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New Cubic Transmuted Pareto Distribution: Properties and Applications*

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Abstract

In this paper, we have proposed a new cubic transmuted Pareto (CTP) distribution by using the cubic Transmuted family of distributions introduced by Rahman et al. [1]. Derivation of statistical properties along with the distribution of order statistics have been discussed for the CTP distribution. Certain characterizations of the CTP distribution are presented. Maximum likelihood estimation of the model parameters has been conducted alongside an extensive Monte Carlo simulation study to assess the performance of the estimation procedure. Finally, two real-life applications have been considered to investigate the applicability of the CTP distribution.

1 Introduction

Vilfredo Pareto (1848-1923) observed that 80% wealth in capitalist countries is handled by its 20% population. He generalized this idea as a Pareto principle. On the basis of this principle, he introduced the Pareto distribution as the distribution of wealth in a society. The density function of the Pareto distribution (for more details see [2]) is given by

$$f(x) = \frac{\theta k^{\theta}}{x^{\theta+1}},\ x \in [k, \infty),$$

where $k \in \mathbb{R}^{+}$ and $\theta \in \mathbb{R}^{+}$ are the scale and shape parameters respectively.

Merovci and Puka [3] have introduced transmuted Pareto (TP) distribution to solve the problems of financial mathematics by applying the quadratic transmutation approach given as

$$F(x) = (1+\lambda)G(x) - \lambda G^{2}(x),\ \lambda \in [-1, 1]. \tag{1}$$

The transmutation approach (1) handles the quadratic behavior in the data. Ansari and Eledum [4] have developed a cubic transmuted Pareto (for short, CTP_{AE}) distribution which turned out to be a member of general transmuted family of distributions, introduced by Rahman et al [5], for $\lambda_2 = -\lambda_1 = -\lambda$. Specifically, the CTP_{AE} distribution is a special case of the CTP distribution introduced by Rahman et al. [6], for $\lambda_2 = -\lambda_1 = -\lambda$.

Rahman et al. [1] have introduced another cubic transmuted family of distributions to capture complex behavior of the data. The *cdf* of this family of distributions has the form

$$F(x) = (1+\lambda_1+\lambda_2)G(x) - (\lambda_1+2\lambda_2)G^{2}(x) + \lambda_2 G^{3}(x),\ x \in \mathbb{R}, \tag{2}$$

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where $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [0, 1]$.

In this article, we have proposed another new CTP distribution by using cubic transmuted family given in (2). The rest of the article is structured as follows: A new CTP distribution is proposed in Section 2. Section 3 contains statistical properties of CTP distribution. The distribution of order statistics for CTP distribution is given in Section 4. Section 5 presents some characterizations results. Parameter estimation has been discussed in Section 6. In Section 7, extensive simulation study has been conducted to assess the performance of estimation method alongside two real-life applications of the proposed distribution to investigate its applicability. Finally, in Section 8, some concluding remarks are given.

2 Cubic Transmuted Pareto Distribution

We develop a new CTP distribution for capturing the complexity of data arising in finance and other areas of life as follows.

The *cdf* of the Pareto distribution is

$$G(x) = 1 - \left(\frac{k}{x}\right)^{\theta}, \; x \in [k, \infty), \tag{3}$$

where $k \in \mathbb{R}^+$ and $\theta \in \mathbb{R}^+$.

Merovci and Puka [3], have developed TP distribution, by using the transmuted family given in (1), which has the *cdf*

$$F(x) = \left[1 - \left(\frac{k}{x}\right)^{\theta}\right]\left[1 + \lambda\left(\frac{k}{x}\right)^{\theta}\right], \tag{4}$$

where $k \in \mathbb{R}^+$, $\theta \in \mathbb{R}^+$ and $\lambda \in [-1, 1]$.

The new CTP distribution is proposed by using the *cdf* of Pareto distribution, from (3), in the cubic transmuted family given in (2). The *cdf* of proposed CTP distribution is

$$F(x) = \left[1 - \left(\frac{k}{x}\right)^{\theta}\right]\left[1 + \left\{\lambda_1 + \lambda_2\left(\frac{k}{x}\right)^{\theta}\right\}\left(\frac{k}{x}\right)^{\theta}\right], \; x \in [k, \infty), \tag{5}$$

where $k \in \mathbb{R}^+$, $\theta \in \mathbb{R}^+$, $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [0, 1]$.

The *pdf* corresponding to (5) is

$$f(x) = \frac{\theta k^{\theta}}{x^{\theta+1}}\left[1 - \lambda_1 + 2(\lambda_1 - \lambda_2)\left(\frac{k}{x}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x}\right)^{2\theta}\right], \; x \in [k, \infty). \tag{6}$$

Some special cases of the proposed CTP distribution are listed below.

1. The *cdf* of CTP distribution given in (5), reduces to the *cdf* of TP distribution given in (4), for $\lambda_2 = 0$.
2. For $\lambda_1 = \lambda_2 = 0$, the *cdf* of CTP distribution given in (5) turns out to be the *cdf* of base Pareto distribution given in (3).

Some possible shapes for the *pdf* and *cdf* of the proposed CTP distribution are presented in Figure 1 for selected values of the model parameters θ and λ_2, setting $k = 1.5$ and $\lambda_1 = -1$.

3 Statistical Properties

In this section, we will present some statistical properties; such as moments, moment generating functions, characteristic function, quantile function, generating random sample, reliability function and Shannon entropy; of the proposed CTP distribution. These properties are given in the following subsections.

