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Robust Numerical Scheme For Solving Singularly Perturbed Differential Equations Involving Small Delays*

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Abstract

In this article, we consider singularly perturbed differential equation containing delay parameter on the convection and reaction terms. The considered problem exhibits boundary layer on the left or right side of the domain, depending on the sign of the coefficient of convective term. The terms with the delay treated using Taylor's approximation. The resulting singularly perturbed boundary value problem is solved using the technique of non-standard finite difference method. The stability of the scheme is analyzed and investigated using maximum principle and solution bound. The formulated scheme converges independent of the perturbation parameter with order of convergence $O(N^{-1})$. The theoretical finding is validated using numerical examples. The obtained result in this article is accurate and parameter uniformly convergent.

1 Introduction

Different mathematical models in science and engineering (such as in control theory, epidemiology, laser optics) take into account not only the present state of a physical system but also it includes the past history. Time delays are natural components of the dynamic processes of biology, ecology, physiology, economics, epidemiology and mechanics [5] and 'to ignore them is to ignore reality'[2]. Some modelers ignore the lag effect and use differential equation model as substitute for delay differential equation model. Kuang ([10], pp. 11) comments on the dangers that researchers risk if they ignore lags (delays) which they think are small.

Delay differential equations (DDEs) model problems where there is after effect affecting the variable of the problem as compared to differential equations which model the problem to current conditions. DDEs is said to be retarded type if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral DDEs. A singularly perturbed delay differential equations is differential equations in which its highest order derivative is multiplied by small perturbation parameter and having delay parameter(s) on the terms different from the highest order derivative. Singularly perturbed DDEs arise in the mathematical modeling of various physical phenomena, for example in micro scale heat transfer [17], fluid dynamics [7], diffusion in polymers [12], reaction-diffusion equations [3], a lot of model in diseases or physiological processes [13, 18] etc.

Notations: In this paper, N is denoted for the number of mesh intervals. The symbol C is denoted for positive constant independent of ε and N. The norm $\|\cdot\|$ is used to denote maximum norm i.e. $\|f\| = \max_x \|f(x)\|$.

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2 Description of the Problem

Singularly perturbed delay differential equations having delay in the convection and reaction terms of the problem have the form

$$-\varepsilon u''(x) + a(x)u'(x-\delta) + \beta(x)u(x) + \omega(x)u(x-\delta) = f(x), x \in \Omega = (0,1), \tag{1}$$

with interval-boundary conditions

$$u(x) = \phi(x), \;\; x \in \Omega_L = [-\delta, 0], \;\; u(1) = \psi, \tag{2}$$

where $\varepsilon, 0 < \varepsilon \ll 1$ is singular perturbation parameter and δ is delay parameter satisfying $\delta = o(\varepsilon)$. The functions $a(x)$, $\beta(x)$, $\omega(x)$ and $f(x)$ are assumed to be smooth, bounded and not a function of ε for guaranteeing the existence of unique solution.

We assume $\beta(x) + \omega(x) \geq \zeta > 0, \forall x \in \bar{\Omega}$ to ensure problem in (1)–(2) exhibits regular boundary layer of thickness $O(\varepsilon)$ and the position of the boundary layer depends on the conditions: For $a(x) - \delta\omega(x) < 0$ left boundary layer exist and for $a(x) - \delta\omega(x) > 0$ right boundary layer exist. In case $a(x) - \delta\omega(x)$ changes sign interior layer occurs. When $\delta = 0$, the problem reduces to singularly perturbed BVPs, in which for small ε it exhibits boundary layers. The layer is maintained for $\delta \neq 0$ but sufficiently small [20].

It is well known that classical numerical methods are inefficient and the computed solution oscillates or diverges as the perturbation parameter $\varepsilon \ll h$, where h is the discretization mesh size [4]. To avoid the oscillations while using classical methods, an unacceptably large number of mesh points are required when ε is very small. This is not practical and leads to rounding error. Therefore, to overcome this drawback associated with classical numerical methods, we developed scheme using non-standard finite difference method, which treat the problem without creating oscillations.

On the review paper by Kadelajoo and Gupta [8] one can find a number of papers dealing with the numerical solution of singularly perturbed BVPs, singularly perturbed problems having delay on the convection or reaction term only. Singularly perturbed differential equations having delay on both the convection and reaction terms are not studied well. To review the numerical schemes developed for solving such problem so far; Kumar and Kadalbajoo in [11] considered a singularly perturbed problem having delays on the convection and reaction terms. The authors used Taylor series approximation for the delay terms and converted the problem into equivalent BVPs. The authors computed the numerical solution using B-spline collocation method on Shishkin mesh. In [1, 6] the authors used Taylor's series approximation for the delay terms and apply fifth and sixth order finite difference approximation for the derivative terms and develop finite difference scheme.

