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Point Canonical Transformation And The Time Independent Fractional Schrödinger Equation With Position Dependent Mass*

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Abstract

Fractional one-dimensional Schrödinger equation is considered within fractional position-dependent mass formalism. Fractional Coulomb type interaction is taken for the study. The entire work is composed of Katugampola fractional derivative. Point canonical transformation (PCT) is used as an analytical tool. The energy spectrum of the bound states and their eigenfunctions are written explicitly for different position-dependent mass profiles. Finally, we furnish a few wave function variations for different fractional parameter values and two separate mass profiles.

1 Introduction

Recently, one-dimensional Schrödinger equation with position-dependent mass has taken a lot of interest among the researchers. There are many physical systems such as semiconductor [1], heterostructures [2], the material of non-uniform chemical compositions [3], superlattice [4] where Schrödinger equation with position-dependent mass plays a vital role to describe the physics behind the system. A few important and notable works on this subject are listed in the references [5–9]. Along with this development, we also see a doughty trend to study quantum mechanics within the framework of fractional calculus [10–18]. Fractional Schrödinger equation is one of the most promising areas of applied mathematical physics which defines the quantum phenomenon from a different angle. This new and beautiful subject is discovered by N. Laskin [19–20]. According to Laskin, fractional quantum mechanics is the result of Feynman path integral with lévy like quantum paths. Since then, fractional quantum mechanics started its journey and at present, many researchers are actively working in this field. The volume of the research article is growing exponentially as the concept of fractional Schrödinger equation helps to study the system with memory effects i.e. quantum states do not depend solely on time and position but also previous states. The non-local character of the fractional derivative gives fractional derivative an inbuilt tool to incorporate the memory effect.

Now the subject of the fractional derivative is not unique for everyone.There are a number of different definitions of fractional derivative [21–24]. Different types of mathematical operations like product rule, chain rule, the fractional derivative of a constant are always not similar and also some of these do not follow the standard integer-order derivative rules as well. In 2014, Khalil et al. [25] introduced a new definition of fractional derivative which was analogous to the standard derivative of integer order. Katugampola [26] generalized the definition further and we are going to use it in this paper (see section-2). So from here to the rest of the paper, the term 'fractional derivative' will stand for Katugampola fractional derivative.

Motivated by these developments, in this paper we are going to study one dimensional fractional Schrödinger equation with position-dependent mass and as far as our knowledge this has not been done yet. As an analytical tool point canonical transformation (PCT) has been used [27–29]. The idea of PCT applies to shape invariant potentials within a specific class. In this approach, first, we need a solution to a reference potential problem then using mapping it is easy to find the solution to other potential problems within the

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same class. In other words, the reference problem is just like a seed for generating new solutions within the same class of other shape invariant potentials. In this study, we have taken fractional Coulomb potential as a reference problem for fractional Schrödinger equation. The solution of the time-independent fractional Schrödinger equation for fractional Coulomb potential is not well known. So we will solve our reference problem first and then using the mapping technique of PCT we will try to solve the one-dimensional fractional Schrödinger equation with the position-dependent mass problem.

To make the paper self-contained, this paper is organized as follows. The next section comes with the brief of Katugampola fractional derivative. In section 3 the general development of formalism will be presented. Section 4 comes with the solution to the reference problem. Section 5 is for application where we will use the results of the reference problem to study the actual problem i.e. one-dimensional fractional Schrödinger equation with the position-dependent mass problem via mapping technique of PCT scheme. Discussion is placed in section 6, where numerical results of the energy spectrum and a few graphical studies of wave functions will be presented. Finally, the conclusion of the present work takes place in section 7.

2 The Katugampola Derivative

The Katugampola derivative [30] is a limit based formalism. Here the Katugampola formalulation of fractional calculus and specifically the operator, D^α, defined as

$$\begin{cases} D^\alpha[f(t)] = \lim_{\epsilon\to 0} \frac{f(te^{\epsilon t^{-\alpha}})-f(t)}{\epsilon}\,, \quad t>0, \\ D^\alpha[f(0)] = \lim_{\epsilon\to 0^+} D^\alpha[f(t)], \end{cases} \tag{1}$$

where $\alpha(0<\alpha<1)$ is called fractional parameter and $t>0$. The following results for D^α are well established

$$D^\alpha[c_1 f + c_2 g] = c_1 D^\alpha[f] + c_2 D^\alpha[g], \quad \text{(linearity)}$$

$$D^\alpha[fg] = fD^\alpha[g] + gD^\alpha[f], \quad \text{(product rule)}$$

$$D^\alpha[f(g)] = \frac{df}{dg} D^\alpha[g], \quad \text{(chain rule)}$$

$$D^\alpha[f] = t^{1-\alpha} f^{'}, \ \text{ where } \ f^{'} = \frac{df}{dt}.$$

The last rule can be used to construct a number of important results of Katugampola derivatives. The most used are

$$D^\alpha[e^{ct}] = ct^{1-\alpha}e^{ct}\,, \tag{2}$$

$$D^\alpha[e^{\frac{t^\alpha}{\alpha}}] = e^{\frac{t^\alpha}{\alpha}}\,, \tag{3}$$

$$D^\beta[D^\alpha[y]] = t^{2-\alpha-\beta}y^{''} + (1-\alpha)t^{1-\alpha-\beta}y^{'}\,. \tag{4}$$

The equation number (4) can be written as

$$D^\beta[D^\alpha[y]] = D^{\beta+\alpha}y + (1-\alpha)t^{1-\alpha-\beta}y'\,,$$

under the condition $0<\alpha,\beta\leq 1$ such that $1<\alpha+\beta\leq 2$. It is clear when α is very close to 1.0 one can write $D^\beta[D^\alpha[y]] \approx D^{\beta+\alpha}y$ or similarly $D^{2\alpha} \equiv D^\alpha D^\alpha$.