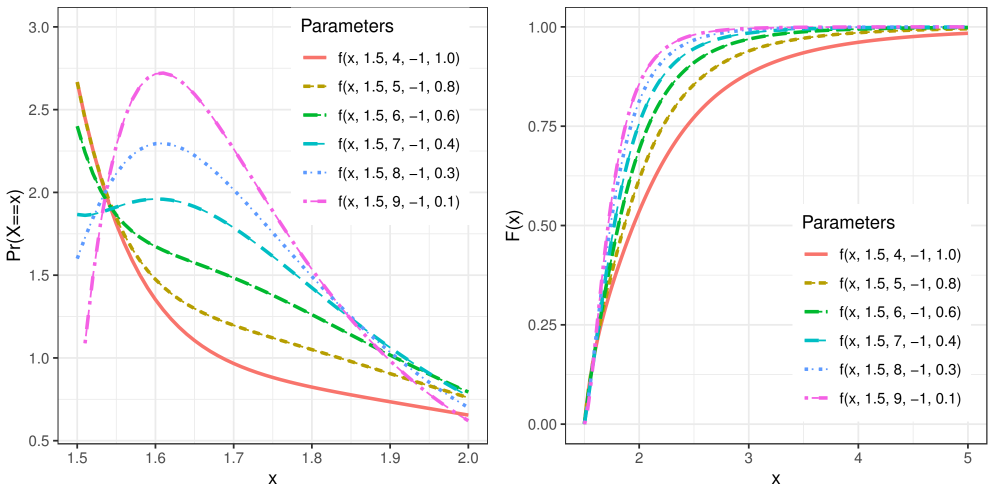

Figure 1: Density and distribution functions plots for the proposed CTP distribution

3.1 Moments

The moments are useful in studying behaviour of any probability distribution. The expression of rth moment for CTP distribution is given in the following theorem.

Theorem 1 *Suppose the random variable X follows the CTP distribution. Then the rth moment of X is then given as*

$$\mu'_r = E(X^r) = \frac{\theta k^r \left[(r-3\theta)(2\theta-\lambda_1 r - r) - \lambda_2 r(r-\theta)\right]}{(r-3\theta)(r-2\theta)(r-\theta)}, \ \theta > r. \tag{7}$$

The mean and variance, respectively, are

$$E(X) = \frac{\theta k \left[(1-3\theta)(2\theta-\lambda_1-1) - \lambda_2(1-\theta)\right]}{(1-3\theta)(1-2\theta)(1-\theta)}, \ \theta > 1,$$

and

$$\sigma^2 = V(X) = \theta k^2 \left[\frac{(2-3\theta)(2\theta-2\lambda_1-2) - 2\lambda_2(2-\theta)}{(2-3\theta)(2-2\theta)(2-\theta)} \right.$$
$$\left. - \frac{\theta\left\{(1-3\theta)(2\theta-\lambda_1-1)-\lambda_2(1-\theta)\right\}^2}{\left\{(1-3\theta)(1-2\theta)(1-\theta)\right\}^2}\right], \ \theta > 2.$$

Proof. The rth moment of a random variable is

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) \mathrm{d}x.$$

Table 1: Mean chart of the CTP distribution

			$\lambda_1=-1$	$\lambda_1=-0.5$	$\lambda_1=0$	$\lambda_1=0.5$	$\lambda_1=1$
$k=1$	$\theta=5$	$\lambda_2=0$	1.389	1.319	1.250	1.181	1.111
		$\lambda_2=0.5$	1.369	1.300	1.230	1.161	1.091
		$\lambda_2=1$	1.349	1.280	1.210	1.141	1.071
	$\theta=10$	$\lambda_2=0$	1.170	1.140	1.111	1.082	1.053
		$\lambda_2=0.5$	1.161	1.131	1.102	1.073	1.044
		$\lambda_2=1$	1.151	1.122	1.093	1.064	1.034
	$\theta=15$	$\lambda_2=0$	1.108	1.090	1.071	1.053	1.034
		$\lambda_2=0.5$	1.102	1.084	1.066	1.047	1.029
		$\lambda_2=1$	1.097	1.078	1.060	1.041	1.023
$k=5$	$\theta=5$	$\lambda_2=0$	6.944	6.597	6.250	5.903	5.556
		$\lambda_2=0.5$	6.845	6.498	6.151	5.804	5.456
		$\lambda_2=1$	6.746	6.399	6.052	5.704	5.357
	$\theta=10$	$\lambda_2=0$	5.848	5.702	5.556	5.409	5.263
		$\lambda_2=0.5$	5.803	5.656	5.510	5.364	5.218
		$\lambda_2=1$	5.757	5.611	5.465	5.319	5.172
	$\theta=15$	$\lambda_2=0$	5.542	5.450	5.357	5.265	5.172
		$\lambda_2=0.5$	5.512	5.420	5.328	5.235	5.143
		$\lambda_2=1$	5.483	5.391	5.298	5.206	5.114
$k=10$	$\theta=5$	$\lambda_2=0$	13.889	13.194	12.500	11.806	11.111
		$\lambda_2=0.5$	13.690	12.996	12.302	11.607	10.913
		$\lambda_2=1$	13.492	12.798	12.103	11.409	10.714
	$\theta=10$	$\lambda_2=0$	11.696	11.404	11.111	10.819	10.526
		$\lambda_2=0.5$	11.605	11.313	11.020	10.728	10.436
		$\lambda_2=1$	11.514	11.222	10.930	10.637	10.345
	$\theta=15$	$\lambda_2=0$	11.084	10.899	10.714	10.530	10.345
		$\lambda_2=0.5$	11.025	10.840	10.656	10.471	10.286
		$\lambda_2=1$	10.966	10.781	10.597	10.412	10.227

Using the density function of CTP distribution, from (6), and on simplification we have

$$\begin{aligned}
\mu_r' &= \int_k^\infty x^r \frac{\theta k^\theta}{x^{\theta+1}}\left[1-\lambda_1+2\left(\lambda_1-\lambda_2\right)\left(\frac{k}{x}\right)^\theta+3\lambda_2\left(\frac{k}{x}\right)^{2\theta}\right]\mathrm{d}x \\
&= \theta k^\theta(1-\lambda_1)\left\{-\frac{k^{r-\theta}}{r-\theta}\right\}+\theta k^{2\theta}2(\lambda_1-\lambda_2)\left\{-\frac{k^{r-2\theta}}{r-2\theta}\right\} \\
&\quad +\theta k^{3\theta}3\lambda_2\left\{-\frac{k^{r-3\theta}}{r-3\theta}\right\} \\
&= \theta k^r\left[\frac{\lambda_1-1}{r-\theta}-\frac{2(\lambda_1-\lambda_2)}{r-2\theta}-\frac{3\lambda_2}{r-3\theta}\right] \\
&= \frac{\theta k^r}{(r-3\theta)(r-2\theta)(r-\theta)}\big[\left(\lambda_1-1\right)(r-2\theta)(r-3\theta) \\
&\quad -2\left(\lambda_1-\lambda_2\right)(r-\theta)(r-3\theta)-3\lambda_2(r-\theta)(r-2\theta)\big] \\
&= \frac{\theta k^r\left[\lambda_2 r(\theta-r)-(r-3\theta)\left(-2\theta+\lambda_1 r+r\right)\right]}{(r-3\theta)(r-2\theta)(r-\theta)} \\
&= \frac{\theta k^r\left[(r-3\theta)\left(2\theta-\lambda_1 r-r\right)-\lambda_2 r(r-\theta)\right]}{(r-3\theta)(r-2\theta)(r-\theta)}. \qquad (8)
\end{aligned}$$