In this paper, we developed uniformly convergent numerical scheme (or robust) using non-standard finite difference method for solving singularly perturbed delay differential equations. In addition, we developed the parameter uniform convergence analysis of the scheme.

When the delay parameter is smaller than the perturbation parameter, treating the delay terms using Taylor's series approximation is acceptable [16]. So, we approximate $u'(x-\delta)$ and $u(x-\delta)$ as

$$\begin{cases} u'(x-\delta) \approx u'(x) - \delta u''^{2}), \\ u(x-\delta) \approx u(x) - \delta u'(x) + \frac{\delta^2}{2} u''^{3}). \end{cases} \tag{3}$$

Substituting the approximations in (3) into (1) results to

$$Lu(x) = -c_\varepsilon(x)u''(x) + p(x)u'(x) + d(x)u(x) = f(x), \;\; x \in (0,1), \tag{4}$$

with the boundary conditions

$$u(0) = \phi(0), \;\; u(1) = \psi, \tag{5}$$

where $c_\varepsilon(x) = \varepsilon + \delta a(x) - \frac{\delta^2}{2}\omega$, $p(x) = a(x) - \delta\omega(x)$ and $d(x) = \beta(x) + \omega(x)$.

Since $\delta = o(\varepsilon)$ which implies that $O(\delta^2) \to 0$. For small ε the problems in (4)–(5) is asymptotically equivalent to (1)–(2). Now, we assume $0 < c_\varepsilon(x) \leq \varepsilon + \delta M_1 - \delta^2 M_2 = c_\varepsilon$ where $a(x) \geq M_1$ and $\omega(x) \geq 2M_2$

for M_1 and M_2 are constants. We consider first the case $p(x) \geq p^* > 0$, which implies the existence of right boundary, the other case $p(x) \leq p^* < 0$, implies to left boundary layer and can be treated similarly.

Setting $c_\varepsilon = 0$ in (4)–(5), we obtain

$$\begin{aligned} p(x)u_0'(x) + d(x)u_0(x) &= f(x), \ \forall x \in \Omega, \\ u_0(0) &= \phi(0). \end{aligned} \tag{6}$$

It is called reduced problem. For small values of c_ε the solution $u(x)$ of the problem in (4)–(5) is very close to the solution $u_0(x)$ of (6).

2.1 Properties of the Continuous Solution

Lemma 1 *Let z be a sufficiently smooth function defined on Ω which satisfies $z(x) \geq 0, \ x \in \{0,1\}$. Then $Lz(x) > 0, \ \forall x \in \Omega$ implies that $z(x) \geq 0, \ \forall x \in \bar{\Omega}$.*

Proof. Let x^* be such that $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$ and suppose that $z(x^*) < 0$. It is clear that $x^* \notin \{0,1\}$. Since $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$ from extrema values in calculus we have $z'(x^*) = 0$ and $z''(x^*) \geq 0$ and implies that $Lz(x^*) < 0$ which is contradiction to the assumption that made above $Lz(x^*) > 0, \ \forall x \in \Omega$. Therefore $z(x) \geq 0, \ \forall x \in \bar{\Omega}$. ∎

Lemma 2 *Let $u(x)$ be the solution of the problem in (4)–(5). Then we obtain the bound*

$$|u(x)| \leq \frac{\|f\|}{\zeta} + \max\{|\phi|, |\psi|\},$$

for $d(x) \geq \zeta > 0$, where ζ is lower bound of $d(x)$.

Proof. Defining barrier functions $\vartheta_\pm(x,t)$ as $\vartheta_\pm(x,t) = \frac{\|f\|}{\zeta} + \max\{|\phi|, |\psi|\} \pm u(x)$ and applying the maximum principle, we obtain the required bound. At the boundary points,

$$\vartheta_\pm(0) = \frac{\|f\|}{\zeta} + \max\{|\phi|, |\psi|\} \pm u(0) \geq 0,$$

$$\vartheta_\pm(1) = \frac{\|f\|}{\zeta} + \max\{|\phi|, |\psi|\} \pm u(1) \geq 0.$$