The solution of fractional order differential equation with Katugampola derivative is useful in this paper. A few such results are

- The fractional differential equation $D^\alpha[y] + \lambda y = 0$ has solution $y = C_1 e^{-\frac{\lambda t^\alpha}{\alpha}}$.
- The fractional differential equation $D^\beta[D^\alpha[y]] = 0$ has solution $y = C_1\frac{t^\alpha}{\alpha} + C_2$. The solution is independent from β. If $\beta=\alpha$, then the solution is again the same i.e. $y = C_1\frac{t^\alpha}{\alpha} + C_2$.

- Under the condition $\beta = \alpha$ the fractional differential equation $D^{\beta}[D^{\alpha}[y]] = \Lambda$ has solutions $y = \frac{\Lambda}{2\alpha^2}t^{2\alpha} + C_1\frac{1}{\alpha}t^{\alpha} + C_2$.
- Under the condition $\beta = \alpha$ the fractional differential equation $D^{\beta}[D^{\alpha}[y]] = -\Lambda y$ has solutions $y = C_1\cos[\frac{\sqrt{\Lambda}}{\alpha}t^{\alpha}] + C_2\sin[\frac{\sqrt{\Lambda}}{\alpha}t^{\alpha}]$.

The last of the bullet items is a special case. If $\beta \neq \alpha$ then the solution of the corresponding fractional differential equation is difficult. Furthermore, inserting $\alpha = 1$ it is easy to achieve conventional known solutions. We think this is enough for our work. If readers need more they can go through references.

3 Action of PCT-General Formalism of the Study

The one dimensional fractional Schrödinger equation with position dependent mass may be expressed as (in natural units $\hbar = c = 1$)

$$D^{2\alpha}\phi(x) + m(x)D^{\alpha}\left\{\frac{1}{m(x)}\right\}D^{\alpha}\phi(x) + 2m(x)[E - V(x)]\phi(x) = 0, \tag{5}$$

where E, $\phi(x)$, $V(x)$ are energy eigenvalue, wave function and potential function. The position dependent mass is expressed by $m(x)$.

Let us take the known problem of fractional Schrödinger equation such as

$$\left[\frac{d^{2\alpha}}{dz^{2\alpha}} - 2\left\{U(z) - \xi_n\right\}\right]\psi(z) = 0. \tag{6}$$

Here the potential $U(z)$, solution $\psi(z)$ and n-th state energy eigenvalue ξ_n are known completely. Now as the scheme of PCT, we will map the known solution $\psi(z)$ with unknown solution $\phi(x)$. To aim that let us take

$$\psi(z) = G(x)\phi(x), \;\; z = P(x), \tag{7}$$

where $G(x)$ is some unknown function too. Taking the transformation into equation (6) we have

$$D^{2\alpha}\phi(x) + AD^{\alpha}\phi(x) + B\phi(x) = 0, \tag{8}$$

where

$$A = \left(\frac{2D^{\alpha}G(x)}{G(x)} + \frac{D^{\alpha}[P^{'}(x)]^{-1}}{[P^{'}(x)]^{-1}}\right), \tag{9}$$

$$B = \left(\frac{D^{2\alpha}G(x)}{G(x)} + \frac{D^{\alpha}G(x)D^{\alpha}[P^{'}(x)]^{-1}}{G(x)[P^{'}(x)]^{-1}} - 2[P^{'}(x)]^2\left\{U(P(x)) - \xi_n\right\}\right), \tag{10}$$

where $P^{'}(x) = \frac{dP}{dx}$.

Proof. Keeping in mind of the transformation (7) we write

$$\begin{aligned} D^{\alpha}\psi(z) &= \frac{d^{\alpha}}{dz^{\alpha}}\psi(z) = \frac{d^{\alpha}}{dz^{\alpha}}[G(x)\phi(x)] \\ &= \left(\frac{dx}{dz}\right)\frac{d^{\alpha}}{dx^{\alpha}}[G(x)\phi(x)] \\ &= [P^{'}(x)]^{-1}[G(x)D^{\alpha}\phi(x) + \phi(x)D^{\alpha}G(x). \end{aligned}$$

So here we have the operator $\frac{d^{\alpha}}{dz^{\alpha}} = [P'(x)]^{-1}\frac{d^{\alpha}}{dx^{\alpha}}$. Applying it twice on $\psi(z)$ the following is easy to derive

$$\frac{d^{2\alpha}}{dz^{2\alpha}}\psi(z) = g_1 + g_2\,,$$

where

$$\begin{aligned} g_1 &= [P'(x)]^{-2} D^\alpha G(x) D^\alpha \phi(x) + [P'(x)]^{-1} G(x) D^\alpha [P'(x)]^{-1} D^\alpha \phi(x) \\ &\quad + G(x)[P'(x)]^{-2} D^{2\alpha} \phi(x), \end{aligned}$$

$$\begin{aligned} g_2 &= [P'(x)]^{-1} D^\alpha [P'(x)]^{-1} \phi(x) D^\alpha G(x) + [P'(x)]^{-2} D^\alpha \phi(x) D^\alpha G(x) \\ &\quad + [P'(x)]^{-2} \phi(x) D^{2\alpha} G(x). \end{aligned}$$

Now arranging further

$$\frac{d^{2\alpha}}{dz^{2\alpha}} \psi(z) = G(x)[P'(x)]^{-2} D^{2\alpha} \phi(x) + g_3 D^\alpha \phi(x) + g_4 \phi(x),$$

where

$$g_3 = 2D^\alpha G(x)[P'(x)]^{-2} + G(x)[P'(x)]^{-1} D^\alpha [P'(x)]^{-1},$$

$$g_4 = [P'(x)]^{-1} D^\alpha [P'(x)]^{-1} D^\alpha G(x) + [P'(x)]^{-2} D^{2\alpha} G(x).$$