The mean can be readily obtained from (8) by using $r = 1$. The variance is obtained by using

$$V(X) = E(X^2) - \{E(X)\}^2,$$

where $E(X^r)$ for $i = 1, 2$ can be obtained from (8). The higher moments of CTP can also be obtained from (8). ■

Table 1 presents values of the mean of CTP distribution for various combination of model parameters.

3.2 Moment Generating Function

Moment generating function (MGF) is a useful function to obtain moments of a random variable. The MGF of CTP is given in the following theorem.

Theorem 2 *Suppose random variable X follows the CTP distribution. Then the MGF is*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\theta k^r \left[(r-3\theta)\left(2\theta - \lambda_1 r - r\right) - \lambda_2 r(r-\theta)\right]}{(r-3\theta)(r-2\theta)(r-\theta)}, \ \theta > r, \tag{9}$$

where $t \in \mathbb{R}$.

Proof. The MGF is defined as

$$M_X(t) = E[e^{tx}] = \int_k^{\infty} e^{tx} f(x) \mathrm{d}x.$$

Using series expansion of e^{tx} (see [8]), the moment generating function can be written as

$$M_x(t) = \int_k^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x) \mathrm{d}t = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r). \tag{10}$$

Now, using (7) in (10), we obtain (9). ■

3.3 Characteristic Function

The characteristic function of a real valued random variable completely defines its density function. The characteristic function of CTP distribution is given in the following theorem.

Theorem 3 *Suppose random variable X follows the CTP distribution. Then the characteristic function of X, $\phi_X(t)$, is given by*

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\theta k^r \left[(r-3\theta)\left(2\theta - \lambda_1 r - r\right) - \lambda_2 r(r-\theta)\right]}{(r-3\theta)(r-2\theta)(r-\theta)}, \ \theta > r,$$

where $i = \sqrt{-1}$ is the imaginary unit and $t \in \mathbb{R}$.

Proof. The proof is simple and hence omitted. ■

3.4 Quantile Function

The quantile function is obtained by inversely solving the *cdf* for x. Now, solving (5) for x the quantile function, x_q, of CTP is (see for example [6, 7])

$$x_q = k e^{-\frac{1}{\theta} ln(y)}, \tag{11}$$

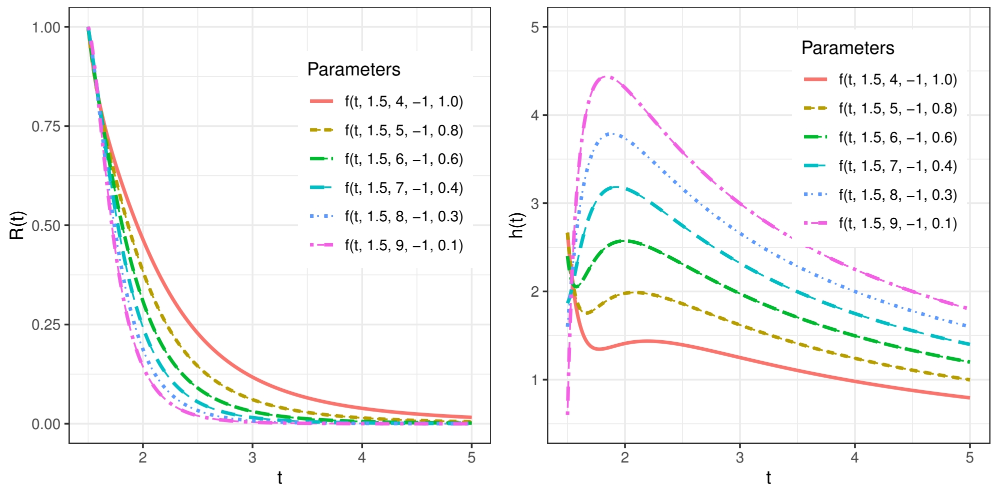

Figure 2: Reliability and hazard functions plots for the proposed CTP distribution

where

$$\left.\begin{array}{l} y = -\frac{b}{3a} - \frac{2^{1/3}\xi_1}{3a\left(\xi_2+\sqrt{4\xi_1^3+\xi_2^2}\right)^{1/3}} + \frac{\left(\xi_2+\sqrt{4\xi_1^3+\xi_2^2}\right)^{1/3}}{3(2^{1/3})a}, \\ \xi_1 = -b^2 + 3ac \geq 0,\ \xi_2 = -2b^3 + 9abc - 27a^2 d, \\ a = -\lambda_2,\ b = \lambda_2 - \lambda_1,\ c = \lambda_1 - 1 \text{ and } d = 1 - q. \end{array}\right\} \tag{12}$$

The lower quartile, median and upper quartile can be obtained by setting $q = 0.25$, 0.50 and 0.75 in (11), respectively.

3.5 Simulating the Random Sample

The random sample from CTP distribution can be easily generated by using its quantile function. Specifically a random observation from the CTP is

$$X = ke^{-\frac{1}{\theta}ln(Y)}, \tag{13}$$

where Y is given in (12). Now using $d = 1 - u$, where u is a uniform random number, we can obtain a random observation from the CTP distribution.

One can easily obtain random data from the proposed CTP distribution by applying (13) and using various combination of the model parameters k, θ, λ_1 and λ_2.