On the differential operator

$$\begin{aligned} \bar{L}\vartheta_\pm(x) &= -c_\varepsilon \vartheta_\pm''(x) + p(x)\vartheta_\pm'(x) + d(x)\vartheta_\pm(x) \\ &= -c_\varepsilon(0 \pm u''(x)) + p(x)(0 \pm u'(x)) + d(x)\left(\frac{\|Lu\|}{\zeta} + \max\{|\phi|, |\psi|\} \pm u(x)\right) \\ &= d(x)\left(\frac{\|Lu\|}{\zeta} + \max\{|\phi|, |\psi|\}\right) \pm f(x) \\ &\geq 0, \ \text{ since } \ d(x) \geq \zeta > 0, \end{aligned}$$

which implies $\bar{L}\vartheta_\pm(x) \geq 0$. Hence, by maximum principle we obtain, $\vartheta_\pm(x) \geq 0, \ \forall x \in \bar{\Omega}$. ∎

Lemma 3 *The bound on the derivative of the solution $u(x)$ of the problem in (4)–(5) is given by*

$$\left|u^{(i)}(x)\right| \leq C\left(1 + c_\varepsilon^{-i}\exp\left(-\frac{p^* x}{c_\varepsilon}\right)\right), \ x \in \bar{\Omega}, \ 0 \leq i \leq 4, \ \textit{for left boundary layer,}$$

$$\left|u^{(i)}(x)\right| \leq C\left(1 + c_\varepsilon^{-i}\exp\left(-\frac{p^*(1-x)}{c_\varepsilon}\right)\right), \ x \in \bar{\Omega}, \ 0 \leq i \leq 4, \ \textit{for right boundary layer.}$$

Proof. See in [9], [15]. ∎

3 Numerical Scheme Formulation

The construction of non-standard finite difference method (NSFDM), depends on the knowledge of the corresponding exact finite difference method.

3.1 Exact Finite Difference

We consider separately for left and right boundary layer problems and develop individual schemes for each. First let us consider the right boundary layer problem.

1. Right boundary layer problems

For the problem of the form in (4)–(5), in order to construct exact finite difference scheme we follow the procedures used in [14]. Consider the constant coefficient sub-equations from (4)–(5) as

$$-c_\varepsilon u''^{*} u'(x) + \zeta u(x) = 0, \tag{7}$$

$$-c_\varepsilon u''^{*} u'(x) = 0, \tag{8}$$

where $p(x) \geq p^*$ and $d(x) \geq \zeta$. Thus the equation in (7) has two independent solutions namely $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$ where

$$\lambda_{1,2} = \frac{-p^* \pm \sqrt{(p^*)^2 + 4c_\varepsilon \zeta}}{-2c_\varepsilon}.$$

Discretizing the domain $\bar{\Omega} = [0, 1]$ as

$$\Omega^N = \left\{ x_i = x_0 + ih,\ i = 1, 2, ..., N,\ x_0 = 0,\ x_N = 1,\ h = \frac{1}{N} \right\},$$

where N is the number of mesh intervals. We denote U_i as the approximate solution of $u(x)$ at mesh point x_i. The target is to calculate a difference equation which has the same general solution as the differential equation in (8) has at the mesh point x_i is given by $U_i = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i)$. Using the theory of difference equations for second order linear difference equations in [14], we obtain

$$\begin{vmatrix} U_{i-1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ U_i & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ U_{i+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{vmatrix} = 0.$$

Substituting the values of λ_1 and λ_2 gives

$$\exp\left(\frac{p^* h}{2c_\varepsilon}\right) U_{i-1} - 2\cosh\left(\frac{h\sqrt{(p^*)^2 + 4c_\varepsilon \zeta}}{2c_\varepsilon}\right) U_i + \exp\left(-\frac{p^* h}{2c_\varepsilon}\right) U_{i+1} = 0 \tag{9}$$

which is an exact difference scheme for (8). For $\varepsilon \to 0$, we use the approximation $\frac{h\sqrt{(p^*)^2+4c_\varepsilon\zeta}}{2c_\varepsilon} \approx \frac{p^* h}{2c_\varepsilon}$ in (9). Multiplying both sides by $\exp\left(\frac{p^* h}{2c_\varepsilon}\right)$, simplifying we obtain

$$U_{i-1} - 2U_i + U_{i+1} = \left(\exp\left(\frac{p^* h}{c_\varepsilon}\right) - 1\right)(U_i - U_{i-1}). \tag{10}$$

Rearranging gives

$$-c_\varepsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{\frac{hc_\varepsilon}{p^*}\left(\exp(\frac{p^* h}{c_\varepsilon}) - 1\right)} + p^* \frac{U_i - U_{i-1}}{h} = 0.$$