Hence inserting in the equation (6) we get equations (8) to (10). Now mapping equation (5) and equation (8) we have

$$m(x) D^\alpha \left\{ \frac{1}{m(x)} \right\} = A, \tag{11}$$

$$2m(x)[E - V(x)] = B. \tag{12}$$

First we have to solve equation (11). Using equation (9), the equation (11) can be written as

$$\frac{2D^\alpha G(x)}{G(x)} = m(x) D^\alpha \left\{ \frac{1}{m(x)} \right\} - \frac{D^\alpha [P'(x)]^{-1}}{[P'(x)]^{-1}} = -\lambda, \tag{13}$$

where λ is a common constant to satisfy the identity. It is easy to split equation (13) and we have

$$D^\alpha G(x) + \frac{\lambda}{2} G(x) = 0, \tag{14}$$

$$D^\alpha \left\{ \frac{1}{m(x)} \right\} + \gamma \left\{ \frac{1}{m(x)} \right\} = 0, \tag{15}$$

$$D^\alpha [P'(x)]^{-1} + (\gamma - \lambda)[P'(x)]^{-1} = 0, \tag{16}$$

where γ is another constant such that $\gamma > \lambda$. Using Section 2 the solutions of $(14, 15, 16)$ may be written as

$$G(x) \sim e^{-\frac{\lambda}{2\alpha} x^\alpha}, \tag{17}$$

$$\frac{1}{m(x)} \sim e^{-\frac{\gamma}{\alpha} x^\alpha}, \tag{18}$$

$$[P'(x)]^{-1} \sim e^{-\frac{\gamma - \lambda}{\alpha} x^\alpha}. \tag{19}$$

Manipulating equations (17, 18, 19) it is easy to achieve

$$G(x) = \sqrt{\frac{P'(x)}{m(x)}}. \tag{20}$$

According to equation (20), with a given $m(x)$ and a choice of transformation function $P(x)$ will determine $G(x)$. The new $G(x)$ will deduce the energy spectrum and potential function $V(x)$ for the target problem. Now using equation (10) and (12) we have

$$E - V(x) = \frac{1}{2m(x)} Q(m(x), m'(x), x) - \frac{[P^{'}(x)]^2}{m(x)} \left[U(P(x)) - \xi_n\right], \tag{21}$$

where

$$Q(m(x), m'(x), x) = \left[\frac{D^{2\alpha}G(x)}{G(x)} + \frac{D^{\alpha}G(x)D^{\alpha}[P'(x)]^{-1}}{G(x)[P'(x)]^{-1}}\right]. \tag{22}$$

The last term of equation (21) is crucial here. Proper manipulation of the last term will provide a additive constant in the equation. This constant term will be identified with energy eigenvalue. To that process the following integral

$$h(x) = \frac{1}{\mu}\int \sqrt{m(x)}dx, \tag{23}$$

is helpful. The term μ is a scaling parameter. It is possible to find a few choices such that the last term of equation (21) will provide an additive constant.

3.1 Choice-I

$$[P^{'}(x)]^2 = m(x) \text{ and hence } \; P(x) = \mu h(x). \tag{24}$$

The choice makes

$$G(x) = [m(x)]^{-\frac{1}{4}}. \tag{25}$$

Inserting $G(x)$ and $[P^{'}(x)]$ in equation (22) we have

$$Q_I(m(x), m'(x), x) = -\frac{1}{4}\left[x^{1-2\alpha}(1-\alpha)\frac{m'(x)}{m(x)} + x^{2-2\alpha}\left\{\frac{m''(x)}{m(x)} - \frac{7}{4}\left(\frac{m'(x)}{m(x)}\right)^2\right\}\right]. \tag{26}$$

and equation (21) takes the simple form

$$V(x) - E = U(\mu h(x)) - \xi_n - \frac{1}{2m(x)}Q_I(m(x), m'(x), x). \tag{27}$$

So from identity of left and right hand side of the equation (27) the target problem has following energy spectrum and effective potential for n-th state

$$E_n = \xi_n, \tag{28}$$

$$V(x) = U(\mu h(x)) + \frac{1}{8m(x)}\left[x^{1-2\alpha}(1-\alpha)\frac{m'(x)}{m(x)} + x^{2-2\alpha}\left\{\frac{m''(x)}{m(x)} - \frac{7}{4}\left(\frac{m'(x)}{m(x)}\right)^2\right\}\right] \tag{29}$$

and the wave function for n-th state immediately emerges from equation (7) as

$$\phi_n(x) = [m(x)]^{\frac{1}{4}}\psi_n(\mu h(x)), \tag{30}$$

where equations (24) and (25) have been used.

3.2 Choice-II

The second possibility is

$$[P^{'}(x)]^2U(P(x)) = \pm\frac{m(x)}{\sigma^2} \text{ and hence } P(x) = W^{-1}\left(\frac{\mu h(x)}{\sigma}\right), \tag{31}$$

where $W(P(x)) = \int\sqrt{\pm U(P(x))}dP(x)$ and the selection of $\pm$ depends on the sign of $U(P(x))$. For the time being we are taking $+$ sign only. This now makes

$$G(x) = [\sigma^2 m(x)U(P(x))]^{-\frac{1}{4}}. \tag{32}$$

As before inserting $G(x)$ and $P'(x) = \frac{1}{\sigma}\sqrt{\frac{m(x)}{U(P(x))}}$ into equation (22) we reach

$$\begin{aligned} Q_{II}(m(x), m'(x), x) \quad = \quad & -\frac{1}{4}x^{2-2\alpha}\left[\left\{\frac{m''(x)}{m(x)} - \frac{7}{4}\left(\frac{m'(x)}{m(x)}\right)^2\right\}\right. \\ & \left. -\frac{m(x)}{4\sigma^2 U(P(x))}\left\{\frac{U^{**}(P(x))}{U(P(x))} - \frac{5}{4}\left(\frac{U^*(P(x))}{U(P(x))}\right)^2\right\}\right] + Q_0, \end{aligned} \tag{33}$$

where

$$U^*(P(x)) = \frac{dU(P(x))}{dP(x)}, \tag{34}$$

$$U^{**}(P(x)) = \frac{d^2U(P(x))}{d[P(x)]^2}, \tag{35}$$

$$Q_0 = -\frac{x^{1-2\alpha}}{4}(1-\alpha)\left[\frac{m'(x)}{m(x)} + \frac{1}{\sigma}U^*(P(x))\left\{U(P(x))\right\}^{-\frac{3}{2}}\left\{m(x)\right\}^{\frac{1}{2}}\right]. \tag{36}$$