3.6 Reliability Analysis

The reliability function is expressed as $R(t) = 1 - F(t)$ and, for CTP distribution it is

$$R(t) = 1 - \left[1 - \left(\frac{k}{t}\right)^{\theta}\right]\left[1 + \left\{\lambda_1 + \lambda_2\left(\frac{k}{t}\right)^{\theta}\right\}\left(\frac{k}{t}\right)^{\theta}\right],\ t \in \mathbb{R}^+.$$

The hazard rate function is

$$h(t) = \frac{\frac{\theta k^{\theta}}{t^{\theta+1}}\left[1 - \lambda_1 + 2\left(\lambda_1 - \lambda_2\right)\left(\frac{k}{t}\right)^{\theta} + 3\lambda_2\left(\frac{k}{t}\right)^{2\theta}\right]}{1 - \left[1 - \left(\frac{k}{t}\right)^{\theta}\right]\left[1 + \left\{\lambda_1 + \lambda_2\left(\frac{k}{t}\right)^{\theta}\right\}\left(\frac{k}{t}\right)^{\theta}\right]},\ t \in \mathbb{R}^+.$$

Figure 2 describes some possible shapes for the reliability and hazard rate functions of the proposed CTP distribution for different combinations of the model parameters θ and λ_2 setting $k = 1.5$ and $\lambda_1 = -1$ respectively.

3.7 Shannon Entropy

Shannon [9] has defined the entropy H to measure the uncertainty of a random variable X and for the CTP distribution it is given as

$$\begin{aligned} H &= -E[log\{f(x)\}] \\ &= -E\left[log\left\{\frac{\theta k^\theta}{x^{\theta+1}}\left(1-\lambda_1+2\left(\lambda_1-\lambda_2\right)\left(\frac{k}{x}\right)^\theta+3\lambda_2\left(\frac{k}{x}\right)^{2\theta}\right)\right\}\right] \\ &= -(I_1+I_2), \end{aligned} \tag{14}$$

where

$$I_1 = E\left[log\left\{\frac{\theta k^\theta}{x^{\theta+1}}\right\}\right],$$

and

$$I_2 = E\left[log\left\{1-\lambda_1+2\left(\lambda_1-\lambda_2\right)\left(\frac{k}{x}\right)^\theta+3\lambda_2\left(\frac{k}{x}\right)^{2\theta}\right\}\right].$$

On further simplification and using the terms I_1 and I_2 in (14), Shannon entropy H can be expressed as

$$\begin{aligned} H = &-\frac{3(\theta+1)\lambda_1+(\theta+1)\lambda_2+6\theta\log\left(\theta k^\theta\right)-6(\theta+1)(\theta\log(k)+1)}{6\theta} \\ &-\int_k^\infty \frac{\theta k^\theta}{x^{\theta+1}}\left[1-\lambda_1+2\left(\lambda_1-\lambda_2\right)\left(\frac{k}{x}\right)^\theta+3\lambda_2\left(\frac{k}{x}\right)^{2\theta}\right] \\ &\times log\left[1-\lambda_1+2\left(\lambda_1-\lambda_2\right)\left(\frac{k}{x}\right)^\theta+3\lambda_2\left(\frac{k}{x}\right)^{2\theta}\right]dx, \end{aligned}$$

and can be obtained numerically.

4 Order Statistics

The *pdf* of the *rth* order statistic for the CTP distribution is

$$\begin{aligned} f_{X_{r:n}}(x) &= \frac{n!}{(r-1)!(n-r)!} \\ &\times\left[\frac{\theta k^\theta}{x^{\theta+1}}\left\{1-\lambda_1+2\left(\lambda_1-\lambda_2\right)\left(\frac{k}{x}\right)^\theta+3\lambda_2\left(\frac{k}{x}\right)^{2\theta}\right\}\right] \\ &\times\left[\left\{1-\left(\frac{k}{x}\right)^\theta\right\}\left\{1+\left(\lambda_1+\lambda_2\left(\frac{k}{x}\right)^\theta\right)\left(\frac{k}{x}\right)^\theta\right\}\right]^{r-1} \\ &\times\left[1-\left\{1-\left(\frac{k}{x}\right)^\theta\right\}\left\{1+\left(\lambda_1+\lambda_2\left(\frac{k}{x}\right)^\theta\right)\left(\frac{k}{x}\right)^\theta\right\}\right]^{n-r}, \end{aligned} \tag{15}$$

where $r = 1, 2, \cdots, n$. The *pdf* of the smallest order statistics for CTP distribution is easily obtained by using $r = 1$ and is

$$f_{X_{1:n}}(x) = \frac{n\theta k^{\theta}}{x^{\theta+1}} \left\{1 - \lambda_1 + 2(\lambda_1 - \lambda_2)\left(\frac{k}{x}\right)^{\theta} + 3\lambda_2 \left(\frac{k}{x}\right)^{2\theta}\right\}$$
$$\times \left[1 - \left\{1 - \left(\frac{k}{x}\right)^{\theta}\right\}\left\{1 + \left(\lambda_1 + \lambda_2 \left(\frac{k}{x}\right)^{\theta}\right)\left(\frac{k}{x}\right)^{\theta}\right\}\right]^{n-1}.$$

Further, using $r = n$, the *pdf* of the largest order statistic $X_{n:n}$, is given by

$$f_{X_{n:n}}(x) = \frac{n\theta k^{\theta}}{x^{\theta+1}} \left\{1 - \lambda_1 + 2(\lambda_1 - \lambda_2)\left(\frac{k}{x}\right)^{\theta} + 3\lambda_2 \left(\frac{k}{x}\right)^{2\theta}\right\}$$
$$\times \left[\left\{1 - \left(\frac{k}{x}\right)^{\theta}\right\}\left\{1 + \left(\lambda_1 + \lambda_2 \left(\frac{k}{x}\right)^{\theta}\right)\left(\frac{k}{x}\right)^{\theta}\right\}\right]^{n-1}.$$

The density function of rth order statistics for Pareto distribution is readily obtained from (15) for $\lambda_1 = \lambda_2 = 0$, and is

$$g_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!} \frac{n\theta k^{\theta}}{x^{\theta+1}} \left[1 - \left(\frac{k}{x}\right)^{\theta}\right]^{r-1} \left[\left(\frac{k}{x}\right)^{\theta}\right]^{n-r}, r = 1, 2, \cdots, n.$$

The kth order moment of $X_{r:n}$ for the CTP distribution is obtained by using

$$E(X_{r:n}^{k}) = \int_{k}^{\infty} x_r^k \cdot f_{X_{r:n}}(x) \cdot \mathrm{d}x,$$

where $f_{X_{r:n}}(x)$ is given in (15).