The required denominator function for second derivative discretization becomes $\gamma^R = \frac{hc_\varepsilon}{p^*}\left(\exp\left(\frac{hp^*}{c_\varepsilon}\right) - 1\right)$. Adopting γ^R for the variable coefficient problem we write as

$$\gamma_i^R = \frac{hc_\varepsilon}{p(x_i)}\left(\exp\left(\frac{hp(x_i)}{c_\varepsilon}\right) - 1\right). \tag{11}$$

Using the denominator function γ_i^R into the scheme in (4), the difference scheme becomes

$$L_R^h U_i = -c_\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\gamma_i^R} + p(x_i)\frac{U_i - U_{i-1}}{h} + d(x_i)U_i = f(x_i),\ i = 1, 2, ..., N-1 \tag{12}$$

with the boundary conditions

$$U_0 = \phi(0),\ \ U_N = \psi(1). \tag{13}$$

2. Left boundary layer problems

In this case $-p(x) \leq -p^* < 0$ in (4)–(5), we consider the constant coefficient sub-equations from (4)–(5) as

$$-c_\varepsilon u''{}^* u'(x) + \zeta u(x) = 0, \tag{14}$$

$$-c_\varepsilon u''{}^* u'(x) = 0, \tag{15}$$

where $d(x) \geq \zeta$. Thus, the equation in (14) has two independent solutions namely $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$ where

$$\lambda_{1,2} = \frac{p^* \mp \sqrt{(p^*)^2 + 4c_\varepsilon \zeta}}{-2c_\varepsilon}.$$

We descretize the domain $\Omega = [0, 1]$, as $\Omega^N = \{x_i\}_{i=0}^N$ with $x_0 = 0,\ x_N = 1,\ h = \frac{1}{N}\}$ where N is mesh interval. We denote the approximate solution of $u(x)$ at mesh point x_i by U_i. Our objective is to calculate a difference equation which has the same general solution as the differential equation in (15) has at the mesh point x_i given by $U_i = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i)$. Using the theory of difference equations in [14] for second order linear difference equations, we have

$$\begin{vmatrix} U_{i-1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ U_i & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ U_{i+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{vmatrix} = 0$$

Substituting the values of λ_1 and λ_2 we obtain

$$\exp\left(-\frac{p^* h}{2c_\varepsilon}\right)U_{i-1} - 2\cosh\left(\frac{h\sqrt{(p^*)^2 + 4c_\varepsilon\zeta}}{2c_\varepsilon}\right)U_i + \exp\left(\frac{p^* h}{2c_\varepsilon}\right)U_{i+1} = 0$$

is an exact difference scheme for (15). For $\varepsilon \to 0$, we use the approximation $\frac{h\sqrt{(p^*)^2+4c_\varepsilon\zeta}}{2c_\varepsilon} \approx \frac{p^* h}{2c_\varepsilon}$. After doing the arithmetic adjustment, we obtain

$$-c_\varepsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{\frac{hc_\varepsilon}{p^*}\left(1 - \exp(\frac{p^* h}{c_\varepsilon})\right)} - p^* \frac{U_{i+1} - U_i}{h} = 0.$$

The denominator function becomes $\gamma^L = \frac{hc_\varepsilon}{p^*}\left(1 - \exp\left(\frac{hp^*}{c_\varepsilon}\right)\right)$. Adopting it for the variable coefficient problem, we write as

$$\gamma_i^L = \frac{hc_\varepsilon}{p(x_i)}\left(1 - \exp\left(\frac{hp(x_i)}{c_\varepsilon}\right)\right). \tag{16}$$

The required finite difference schemes becomes

$$L_L^h U_i = -c_\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{\gamma_i^L} - p(x_i)\frac{U_{i+1} - U_i}{h} + d(x_i)U_i = f(x_i),\ i = 1, 2, ..., N-1 \tag{17}$$

with the boundary conditions $U_0 = \phi(0),\ \ U_N = \psi(1)$.

3.2 Uniform Convergence Analysis

In this section, we need to show the discrete scheme in (12) or in (17) satisfies the discrete maximum principle, uniform stability estimates and parameter uniform convergence. Here, we prove for right boundary layer problem only and similarly shown for the left boundary layer problem.