Now equation (21) provides

$$V(x) - E = \frac{1}{\sigma^2} - \frac{\xi_n}{\sigma^2 U(P(x))} - \frac{1}{2m(x)}Q_{II}(m(x), m'(x), x). \tag{37}$$

So the target problem has following energy spectrum, effective potential and wave function for n-th state

$$E_n = -\frac{1}{\sigma^2},$$

$$V(x) = -\frac{\xi_n}{\sigma^2 U(P(x))} - \frac{1}{2m(x)}Q_{II}(m(x), m'(x), x),$$

$$\phi_n(x) = [\sigma^2 m(x)U(P(x))]^{\frac{1}{4}}\psi_n(P(x)).$$

4 Reference Potential-Fractional Coulomb Type

According to the PCT scheme, a known problem or reference problem is very important as said while introducing equation (6). In this paper, we have chosen fractional Schrödinger equation problem with fractional Coulomb interaction. It is hard to find enough research on this topic, so in this section, we will solve the one-dimensional fractional Schrödinger equation with fractional Coulomb potential in short. Let us take the fractional Coulomb potential for equation (6) as

$$U(z) = \frac{k}{z^\alpha},$$

where k acts like a constant. It maybe positive or negative depending on the interaction, whether it is repulsive or attractive. Now the equation (6) converts into

$$\left[D^{2\alpha} - \frac{k}{z^{\alpha}} + 2\xi_n\right]\psi(z) = 0. \tag{38}$$

The asymptotic solution ($z \to \infty$) demands

$$\left[D^{2\alpha} + 2\xi_n\right]\psi(z) = 0. \tag{39}$$

It can be shown that the solution of equation (39) may be written in the following form

$$\psi(z) = C_{01}e^{\frac{b}{\alpha}z^{\alpha}} + C_{02}e^{-\frac{b}{\alpha}z^{\alpha}}, \tag{40}$$

where $C_{01,02}$ are constants and $b = \sqrt{-2\xi_n}$.

4.1 Derivation of (40)

It is easy to proof that $e^{\pm\frac{b}{\alpha}z^{\alpha}}$ is the eigenfunction of Katugampola derivative operator D^{α}.

$$D^{\alpha}[e^{\pm\frac{b}{\alpha}z^{\alpha}}] = z^{1-\alpha}\frac{d}{dz}e^{\pm\frac{b}{\alpha}z^{\alpha}} = z^{1-\alpha}e^{\pm\frac{b}{\alpha}z^{\alpha}}(\frac{b}{\alpha})\alpha z^{\alpha-1} = \pm be^{\pm\frac{b}{\alpha}z^{\alpha}}.$$

We will show that $\psi(z) = C_{01}e^{\frac{b}{\alpha}z^{\alpha}} + C_{02}e^{-\frac{b}{\alpha}z^{\alpha}}$ will generate the fractional Differential equation similar to equation (39).

$$\begin{aligned}
D^{2\alpha}\psi(z) &= D^{\alpha}D^{\alpha}\psi(z)\,, \\
&= D^{\alpha}D^{\alpha}[C_{01}e^{\frac{b}{\alpha}z^{\alpha}} + C_{02}e^{-\frac{b}{\alpha}z^{\alpha}}]\,, \\
&= D^{\alpha}b[C_{01}e^{\frac{b}{\alpha}z^{\alpha}} - C_{02}e^{-\frac{b}{\alpha}z^{\alpha}}]\,, \\
&= b^2[C_{01}e^{\frac{b}{\alpha}z^{\alpha}} + C_{02}e^{-\frac{b}{\alpha}z^{\alpha}}]\,, \\
&= b^2\psi(z).
\end{aligned}$$

So we can say $C_{01}e^{\frac{b}{\alpha}z^{\alpha}} + C_{02}e^{-\frac{b}{\alpha}z^{\alpha}}$ is a solution of $[D^{2\alpha} - b^2]\psi(z) = 0$. Now comparing with equation (39), it is straight forward to write the parameter $b = \sqrt{-2\xi_n}$.

The first part of the solution does not go with physical situation because it blows up the solution at infinity. So to get a finite solution the obvious choice is the second part. Now we can assume the complete solution of equation (38) as

$$\psi(z) = C_{02}f(z)e^{-\frac{b}{\alpha}z^{\alpha}}. \tag{41}$$

Substituting (41) into equation (38) following fractional differential equation emerges

$$z^{\alpha}D^{2\alpha}f(z) - 2bz^{\alpha}D^{\alpha}f(z) - 2kf(z) = 0. \tag{42}$$

The solution of (42) has been done with the help of power series method. The wave function and energy eigenvalue are

$$\psi_n(z) = \sum_{n=1}^{\infty} z^{\alpha n}e^{-\frac{b}{\alpha}z^{\alpha}}, \tag{43}$$

$$\xi_n = -\frac{k^2}{2\alpha^2 n^2}\,,\ \ [n \neq 0]. \tag{44}$$

4.2 Solution of (42)

Using $D^\alpha[f] = t^{1-\alpha} f^{'}$ and equation (4), the equation (38) can be written as

$$z^{2-\alpha} f^{''}(z) + [(1-\alpha)z^{1-\alpha} - 2bz]f^{'}(z) - 2kf(z) = 0,$$

where $f''(z) = \frac{d^2 f}{dz^2}$ and $f'(z) = \frac{df}{dz}$. Let us use power series method to solve this. Introducing the solution as

$$f(z) = \sum_{l=0}^{\infty} c_l z^{\alpha l + j},$$

and inserting $f''(z)$ and $f'(z)$

$$\sum_{l=0}^{\infty} c_l (\alpha l + j)(\alpha l + j - \alpha) z^{\alpha(l-1)+j} + \sum_{l=0}^{\infty} c_l [-2b(\alpha l + j) - 2k] z^{\alpha l + j} = 0.$$

The indicial equation (coefficient of lowest power z i.e. $z^{j-\alpha}$) corresponds to $l = 0$ and provides $c_0 j(j-\alpha) = 0$.