5 Characterization Results

This section is devoted to the characterizations of the CTP distribution in different directions: (i) based on the ratio of two truncated moments; (ii) in terms of the hazard function; (iii) in terms of the reverse hazard function and (iv) based on the conditional expectation of certain function of the random variable. Note that (i) can be employed also when the cdf does not have a closed form. We present our characterizations (i) - (iv) in four subsections.

5.1 Characterizations Based on Two Truncated Moments

This subsection deals with the characterizations of the CTP distribution based on the ratio of two truncated moments. Our first characterization employs a theorem due to Glänzel [13], see Theorem 4 of Appendix A. The result, however, holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

Proposition 1 *Let $X : \Omega \to (k, \infty)$ be a continuous random variable and let $q_1(x) = \Big[1 - \lambda_1 + 2(\lambda_1 - \lambda_2) \times \left(\frac{k}{x}\right)^{\theta} + 3\lambda_2 \left(\frac{k}{x}\right)^{2\theta}\Big]$ and $q_2(x) = x^{-1} q_1(x)$ for $x > k$. The random variable X has pdf* (6) *if and only if the function η defined in Theorem 4 is of the form*

$$\eta(x) = \frac{\theta}{\theta+1} x^{-1}, \quad x > k.$$

Proof. Suppose the random variable X has *pdf* (6). Then

$$(1 - F(x)) E[q_1(X) \mid X \geq x] = \left(\frac{k}{x}\right)^{\theta}, \quad x > k,$$

and

$$(1 - F(x)) E[q_2(X) \mid X \geq x] = \frac{\theta}{\theta + 1} x^{-1} \left(\frac{k}{x}\right)^{\theta}, \quad x > k.$$

Further,

$$\eta(x) q_1(x) - q_2(x) = -\frac{1}{\theta + 1} x^{-1} q_1(x) < 0 , \quad for \quad x > k.$$

Conversely, if η is of the above form, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \theta x^{-1}, \quad x > k,$$

and consequently

$$s(x) = \log\left\{x^{\theta}\right\}, \quad x > k.$$

Now, according to Theorem 4, X has density (6). ■

Corollary 1 *Let $X : \Omega \rightarrow (k, \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 1. The random variable X has pdf (6) if and only if there exist functions q_2 and η defined in Theorem 4 satisfying the following differential equation*

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \theta x^{-1}, \quad x > k.$$

Corollary 2 *The general solution of the differential equation in Corollary 1 is*

$$\eta(x) = x^{\theta} \left[-\int \theta x^{-(\theta+1)} (q_1(x))^{-1} q_2(x) \, dx + D\right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 1 with $D = 0$. Clearly, there are other triplets (q_1, q_2, η) which satisfy conditions of Theorem 4.

5.2 Characterization in Terms of Hazard Function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of CTP distribution, for $\lambda = 1$, $\lambda_2 = 0$, in terms of the hazard function.

Proposition 2 *Let $X : \Omega \rightarrow (0, k)$ be a continuous random variable. The random variable X has pdf (6) if and only if its hazard function $h_F(x)$ satisfies the following differential equation*

$$h'_F(x) + x^{-1} h_F(x) = 0, \quad x > k.$$

Proof. Is straightforward and hence omitted. ■

5.3 Characterization in Terms of the Reverse Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in support\ of\ F.$$

In this subsection we present a characterization of CTP distribution, for $\lambda = -1$, $\lambda_2 = 0$, in terms of the reverse hazard function.

Proposition 3 *Let $X : \Omega \to (0, k)$ be a continuous random variable. The random variable X has pdf (6) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation*

$$r_F'(x) + (\theta + 1)\, x^{-1} r_F(x) = -2\theta^2 k^{2\theta} x^{-2(\theta+1)} \left[1 - \left(\frac{k}{x}\right)^{\theta}\right]^{-2}, \; x > k.$$

Proof. Is straightforward and hence omitted. ■

5.4 Characterization Based on the Conditional Expectation of Certain Function of the Random Variable

In this subsection we employ a single function ψ (or ψ_1) of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$ (or $\psi_1(X)$). The following propositions have already appeared in Hamedani's previous work [14], so we will just state them here which can be used to characterize some of the CTP distribution for the special cases: $\lambda = 1$, $\lambda_2 = 0$ and $\lambda = -1$, $\lambda_2 = 0$, respectively.

Proposition 4 *Let $X : \Omega \to (e, f)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (e, f) with $\lim_{x \to e^+} \psi(x) = 1$. Then for $\delta \neq 1$,*

$$E[\psi(X) \mid X \geq x] = \delta \psi(x), \qquad x \in (e, f),$$

if and only if

$$\psi(x) = (1 - F(x))^{\frac{1}{\delta} - 1}, \qquad x \in (e, f).$$

Proposition 5 *Let $X : \Omega \to (e, f)$ be a continuous random variable with cdf F. Let $\psi_1(x)$ be a differentiable function on (e, f) with $\lim_{x \to f^-} \psi_1(x) = 1$. Then for $\delta_1 \neq 1$,*

$$E[\psi_1(X) \mid X \leq x] = \delta_1 \psi_1(x), \qquad x \in (e, f),$$

implies

$$\psi_1(x) = (F(x))^{\frac{1}{\delta_1} - 1}, \qquad x \in (e, f).$$

Remarks (A) For $(e, f) = (k, \infty)$, $\lambda = 1$, $\lambda_2 = 0$, $\psi(x) = \frac{k}{x}$ (or $\psi(x) = \left(\frac{k}{x}\right)^2$) and $\delta = \frac{2\theta}{2\theta+1}$ (or $\delta = \frac{\theta}{\theta+1}$) , Proposition 4 provides a characterization of CTP distribution. (B) For $(e, f) = (k, \infty)$, $\lambda = -1$, $\lambda_2 = 0$, $\psi_1(x) = 1 - \left(\frac{k}{x}\right)^{\theta}$ and $\delta_1 = \frac{2}{3}$, Proposition 5 provides a characterization of CTP distribution.