Lemma 4 *The operator defined by the discrete scheme in (12) satisfies a discrete maximum principle. i.e. Let U_i be any mesh function satisfying $U_0 \geq 0,\ U_N \geq 0$. Then $L_R^h U_i \geq 0,\ \forall i = 1, 2, ..., N-1$ implies that $U_i \geq 0,\ \forall i = 0, 1, ..., N$.*

Proof. Suppose there exists $k \in \{0, 1, ..., N\}$ such that $U_k = \min_{0 \leq i \leq N} U_i$. Suppose that $U_k < 0$ which implies $k \neq 0, N$. Also we assume that $U_{k+1} - U_k > 0$ and $U_k - U_{k-1} < 0$. Now we have

$$L_R^h U_k = -c_\varepsilon \frac{U_{k+1} - 2U_k + U_{k-1}}{\gamma_k^R} + p(x_k)\frac{U_k - U_{k-1}}{h} + d(x_k)U_k.$$

Using the assumptions made above, we obtain $L_R^h U_k < 0$ for $k = 1, 2, 3, ..., N-1$. Thus the supposition $U_i < 0$, for $i = 0, 1, ..., N$ is wrong. Hence, we obtain $U_i \geq 0,\ \ \forall i = 0, 1, ..., N$. ■

Using the results in Lemma, we next prove the discrete scheme in (12) satisfies the uniform stability bound given in the next lemma.

Lemma 5 *The solution U_i of the discrete scheme in (12) satisfy the following bound.*

$$|U_i| \leq \frac{\max|f_i|}{\zeta} + \max\{|\phi|, |\psi|\}.$$

Proof. Let $p = \frac{\max|f_i|}{\zeta} + \max\{|\phi|, |\psi|\}$ and define the barrier functions $\vartheta_i^{\pm}$ as $\vartheta_i^{\pm} = p \pm U_i$. At the boundary points, we obtain

$$\vartheta_0^{\pm} = p \pm U_0 = \frac{\max|f_i|}{\zeta} + \max\{|\phi|, |\psi|\} \pm \phi \geq 0,$$

$$\vartheta_N^{\pm} = p \pm U_N = \frac{\max|f_i|}{\zeta} + \max\{|\phi|, |\psi|\} \pm \psi \geq 0.$$

For the operator on the discretized domain $x_i,\ 0 < i < N$, we obtain

$$\begin{aligned}
\bar{L}_R^h \vartheta_i^{\pm} &= -c_\varepsilon \left(\frac{p \pm U_{i+1} - 2(p \pm U_i) + p \pm U_{i-1}}{\gamma_i^R}\right) + p(x_i)\left(\frac{p \pm U_i - p \pm U_{i-1}}{h}\right) + d(x_i)(p \pm U_i) \\
&= d(x_i)p \pm L_R^h U_i \\
&= d(x_i)\left(\frac{\max|f_i|}{\zeta} + \max\{\phi, \psi\}\right) \pm f(x_i) \\
&\geq 0, \quad \text{since} \quad d(x_i) \geq \zeta.
\end{aligned}$$

Using the discrete maximum principle in Lemma 4, we obtain $\vartheta_i^{\pm} \geq 0,\ \forall x_i \in \bar{\Omega}^N$. Hence the required bound is obtained. ■

Let us define the first and second derivative finite differences operators as

$$D^+ z(x_i) = \frac{z(x_{i+1}) - z(x_i)}{h},\quad D^- z(x_i) = \frac{z(x_i) - z(x_{i-1})}{h},\ \text{and}\ D^+D^- z(x_i) = \frac{D^+ z(x_i) - D^- z(x_i)}{h}, \tag{18}$$

respectively.

Next, let us analyze the parameter uniform convergence. We proved above the discrete operator L_R^h satisfy the maximum principle and uniform stability estimate.

Theorem 6 *Let the coefficients functions $p(x)$, $d(x)$ and the function $f(x)$ in (4) be sufficiently smooth so that $u(x) \in C^4[0,1]$. Then the discrete solution U_i of the scheme in (12) satisfies*

$$\left|L_R^h\left(u(x_i)-U_i\right)\right| \leq Ch\left(1+\sup_{x\in(0,1)}\frac{\exp\left(-p^*(1-x)/c_\varepsilon\right)}{c_\varepsilon^3}\right).$$

Proof. Considering the local truncation error as

$$\begin{aligned}
& \left|\, L_R^h\big(u(x_i)-U_i\big)\,\right| \\
= & \left|\, L^h u(x_i)-L_R^h U_i\,\right| \\
\leq & \; C\left|\, -c_\varepsilon\big(u''(x_i)-\frac{D^+D^-h^2}{\gamma_i^R}u(x_i)\big)+p_i\big(u'(x_i)-D^-u(x_i)\big)\,\right| \\
\leq & \; Cc_\varepsilon\left|\,u''(x_i)-D^+D^-u(x_i)\,\right|+Cc_\varepsilon\left|\,\big(\frac{h^2}{\gamma_i^R}-1\big)D^+D^-u(x_i)\,\right|+Ch\left|\,u''(x_i)\,\right| \\
\leq & \; Cc_\varepsilon h^2\left|\,u^{(4)}(x_i)\,\right|+Ch\left|\,u''(x_i)\,\right|.
\end{aligned}$$