Taking $c_0 \neq 0$ we have $j = 0,\ \alpha$. The choice of $j = \alpha$ makes the coefficient (c_1) of next higher power z i.e. z^j a zero. So we restrict ourselves to the first choice for $j = 0$, $c_1 \neq 0$. The recurrence relation comes out as

$$c_{l+1} = c_l \frac{2b\alpha l + 2k}{l(l+1)\alpha^2}.$$

Now for physical cases the series must terminate for $l = n$. Here n is integer and in quantum cases it is regarded as principle quantum number.

$$[c_{l+1}]_{l=n} = 0 \implies b = -\frac{k}{\alpha n}.$$

Using $b = \sqrt{-2\xi}$ we have the wave function (43) and energy eigenvalue (44).

5 Application on the Actual Problem

In this section, we will use the results of Section 4 to find out the actual problem that was addressed in Section 3.

5.1 In Case of Choice-I

Here $z = P(x) = \mu h(x)$. Therefore, $U(\mu h(x)) = \frac{\tilde{k}}{[h(x)]^\alpha}$, where $\tilde{k} = \frac{k}{\mu^\alpha}$. So under the fractional Coulomb type interaction, the one dimensional fractional Schrödinger equation with position dependent mass provides the following energy spectrum, reference potential and wave function for n-th state

$$E_n = -\frac{\tilde{k}^2 \mu^{2\alpha}}{2\alpha^2 n^2},$$

$$V(x) = \frac{\tilde{k}}{[h(x)]^\alpha} + \frac{1}{8m(x)} \left[x^{1-2\alpha}(1-\alpha)\frac{m'(x)}{m(x)} + x^{2-2\alpha}\frac{m''(x)}{m(x)} - \frac{7}{4}x^{2-2\alpha}\left(\frac{m'(x)}{m(x)}\right)^2 \right],$$

$$\phi_n(x) = [m(x)]^{\frac{1}{4}} \sum_{n=1}^{\infty} c_n [\mu h(x)]^{\alpha n} e^{-\frac{b}{\alpha}[\mu h(x)]^\alpha},$$

where we have used equations (28, 29) and equation (30).

5.1.1 Mass Profile: $m(x) = \frac{1}{(1+\delta x)^2}$, where δ is a real constant

This mass profile has been studied in reference [31]. Here $h(x) = \frac{1}{\mu\delta} ln(1+\delta x)$. Now for this mass profile we have

$$E_n = -\frac{\tilde{k}^2\mu^{2\alpha}}{2\alpha^2 n^2},$$

$$V(x) = \frac{\tilde{k}\mu^\alpha\delta^\alpha}{[\ln(1+\delta x)]^\alpha} - \frac{1}{8}[x^{1-2\alpha}(1-\alpha)2\delta(1+\delta x) + \delta^2 x^{2-2\alpha}],$$

$$\phi_n(x) = \frac{1}{\sqrt{1+\delta x}}\sum_{n=1}^{\infty} c_n[\frac{1}{\delta}\ln(1+\delta x)]^{\alpha n} e^{-\frac{b}{\alpha}[\frac{1}{\delta}\ln(1+\delta x)]^\alpha}.$$

5.1.2 Mass Profile: $m(x) = m_0 x^{2\delta}$, where δ and m_0 are real constants

In this case the energy eigenvalue remains same but the reference potential and wave function are

$$V(x) = \frac{\tilde{k}}{[\theta x^{1+\delta}]^\alpha} - \frac{1}{8m_0 x^{2\alpha+2\delta}}(3\delta^2 + 2\alpha\delta),$$

$$\phi_n(x) = m_0^{\frac{1}{4}} x^{\frac{\delta}{2}} \sum_{n=1}^{\infty} c_n[\mu\theta x^{1+\delta}]^{\alpha n} e^{-\frac{b}{\alpha}[\mu\theta x^{1+\delta}]^\alpha}$$

where $\theta = \frac{\sqrt{m_0}}{\mu(1+\delta)}$ and $h(x) = \theta x^{1+\delta}$.

5.2 In Case of Choice-II

In this case $U(P(x)) = \frac{k}{[P(x)]^\alpha}$, the integration $\int \sqrt{U(P(x))}dP(x)$ delivers the function

$$W(P(x)) = \frac{\mu h(x)}{\sigma} = \left[\frac{2\sqrt{k}}{2-\alpha}\right][P(x)]^{\frac{2-\alpha}{2}},$$

and hence $P(x) = \left[\frac{2-\alpha}{2\sqrt{k}}\frac{\mu h(x)}{\sigma}\right]^{\frac{2}{2-\alpha}}$. Here the potential may be expressed as

$$\begin{aligned} V(x) &= -\frac{\xi_n}{\sigma^2}\frac{[P(x)]^\alpha}{k} - \frac{1}{2m(x)}Q_{II}(m(x), m'(x), x) \\ &= -\frac{\xi_n}{\sigma^{\frac{4}{2-\alpha}} k}\left[\frac{2-\alpha}{2\sqrt{k}}\mu h(x)\right]^{\frac{2\alpha}{2-\alpha}} - \frac{1}{2m(x)}Q_{II}(m(x), m'(x), x). \end{aligned}$$

The target potential term contains the principle quantum number n and according to quantum mechanics it is not acceptable. So we impose a condition on the scaling parameter σ such that

$$\sigma^{\frac{4}{2-\alpha}} \propto -\xi_n \quad \text{or} \quad \sigma^{\frac{4}{2-\alpha}} = -\mathcal{N}\xi_n$$

where $\mathcal{N}$ is a simple constant. This provides the energy eigenvalue of the target problem as

$$E_n = -\frac{1}{\sigma^2} = -\left(\frac{\alpha n}{k}\sqrt{\frac{2}{\mathcal{N}}}\right)^{2-\alpha},$$

and the wave function

$$\phi_n(x) = [\sigma^2 m(x) U(P(x))]^{\frac{1}{4}} \psi_n(P(x)),$$

where $P(x)$ should be used as

$$P(x) = \left[\frac{2-\alpha}{2\sqrt{k}}\frac{\mu h(x)}{\sigma}\right]^{\frac{2}{2-\alpha}}.$$