6 Parameter Estimation and Inference

The most popular method of estimating the parameters is the maximum likelihood estimation (MLE). In this section, we have discussed the maximum likelihood estimation of parameters for CTP distribution. For this, we first see that the likelihood function for the CTP distribution is

$$L = \frac{\theta^n k^{n\theta}}{\prod_{i=1}^{n} x_i^{\theta+1}} \prod_{i=1}^{n} \left[1 - \lambda_1 + 2(\lambda_1 - \lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2 \left(\frac{k}{x_i}\right)^{2\theta}\right].$$

The log-likelihood function is

$$\begin{aligned} l &= n \cdot ln(\theta) + n\theta \cdot ln(k) - (\theta+1)\sum_{i=1}^{n} ln(x_i) \\ &+ \sum_{i=1}^{n} ln\left[1-\lambda_1+2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}+3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}\right]. \end{aligned} \tag{16}$$

Since, $x \in [k, \infty)$, so the MLE of k is first-order statistic $x_{(1)}$. The parameters θ, λ_1 and λ_2 are determined by maximizing (16). The derivatives of (16) with respect to the unknown model parameters are

$$\begin{aligned} \frac{\delta l}{\delta \theta} &= \frac{n}{\theta} + n\log(k) - \sum_{i=i}^{n}\log(x_i) \\ &+ \sum_{i=1}^{n} \frac{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}\log\left(\frac{k}{x_i}\right)+6\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}\log\left(\frac{k}{x_i}\right)}{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}+3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}-\lambda_1+1}, \end{aligned}$$

$$\frac{\delta l}{\delta \lambda_1} = \sum_{i=1}^{n} \frac{2\left(\frac{k}{x_i}\right)^{\theta}-1}{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}+3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}-\lambda_1+1}$$

and

$$\frac{\delta l}{\delta \lambda_2} = \sum_{i=1}^{n} \frac{3\left(\frac{k}{x_i}\right)^{2\theta}-2\left(\frac{k}{x_i}\right)^{\theta}}{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}+3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}-\lambda_1+1}.$$

Further, setting $\frac{\delta l}{\delta \theta}=0$, $\frac{\delta l}{\delta \lambda_1}=0$ and $\frac{\delta l}{\delta \lambda_2}=0$, and solving the resulting nonlinear system of equations simultaneously gives the MLE, $\hat{\Theta} = \left(\hat{\theta}, \hat{\lambda_1}, \hat{\lambda_2}\right)'$ of $\Theta = (\theta, \lambda_1, \lambda_2)'$. Also as $n \to \infty$, the asymptotic distribution of the $MLE's$ $\left(\hat{\theta}, \hat{\lambda_1}, \hat{\lambda_2}\right)$ is given as, see for example [10, 11, 12],

$$\begin{pmatrix} \hat{\theta} \\ \hat{\lambda_1} \\ \hat{\lambda_2} \end{pmatrix} \sim N\left[\begin{pmatrix} \theta \\ \lambda_1 \\ \lambda_2 \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix}\right],$$

where $\hat{V}_{ij} = V_{ij}|_{\Theta=\hat{\Theta}}$. The asymptotic variance-covariance matrix V, for the estimates $\hat{\theta}$, $\hat{\lambda_1}$ and $\hat{\lambda_2}$ is obtained by inverting Hessian matrix; see Appendix B. An approximate $100(1-\alpha)\%$ two sided confidence intervals for θ, λ_1 and λ_2 are, respectively, given as

$$\hat{\theta} \pm Z_{\alpha/2}\sqrt{\hat{V}_{11}}, \quad \hat{\lambda}_1 \pm Z_{\alpha/2}\sqrt{\hat{V}_{22}} \quad \text{and} \quad \hat{\lambda}_2 \pm Z_{\alpha/2}\sqrt{\hat{V}_{33}},$$

where Z_α is the αth percentile of the standard normal distribution.

7 Numerical Studies

In this section, a Monte Carlo simulation study is carried out to assess the performance of the estimation procedure. We have also fitted the proposed CTP on two real data sets for comparison purpose.

7.1 Simulation Study

The Monte Carlo simulation study has been conducted by drawing random samples of sizes 50, 100, 200, 500 and 1000 from the proposed CTP distribution for $k = 1$, $\theta = 3.5$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.7$. Assuming $k = x_{(1)}$, the maximum likelihood estimates of the model parameters θ, λ_1 and λ_2 was obtained. We repeated the whole procedure for 10000 times and have computed the average value of estimates alongside their mean square errors ($MSE's$). Table 2 presents the results of this simulation study. The results show that the estimated values of the model parameters are very close to their true values. From the table, we have also observed that the estimated $MSE's$ consistently decreases by increasing the sample sizes, hence establishing consistency of the estimation procedure.

Table 2: Average estimates of model parameters and corresponding $MSE's$

Sample	Estimate				MSE			
Size	k	θ	λ_1	λ_2	k	θ	λ_1	λ_2
50	1.003	4.044	0.358	0.772	1.52×10^{-5}	3.264	0.273	0.963
100	1.001	3.844	0.381	0.794	2.63×10^{-6}	1.763	0.159	0.462
200	1.001	3.697	0.425	0.776	1.45×10^{-6}	0.691	0.075	0.152
500	1.000	3.591	0.458	0.751	8.83×10^{-8}	0.242	0.027	0.052
1000	1.000	3.538	0.482	0.721	1.50×10^{-8}	0.119	0.013	0.023

7.2 ICU Data Set

We use the ICU data set to show the flexibility and applicability of the proposed CTP distribution. The data set has recently been used by Khan et al. [15], which assess intensive care unit (ICU) patients agitation-sedation (A-S) status. The values are 9, 3, 27, 8, 4, 3, 4, 3, 23, 3, 3, 4, 28, 18, 19, 6, 3, 26, 3, 12, 6, 9, 43, 4, 4, 3, 5, 12, 4, 36, 6, 8, 6, 5, 3, 3 and 33. Table 3 describes the summary statistics for the data set.

Table 3: Summary statistics for selected data sets

	Min.	Q_1	Median	Mean	Q_3	Max.
ICU Data Set	3.00	3.00	6.00	10.78	12.00	43.00
Leukemia Data Set	1.00	4.50	11.00	13.60	18.50	44.00

We have considered CTP_{AE} distribution, TP distribution and baseline Pareto distribution for assessing the performance of proposed CTP distribution. Table 4 describes the estimated $MLE's$ of the model parameters with their standard errors. Figure 3 (top), shows the estimated *pdf* and *cdf* for the ICU data set. The results of various selection criteria; like log-likelihood (LL), Akaike's information criterion (AIC), corrected Akaike's information criterion ($AICc$) and Bayesian information criterion (BIC); are presented in Table 5. We have investigated the results carefully and have observed that the proposed CTP distribution perform better than other models used in this study.