We used the estimate $c_\varepsilon\left|\,\frac{h^2}{\gamma_i^R}-1\,\right|\leq Ch$ in above expression is depending on the behavior of denominator function used in non-standard FDM. To make it clear let us define $\rho=p_ih/c_\varepsilon,\ \rho\in(0,\infty)$. Then using the expression for γ_i^R, we obtain

$$c_\varepsilon\left|\,\frac{h^2}{\gamma_i^R}-1\,\right| = p_ih\left|\,\frac{1}{\exp(\rho)-1}-\frac{1}{\rho}\,\right| =: p_ihQ(\rho), \tag{19}$$

where $Q(\rho)=\frac{\exp(\rho)-1-\rho}{\rho(\exp(\rho)-1)}$. Next, let us set a bound for $Q(\rho)$. Using the limit we obtain

$$\lim_{\rho\to 0}Q(\rho)=\frac{1}{2},\quad \lim_{\rho\to\infty}Q(\rho)=0. \tag{20}$$

Therefore, $Q(\rho)\leq C_2,\ \rho\in(0,\infty)$. Hence, from (19) and (20) the estimate $c_\varepsilon|\frac{h^2}{\gamma_i^R}-1|\leq Ch$. So, the truncation error bound becomes

$$\left|\, L_R^h\big(u(x_i)-U_i\big)\,\right| \leq Cc_\varepsilon h^2\left|\,u^{(4)}(x_i)\,\right|+Ch\left|\,u''(x_i)\,\right|. \tag{21}$$

Using the boundedness of the derivatives of the solution in Lemma 3 into (21), we obtain

$$\begin{aligned}
& \left|\, L_R^h\big(u(x_i)-U_i\big)\,\right| \\
\leq & \; Cc_\varepsilon h^2\left|\,1+c_\varepsilon^{-4}\exp\big(\frac{-p^*(1-x_i)}{c_\varepsilon}\big)\,\right|+Ch\left|\,1+c_\varepsilon^{-2}\exp\big(\frac{-p^*(1-x_i)}{c_\varepsilon}\big)\,\right| \\
\leq & \; Ch^2\left|\,c_\varepsilon+c_\varepsilon^{-3}\exp\big(\frac{-p^*(1-x_i)}{c_\varepsilon}\big)\,\right|+Ch\left|\,1+c_\varepsilon^{-2}\exp\big(\frac{-p^*(1-x_i)}{c_\varepsilon}\big)\,\right| \\
\leq & \; Ch\big(1+\sup_{x_i\in(0,1)}c_\varepsilon^{-3}\exp\big(\frac{-p^*(1-x_i)}{c_\varepsilon}\big)\big),\ \text{since } c_\varepsilon^3\leq c_\varepsilon^2.
\end{aligned}$$

■

Lemma 7 *For $c_\varepsilon\to 0$ and for given fixed mesh number N, we obtain*

$$\lim_{c_\varepsilon\to 0}\max_j\frac{\exp\big(-\frac{p^*x_j}{c_\varepsilon}\big)}{c_\varepsilon^m}=0,\ \lim_{c_\varepsilon\to 0}\max_j\frac{\exp\big(-\frac{p^*(1-x_j)}{c_\varepsilon}\big)}{c_\varepsilon^m}=0,\quad m=1,2,3,... \tag{22}$$

where $x_i=ih$, $h=1/N$, $\forall i=1,2,...,N-1$.

Proof. See in [19]. ■

Theorem 8 *Under the hypothesis of boundedness of discrete solution, the solution of the discrete schemes in* (12) *satisfy the following parameter uniform bound.*

$$\sup_{0<c_\varepsilon\ll 1} \|u(x_i) - U_i\|_{\Omega^N} \leq CN^{-1}. \tag{23}$$

Proof. The parameter uniform convergence of the discrete scheme, follows from the results of Theorem 6, Lemma 7 and using the discrete maximum principle in Lemma 4. ■

4 Numerical Examples and Results

Table 1: Maximum absolute error of Example 9 using the proposed scheme for $\delta = 0.3\varepsilon$.