Here the modified target potential takes the form

$$V(x) = \frac{1}{\mathcal{N}k}\left[\frac{2-\alpha}{2\sqrt{k}}\mu h(x)\right]^{\frac{2\alpha}{2-\alpha}} - \frac{1}{2m(x)}Q_{II}(m(x), m'(x), x).$$

5.2.1 Mass Profile: $m(x) = \frac{1}{(1+\delta x)^2}$, where δ is a real constant

In this situation $h(x) = \frac{1}{\mu\delta} ln(1+\delta x)$,

$$P(x) = \left[\frac{2-\alpha}{2\sqrt{k}\sigma\delta} ln(1+\delta x)\right]^{\frac{2}{2-\alpha}}.$$

Using $U(P(x)) = \frac{k}{[P(x)]^\alpha}$ it is not hard to derive

$$E_n = -\frac{1}{\sigma^2} = -\left(\frac{\alpha n}{k}\sqrt{\frac{2}{\mathcal{N}}}\right)^{2-\alpha}, \tag{46}$$

$$V(x) = \frac{1}{\mathcal{N}k}\left[\frac{2-\alpha}{2\delta\sqrt{k}}\ln(1+\delta x)\right]^{\frac{2\alpha}{2-\alpha}} - \frac{1}{2}(1+\delta x)^2 Q_{II}(m(x), m'(x), x), \tag{47}$$

$$\phi_n(x) = \left[\frac{\sigma^2}{(1+\delta x)^2}U(P(x))\right]^{\frac{1}{4}}\psi_n(P(x)). \tag{48}$$

To express the equation (47) explicitly, we need to manipulate equations (33), (34)–(36) using the present form of $P(x)$ as well as $m(x)$, $m'(x)$, $m''(x)$.

5.2.2 Mass Profile: $m(x) = m_0 x^{2\delta}$, where δ and m_0 are real constants

As we did this mass profile earlier and hence we have $h(x) = \theta x^{1+\delta}$ where $\theta = \frac{\sqrt{m_0}}{\mu(1+\delta)}$. Now

$$P(x) = [\frac{2-\alpha}{2\sqrt{k}}\frac{\mu\theta x^{1+\delta}}{\sigma}]^{\frac{2}{2-\alpha}}.$$

These help us to get

$$E_n = -\frac{1}{\sigma^2} = -\left(\frac{\alpha n}{k}\sqrt{\frac{2}{\mathcal{N}}}\right)^{2-\alpha},$$

$$V(x) = \frac{1}{\mathcal{N}k}\left[\frac{2-\alpha}{2\sqrt{k}}\mu\theta x^{1+\delta}\right]^{\frac{2\alpha}{2-\alpha}} - \frac{1}{2m_0}x^{-2\delta}Q_{II}(m(x), m'(x), x),$$

$$\phi_n(x) = \left[\sigma^2 m_0 x^{2\delta}U(P(x))\right]^{\frac{1}{4}}\psi_n(P(x)).$$

The calculation of $Q_{II}(m(x), m'(x), x)$ should be done accordingly as previous.

6 Result and Discussion

Mathematically choice-II is much complicated. So in this section, we shall furnish numerical data of the energy spectrum and variation of wave functions for the choice-I case only with the mass profile $m(x) = \frac{1}{(1+\delta x)^2}$ and $m(x) = m_0 x^{2\delta}$. These two mass profiles provide the same energy eigenvalues that mean these two cases are iso-spectral. The energy spectrum for these two mass profiles is given in Table 1. We also provide the wave functions for $n = 1, 2$ states both for mass profile 5.1.1 and 5.1.2. Figures 1 and 2 show the variations of wave functions for $n = 1, 2$ state with different fractional parameter α and the mass profile 5.1.1. The same two states are shown in Figures 3 and 4 for the mass profile 5.1.2. The result of the wave functions for these two mass profiles is opposite to each other. In the case of a singular type of mass profile, i.e. 5.1.1, the fractional Schrödinger equation with position-dependent mass indicates for lower α the probability of finding the particle in a specified region becomes lesser than the higher α cases. On the other hand, the situation is opposite to the mass profile 5.1.2. Figures 5 and 6 are for target potential for different α and they are just similar to the Coulomb interaction with only a slight shift.

Table 1: **Energy spectrum for the mass profile 5.1.1 and 5.1.2** ($\mu = 1, k = \sqrt{27.2}$)

State	α	$\xi_n = E_n$ (eV)
1	0.80	-21.2500
	0.85	-18.8235
	0.90	-16.7901
	0.95	-15.0693
	1.00	-13.60
2	0.80	-5.3125
	0.85	-4.7059
	0.90	-4.1975
	0.95	-3.7673
	1.00	-3.40
3	0.80	-2.3611
	0.85	-2.0915
	0.90	-1.8656
	0.95	-1.6744
	1.00	-1.5111

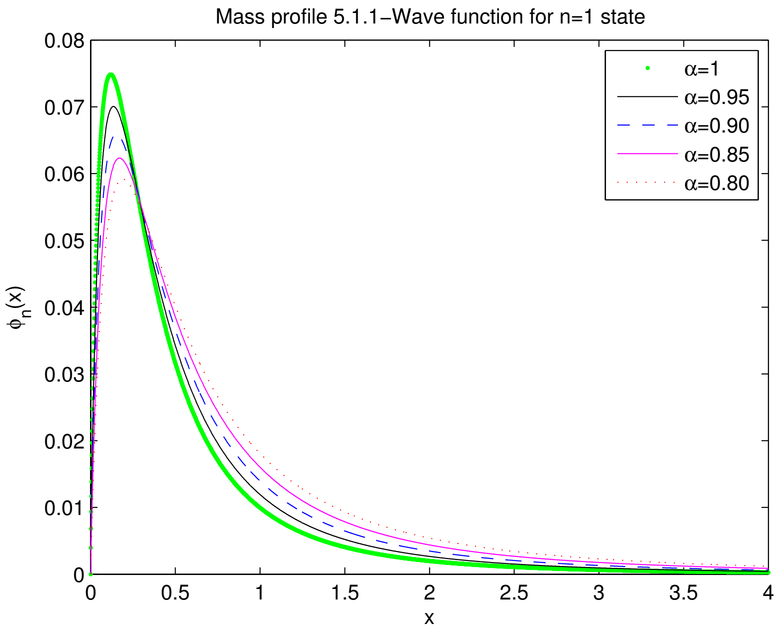

Figure 1: $n = 1$ state eigenfunctions for $\alpha = 0.80, 0.85, 0.90, 0.95, 1.00$. The other parameter values are used $c_1 = 1, \delta = 2, k = \sqrt{27.2}$.