7.3 Leukemia Data Set

The data set represents the remission times, in weeks, for 35 leukemia patients and is 1, 3, 3, 6, 7, 7, 10, 12, 14, 15, 18, 19, 22, 26, 29, 34, 40, 1, 1, 2, 2, 3, 4, 5, 8, 8, 9, 11, 12, 14, 16, 18, 21, 31 and 44, taken from Lawless (p.346) [16]. Summary statistics for the data are given in Table 3.

The estimated values of the model parameters along with corresponding standard errors for CTP_{AE}, TP, Pareto and proposed CTP distributions are presented in Table 6. Estimated *pdf* and *cdf* of the proposed

Table 4: $MLE's$ of the parameters and respective $SE's$ for selected models

Distribution	Parameter	Estimate	SE
CTP	k	$x_{(1)} = 3.000$	–
	θ	1.342	0.289
	λ_1	-0.600	0.657
	λ_2	1.000	0.781
CTP_{AE}	x_0	$x_{(1)} = 3.000$	–
	α	1.108	0.199
	λ	0.848	0.444
TP	k	$x_{(1)} = 3.000$	–
	θ	1.041	0.261
	λ	0.211	0.346
Pareto	k	$x_{(1)} = 3.000$	–
	θ	1.147	0.189

Table 5: Selection criteria estimated for selected models

Distribution	LL	AIC	$AICc$	BIC
CTP	-101.150	208.300	209.027	213.133
CTP_{AE}	-102.652	209.304	209.657	212.525
TP	-104.622	213.244	213.597	216.466
Pareto	-104.841	211.681	211.795	213.292

Table 6: $MLE's$ of the parameters and respective $SE's$ for selected models

Distribution	Parameter	Estimate	SE
CTP	k	$x_{(1)} = 1.000$	–
	θ	0.670	0.122
	λ_1	-1.000	0.871
	λ_2	0.518	0.954
CTP_{AE}	x_0	$x_{(1)} = 1.000$	–
	α	0.469	0.070
	λ	-0.578	0.231
TP	k	$x_{(1)} = 1.000$	–
	θ	0.606	0.087
	λ	-0.696	0.171
Pareto	k	$x_{(1)} = 1.000$	–
	θ	0.460	0.078

Table 7: Selection criteria estimated for selected models

Distribution	LL	AIC	$AICc$	BIC
CTP	-133.318	272.637	273.411	277.303
CTP_{AE}	-136.370	276.740	277.115	279.850
TP	-134.937	273.874	274.249	276.985
Pareto	-138.235	278.470	278.591	280.026

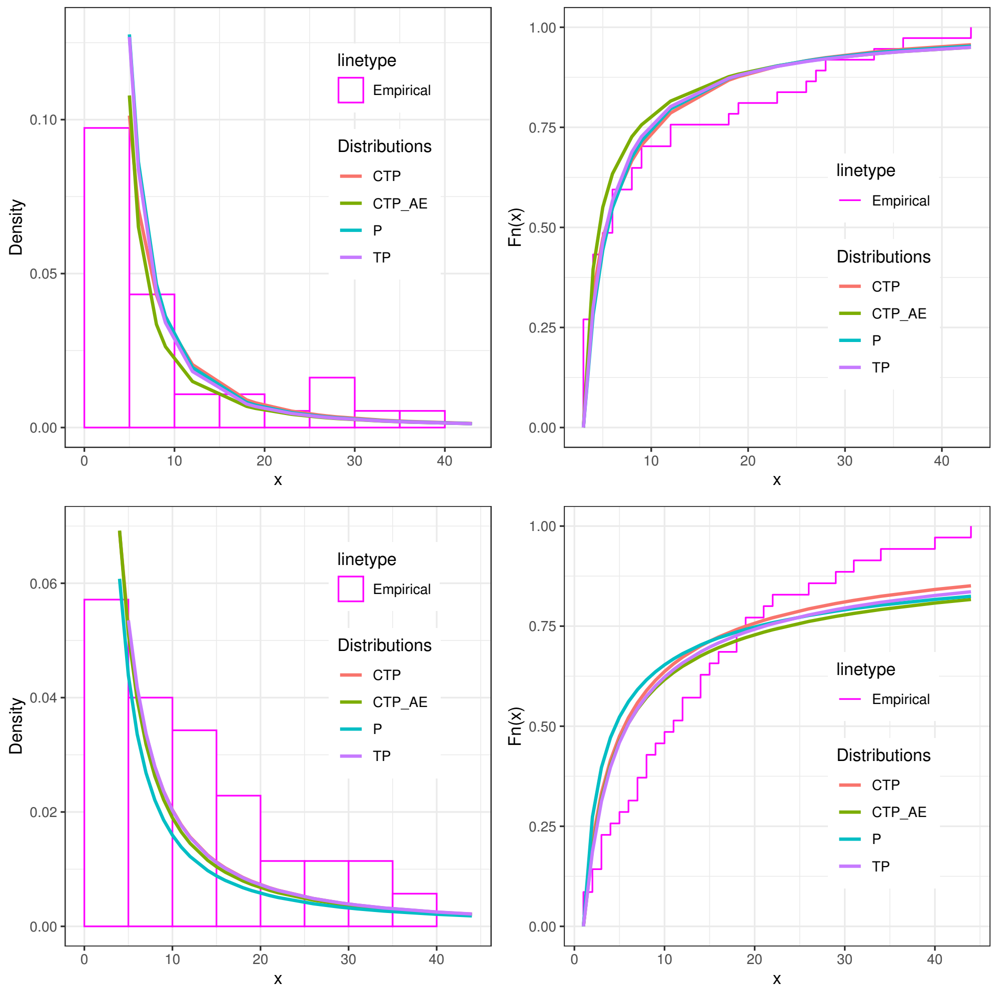

Figure 3: Estimated *pdf* and *cdf* for the ICU data (top) and Leukemia data (bottom)

CTP distribution for the leukemia data set are presented in Figure 3 (bottom). The results for above mentioned selection criteria like LL, AIC, $AICc$ and BIC values are described in Table 7. We observed from the results that the criterion's provides conformation of better fit in favor of proposed CTP distribution.