ε ↓	N= 32	64	128	256	512	1024
10^{-3}	1.2360e-02	6.2211e-03	3.1207e-03	1.5628e-03	7.8204e-04	3.9118e-04
10^{-4}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
10^{-5}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
10^{-6}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
10^{-7}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
10^{-8}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
10^{-9}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
10^{-10}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
E^N	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
r^N	0.9904	0.9953	0.9977	0.9988	0.9994	-

Table 2: Maximum absolute error of Example 9 for different delay values for $\varepsilon = 10^{-5}$.

δ ↓	N= 32	64	128	256	512	1024
0.2ε	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
0.4ε	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
0.6ε	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
0.8ε	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04

Table 3: Maximum absolute error of Example 10, proposed scheme and result in [11] for $\delta = 0.3\varepsilon$.

ε ↓	N= 32	64	N= 32	64
	Proposed	Scheme	Result in	[11]
2^{-8}	1.0769e-03	5.5383e-04	3.7010e-02	1.2611e-02
2^{-12}	1.0771e-03	5.5397e-04	3.6669e-02	1.2541e-02
2^{-16}	1.0771e-03	5.5398e-04	3.6701e-02	1.2463e-02
2^{-20}	1.0771e-03	5.5398e-04	3.6709e-02	1.2478e-02
2^{-24}	1.0771e-03	5.5398e-04	3.6710e-02	1.2479e-02
2^{-28}	1.0771e-03	5.5398e-04	3.6710e-02	1.2479e-02
2^{-32}	1.0771e-03	5.5398e-04	3.6710e-02	1.2479e-02

We consider numerical examples to elaborate the theoretical analysis made in above sections. Here we consider and solved three examples having boundary layer behaviour.

Table 4: Maximum absolute error of Example 10 for different values of delay for $\varepsilon = 10^{-5}$.

$\delta \downarrow$	N= 32	64	128	256	512	1024
0.2ε	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
0.4ε	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
0.6ε	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
0.8ε	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05

Table 5: Maximum absolute error of Example 11 using the proposed scheme for $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow$	N= 32	64	128	256	512	1024
10^{-3}	1.5245e-04	7.7883e-05	9.9188e-06	1.9785e-05	9.8676e-06	4.9660e-06
10^{-4}	1.5235e-04	7.7835e-05	3.9336e-05	1.9773e-05	9.9128e-06	4.9630e-06
10^{-5}	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
10^{-6}	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
10^{-7}	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
10^{-8}	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
10^{-9}	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
10^{-10}	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
E^N	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
r^N	0.9689	0.9846	0.9923	0.9962	0.9981	-

Example 9 *Consider the problem*

$$\varepsilon u''(x) + (1+x)u'(x-\delta) + \sin(2x)u(x-\delta) - \exp(-x)u(x) = \sin(2x) + 3\exp(-x)$$

with interval boundary conditions $u(x) = -1,\ -\delta \le x < 0$ *and* $u(1) = 1$.

Example 10 *Consider the problem*

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) - \exp(-2x)u(x-\delta) + \exp(-x) = 0$$

with interval boundary conditions $u(x) = 1,\ -\delta \le x < 0$ *and* $u(1) = -1$.

Example 11 *Consider the problem*

$$\varepsilon u''(x) + (1+x)u'(x-\delta) + \exp(-2x)u(x-\delta) - 2\exp(-x)u(x) = 1$$

with interval boundary conditions $u(x) = 1,\ -\delta \le x < 0$ *and* $u(1) = 0$.

Since the exact solution of these three problems are not known, the maximum absolute errors are estimated by using the double mesh principle given in [19] and defined by

$$E_\varepsilon^N = \max_{0\le i\le N} |U_i^N - U_i^{2N}|,$$

where U_i^N stands for the numerical solution of the problem on N number of mesh points and U_i^{2N} stands for the numerical solution of the problem on $2N$ number of mesh points by including the mid-points $x_{i+1/2}$ into the mesh. The ε-uniform error is defined as

$$E^N = \max_{\varepsilon} |E_\varepsilon^N|.$$

The rate of convergence of the scheme is obtained as

$$r_\varepsilon^N = \frac{\log(E_\varepsilon^N) - \log(E_\varepsilon^{2N})}{\log(2)},$$

Table 6: Maximum absolute error of Example 11 for different values of delay for $\varepsilon = 10^{-6}$.