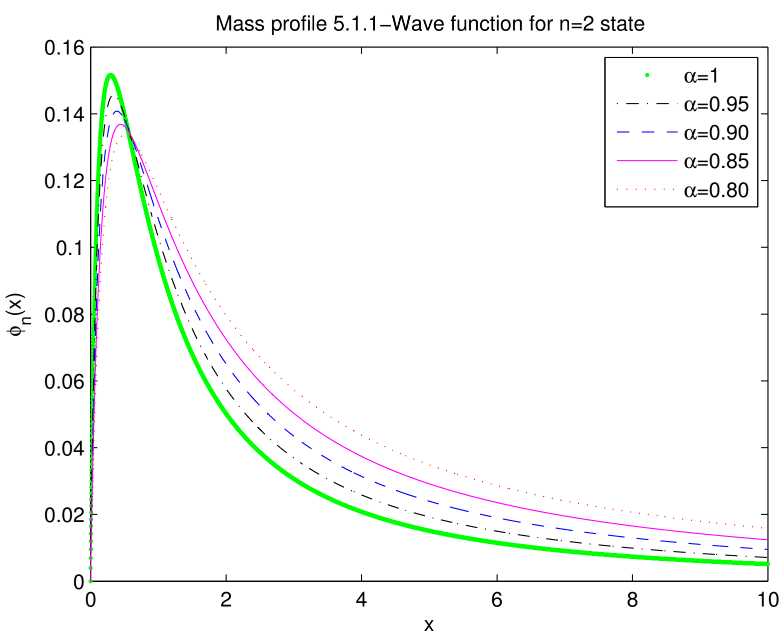

Figure 2: $n = 2$ state eigenfunctions for $\alpha = 0.80, 0.85, 0.90, 0.95, 1.00$. The other parameter values are used $c_1 = c_2 = 1, \delta = 2, k = \sqrt{27.2}$.

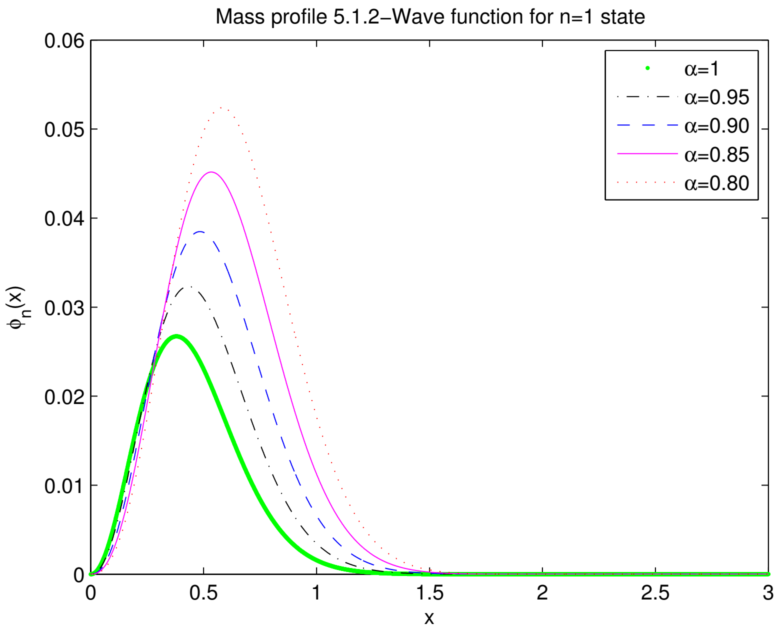

Figure 3: $n = 1$ state eigenfunctions for $\alpha = 0.80, 0.85, 0.90, 0.95, 1.00$. The other parameter values are used $c_1 = m_0 = 1 = \mu = \delta = 1, k = \sqrt{27.2}$.

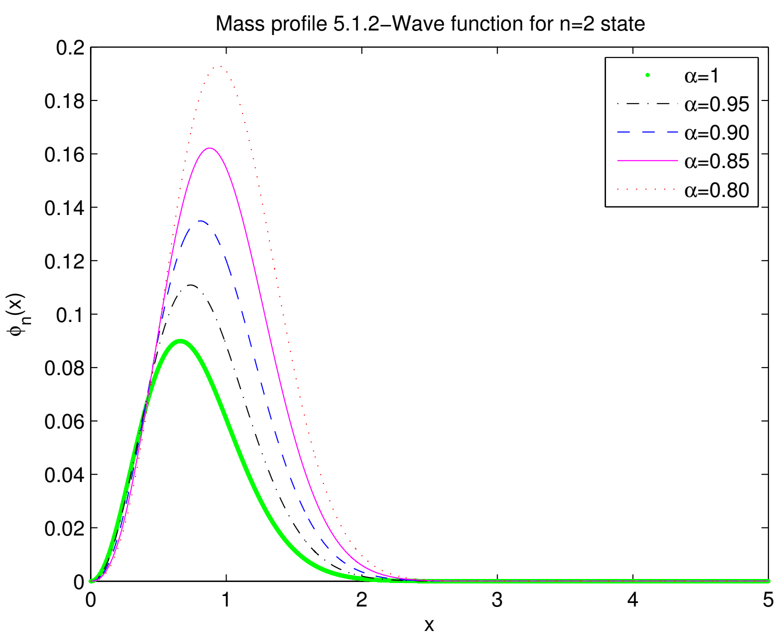

Figure 4: $n = 2$ state eigenfunctions for $\alpha = 0.80, 0.85, 0.90, 0.95, 1.00$. The other parameter values are used $c_1 = c_2 = 1 = m_0 = 1 = \delta = 1, k = \sqrt{27.2}$.

