frac{1}{2} r_1 - r_2$, for $r_1, r_2 \in \mathbb{R}^+$, we obtain that for all $x \in C$, $gy \in Tx$ with $gx \neq gy$,

$$\begin{aligned}\varphi\left(d(gx, Tx), d(gx, gy)\right) &= \frac{1}{2} d(gx, Tx) - d(gx, gy)\\ &\le \frac{1}{2} d(gx, gy) - d(gx, gy)\\ &= -\frac{1}{2} d(gx, gy)\\ &< 0.\end{aligned}$$

By considering $\zeta(t, s) = \psi(s) - t,\ \forall\ t,\ s \ge 0$, it follows from condition (12) that

$$\zeta\left(\alpha(gx, gy)\, H(Tx \cap C, Ty \cap C),\ \eta(d(gx, gy))\right) \ge 0$$

for all $x \in C$, $gy \in Tx \cap C$ with $gx \neq gy$ and $d(gx, gy) \in J$. Consequently, (T, g) becomes a hybrid pair of Suzuki-type (α, φ, ζ)-contraction. Thus all the hypotheses of Theorem 2 are fulfilled and the conclusion of the corollary can be obtained by applying Theorem 2. ■

Corollary 3 *Let (X, d) be a complete metric space, C a closed subset of X and let η be a Bianchini-Grandolfi gauge function on an interval J. Suppose that $T : C \to CL(X)$ is a multivalued Suzuki-type (α, φ, ζ)-contraction. Also assume that there exists $x_0 \in C$ with $d(x_0, z) \in J$ for some $z \in Tx_0 \cap C$ and $\alpha(x_0, z) \ge 1$. Then,*

(a) *there exist an orbit $\{x_0, x_1, \cdots\}$ of T at x_0 in C and $u \in C$ such that $\lim\limits_{n\to\infty} x_n = u$;*

(b) *u is a fixed point of T if the function $h(x) = d(x, Tx \cap C)$ is T-orbitally lower semicontinuous at u.*

Proof. The proof follows from Theorem 2 by taking $g = I$, the identity map on C. ■

Remark 2 *Several special cases of Theorem 2 can be obtained by particular choices of η, φ and ζ.*

Now we present an example to examine the validity of our main result. It should be noticed that a generalized version of Nadler's Theorem can not assure the existence of a point of coincidence in the following example.

Example 2 *Let $X = [0, \infty)$ with usual metric $d(x, y) =| x - y |$ for all $x, y \in X$. Let $C = [0, 1]$ and $T : C \to CL(X)$ be defined by $Tx = [1, x+1]$, $\forall\, x \in C$ and $g : C \to X$ by $gx = x + 1$ for all $x \in C$. Obviously, $T(C) = g(C) = [1, 2]$ and $g(C)$ is a complete subspace of the metric space (X, d).*

For $x = 0,\ y = 1$, we have $gx = 1,\ gy = 2,\ Tx = \{1\},\ Ty = [1, 2]$. Therefore,

$$H(Tx, Ty) = 1 = d(gx, gy) > rd(gx, gy)$$

for any $r \in (0, 1)$ and hence condition (2) of Theorem 1 does not hold true. So, Theorem 1 can not assure the existence of a point of coincidence of g and T.

Let $J = [0, \infty)$, η a Bianchini-Grandolfi gauge function on J and let $\alpha : g(C) \times g(C) \to [0, \infty)$ be defined by $\alpha(x, y) = 1$ for all $x,\ y \in [1, 2]$. Obviously, T is α-admissible w.r.t. g. Moreover, $x_0 = 0 \in C$ such that $d(gx_0, z) \in J$ for $z = 1 \in Tx_0 \cap C$ and $\alpha(gx_0, z) = 1$.

Let $\zeta(t, s) = \psi(s) - t$, $\forall\ t,\ s \geq 0$, where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function such that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. Take $\varphi(r_1, r_2) = \frac{1}{2} r_1 - r_2$, for $r_1,\ r_2 \in \mathbb{R}^+$.

We now show that (T, g) is a hybrid pair of Suzuki-type (α, φ, ζ)-contraction.

Case-I: *For $x = 1$, we have $Tx = [1, 2]$ and $Tx \cap C = \{1\}$. We note that $gy = g0 = 1 \in Tx \cap C$ with $gx \neq gy$ and $Ty \cap C = \{1\}$. Then, $H(Tx \cap C, Ty \cap C) = 0$. Therefore,*

$$\alpha(gx, gy)\, H(Tx \cap C, Ty \cap C) \leq \psi(\eta(d(gx, gy))).$$

Case-II: *For $x = 0$, we have $Tx = \{1\}$ and $Tx \cap C = \{1\}$. In this case, there exists no $gy (\neq gx) \in Tx \cap C$.*

Case-III: *For $0 < x < 1$, we have $Tx = [1, x+1]$ and $Tx \cap C = \{1\}$. This case is also similar to Case-I.*

Thus,

$$\alpha(gx, gy)\, H(Tx \cap C, Ty \cap C) \leq \psi(\eta(d(gx, gy)))$$

for all $x \in C$, $gy \in Tx \cap C$ with $gx \neq gy$ and $d(gx, gy) \in J$. Since $\zeta(t, s) = \psi(s) - t$ for all $t,\ s \geq 0$, it follows that

$$\zeta\left(\alpha(gx, gy)\, H(Tx \cap C, Ty \cap C),\ \eta(d(gx, gy))\right) \geq 0$$

for all $x \in C$, $gy \in Tx \cap C$ with $gx \neq gy$ and $d(gx, gy) \in J$. Consequently, (T, g) becomes a hybrid pair of Suzuki-type (α, φ, ζ)-contraction.

Therefore, all the hypotheses of Theorem 2 are fulfilled and we observe that there exist an orbit $\{gx_0, gx_1, \cdots\}$ of T at $x_0 = 0$ in $g(C)$, where $gx_n = 1$, for $n = 0,\ 1,\ 2,\ 3, \cdots$ and $1 \in g(C)$ such that $\lim_{n \to \infty} gx_n = 1 = g0$.

Furthermore, $h(x) = d(gx, Tx \cap C) = d(gx, \{1\}) = x$ is T-orbitally lower semicontinuous w.r.t. g at $x = 0$. Now applying Theorem 2, we find that 1 is a point of coincidence of g and T in $g(C)$.

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