δ $\downarrow$	N= 32	64	128	256	512	1024
0.2ε	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
0.4ε	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
0.6ε	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06
0.8ε	1.5234e-04	7.7830e-05	3.9333e-05	1.9772e-05	9.9122e-06	4.9627e-06

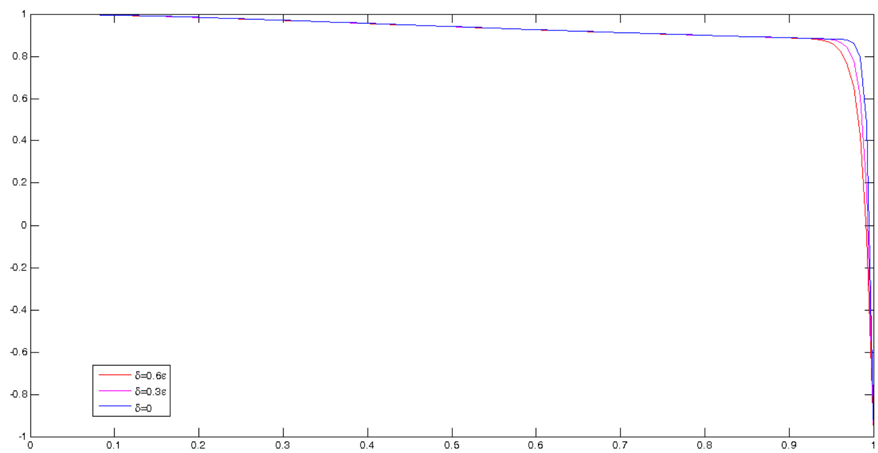

Figure 1: Effect of delay parameter on the solution of Example 10 for $\varepsilon = 0.01$.

and the ε-uniform rate of convergence of the scheme is given as

$$r^N = \frac{\log(E^N) - \log(E^{2N})}{\log(2)}.$$

5 Discussion and Conclusion

In this paper, we consider three examples exhibiting boundary layer. Example 9 and 11 exhibit left boundary layer and Example 10 exhibit right boundary layer. In the computed solutions we used the perturbation parameter ε very small compared to the number of mesh points N. For each examples, we computed the maximum absolute error, parameter uniform error and uniform rate of convergence. In each column of Tables 1, 3 and 5 one can observe that the maximum absolute error is independent of the perturbation parameter ε, as ε goes small. This means that, as the perturbation parameter goes small, the maximum absolute error of the scheme is bounded and it becomes uniformly convergent. On the last two rows of these tables the parameter uniform error and the parameter uniform rate of convergence is given. The scheme gives first order of convergence. In Table 3, we give the comparison of the obtained result with the result given in paper [11]. As one can see, the obtained result is more accurate than the one in [11].

The results in Table 2, 4 and 6, gives the maximum absolute error of Example 9, 10 and 11 respectively for different values of delay parameter by taking fixed value for ε. The result in this tables shows that the developed scheme is also independent of the delay parameters(i.e. as the delay parameter varies the maximum absolute error remains constant for each N).

For left boundary layer problems, one can observe from Figure 2 as the values of the delay parameters increases the size of the boundary layer decreases. For the case of the right boundary layer problems as the values of the delay parameter increases the size of the boundary layer increases as it is seen on Figure 1.

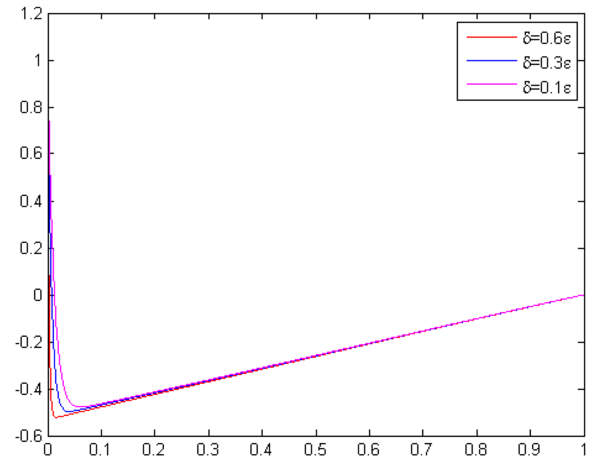

Figure 2: Effect of delay parameter on the solution of Example 11 for $\varepsilon = 0.01$.

In this paper, parameter uniformly convergent(robust) numerical scheme is developed using non-standard finite difference technique. The stability of the scheme is investigated using the maximum principle and by constructing barrier function to show the bound on the solution. The detail convergence analysis is carried out by considering the truncation error of the discretization. The results obtained by the proposed scheme gives accurate and parameter uniform convergence with order of convergence one.

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