8 Concluding Remarks

In this paper, we have proposed CTP distribution for capturing the complexity of the data. The expressions for the moments, moment generating function, characteristic function, quantile function, random data generation, reliability function, Shannon entropy for the proposed CTP have been obtained alongside the distributions of order statistics. The model parameters have been estimated via MLE technique. The proposed model has been fitted on two real data sets and is compared with different other models. We have observed that our proposed CTP distribution fits the data well as compared with the other models used in the study.

Appendix A

Theorem 4 *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a given probability space and let* $H = [a, b]$ *be an interval for some* $d < b$ $(a = -\infty,\ b = \infty$ *might as well be allowed*$)$. *Let* $X : \Omega \to H$ *be a continuous random variable with the distribution function* F *and let* q_1 *and* q_2 *be two real functions defined on* H *such that*

$$\mathbf{E}\left[q_2\left(X\right)\ \mid X \geq x\right] = \mathbf{E}\left[q_1\left(X\right)\ \mid X \geq x\right]\eta\left(x\right), \quad x \in H,$$

is defined with some real function η. *Assume that* $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ *and* F *is twice continuously differentiable and strictly monotone function on the set* H. *Finally, assume that the equation* $\eta q_1 = q_2$ *has no real solution in the interior of* H. *Then* F *is uniquely determined by the functions* q_1, q_2 *and* η, *particularly*

$$F\left(x\right) = \int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right)q_1\left(u\right) - q_2\left(u\right)}\right|\exp\left(-s\left(u\right)\right)\ du\ ,$$

where the function s *is a solution of the differential equation* $s' = \frac{\eta'\ q_1}{\eta q_1 - q_2}$ *and* C *is the normalization constant, such that* $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel [17]), in particular, let us assume that there is a sequence $\{X_n\}$ *of random variables with distribution functions* $\{F_n\}$ *such that the functions* q_{1n}, q_{2n} *and* η_n $(n \in \mathbb{N})$ *satisfy the conditions of Theorem 4 and let* $q_{1n} \to q_1$, $q_{2n} \to q_2$ *for some continuously differentiable real functions* q_1 *and* q_2. *Let, finally,* X *be a random variable with distribution* F. *Under the condition that* $q_{1n}(X)$ *and* $q_{2n}(X)$ *are uniformly integrable and the family* $\{F_n\}$ *is relatively compact, the sequence* X_n *converges to* X *in distribution if and only if* η_n *converges to* η, *where*

$$\eta\left(x\right) = \frac{E\left[q_2\left(X\right) \mid\ X \geq x\right]}{E\left[q_1\left(X\right) \mid X \geq x\right]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1 , q_2 *and* η, *respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if* $\alpha \to \infty$.

A further consequence of the stability property of Theorem 4 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 *and, specially,* η *should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose* η *as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.*

Appendix B

The Hessian matrix is given as

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},$$

where the variance-covariance matrix V is obtained by

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}^{-1},$$

with the elements of Hessian matrix are obtained as

$$H_{11} = -\frac{\delta^2 l}{\delta\theta^2} = \frac{n}{\theta^2} - \sum_{i=1}^{n}\left[\frac{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}\log^2\left(\frac{k}{x_i}\right) + 12\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}\log^2\left(\frac{k}{x_i}\right)}{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1}\right.$$
$$\left. - \frac{\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}\log\left(\frac{k}{x_i}\right) + 6\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}\log\left(\frac{k}{x_i}\right)\right\}^2}{\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1\right\}^2}\right],$$

$$H_{12} = -\frac{\delta^2 l}{\delta\theta\cdot\delta\lambda_1} = -\sum_{i=1}^{n}\left[\frac{2\left(\frac{k}{x_i}\right)^{\theta}\log\left(\frac{k}{x_i}\right)}{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1}\right.$$
$$\left. - \frac{\left\{2\left(\frac{k}{x_i}\right)^{\theta} - 1\right\}\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}\log\left(\frac{k}{x_i}\right) + 6\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}\log\left(\frac{k}{x_i}\right)\right\}}{\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1\right\}^2}\right],$$

$$H_{13} = -\frac{\delta^2 l}{\delta\theta\cdot\delta\lambda_2} = -\sum_{i=1}^{n}\left[\frac{6\left(\frac{k}{x_i}\right)^{2\theta}\log\left(\frac{k}{x_i}\right) - 2\left(\frac{k}{x_i}\right)^{\theta}\log\left(\frac{k}{x_i}\right)}{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1}\right.$$
$$\left. - \frac{\left\{3\left(\frac{k}{x_i}\right)^{2\theta} - 2\left(\frac{k}{x_i}\right)^{\theta}\right\}\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta}\log\left(\frac{k}{x_i}\right) + 6\lambda_2\left(\frac{k}{x_i}\right)^{2\theta}\log\left(\frac{k}{x_i}\right)\right\}}{\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1\right\}^2}\right],$$

$$H_{22} = -\frac{\delta^2 l}{\delta\lambda_1^2} = \sum_{i=1}^{n}\frac{\left\{2\left(\frac{k}{x_i}\right)^{\theta} - 1\right\}^2}{\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1\right\}^2},$$

$$H_{23} = -\frac{\delta^2 l}{\delta\lambda_1\cdot\delta\lambda_2} = \sum_{i=1}^{n}\frac{\left\{2\left(\frac{k}{x_i}\right)^{\theta} - 1\right\}\left\{3\left(\frac{k}{x_i}\right)^{2\theta} - 2\left(\frac{k}{x_i}\right)^{\theta}\right\}}{\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1\right\}^2},$$

and

$$H_{33} = -\frac{\delta^2 l}{\delta\lambda_2^2} = \sum_{i=1}^{n}\frac{\left\{3\left(\frac{k}{x_i}\right)^{2\theta} - 2\left(\frac{k}{x_i}\right)^{\theta}\right\}^2}{\left\{2(\lambda_1-\lambda_2)\left(\frac{k}{x_i}\right)^{\theta} + 3\lambda_2\left(\frac{k}{x_i}\right)^{2\theta} - \lambda_1 + 1\right\}^2}.$$

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