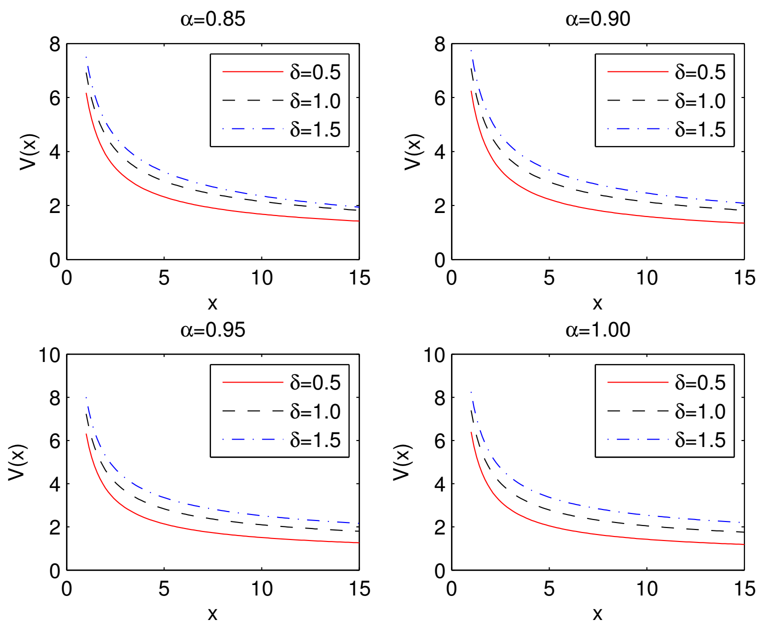

Figure 5: Target potential function (5.1.1) profile for $\mu = 1$, $k = \sqrt{27.2}$.

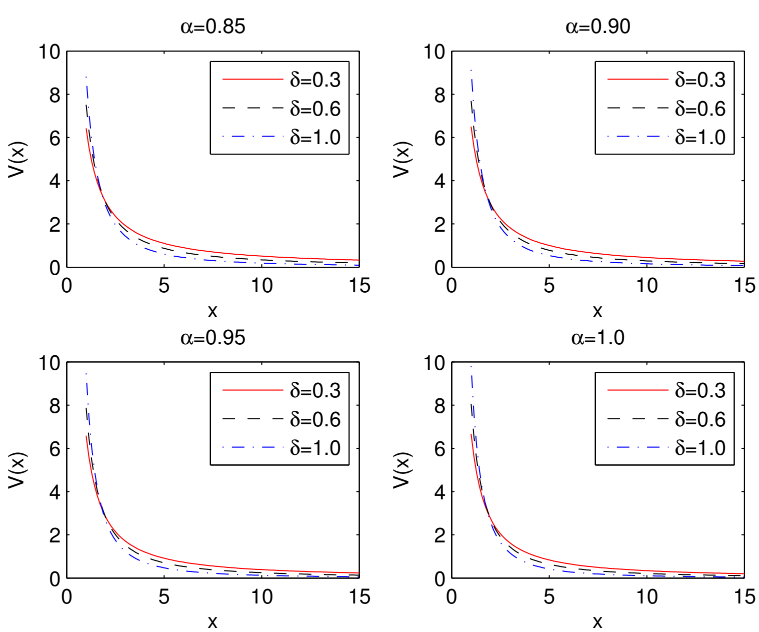

Figure 6: Target potential function (5.1.2) profile for $m_0 = \mu = 1$, $k = \sqrt{27.2}$.

7 Conclusion

In this paper, we have studied the fractional Schrödinger equation with position-dependent mass. The Katugampola fractional derivative has been taken to define the fractional Schrödinger equation with position-dependent mass. As an example two mass profiles have been considered viz $m(x) = \frac{1}{(1+\delta x)^2}$ and $m(x) = m_0 x^{2\delta}$. Point canonical transformation technique has been used as an analytical tool to solve the projected equation. The results are the same if the fractional parameter α is set to unity. The energy spectrum and a few wave functions ($n = 1, 2$ states) are discussed for the two selected mass profiles. We need further investigation to extract the hidden physical facts behind the problem.

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References

[1] O. V. Roos, Position-dependent effective masses in semiconductor theory, Physical Review B, 27(1983), 7547–7552.

[2] G. T. Einevoll, Operator ordering in effective-mass theory for heterostructures, II. Strained systems, Physical Review B, 42(1990), 3497–3496.

[3] M. Barranco, M. Pi, S. M. Gatica, E. S. Hernandez and J. Navarro, Structure and energetics of mixed ^{4}He-^{3}He drops, Phys. Rev. B, 56(1997), 8997–9003.

[4] D. L. Smith and C. Mailhiot, Theory of semiconductor superlattice electronic structure, Reviews of Modern Physics, 62(1990), 173–234.

[5] L. Chetouani, L. Dekar and Th. F. Hammann, Green's functions via path integrals for systems with position-dependent masses, Phys. Rev. A, 52(1995), 82–91.

[6] A. R. Plastino, A. Rigo, M. Casas, F. Garcias and A. Plastino, Supersymmetric approach to quantum systems with position-dependent effective mass, Phys. Rev. A, 60(1999), 4318–4325.

[7] R. Renan, J. M. Pereira, J. Ribeiro, V. N. Freire and G. A. Farias, The effect of a position dependent effective mass on the transmission of electrons through a double graded barrier, Brazilian Journal of Physics, 24(1994), 192–196.

[8] T. Tanaka, N-fold supersymmetry in quantum systems with position-dependent mass, Journal of Physics A: Mathematical and General, 39(2005), 219.

[9] Y. Li, H. M Lu, O. Voskoboynikov, C. P. Lee, S. M. Sze, Dependence of energy gap on magnetic field in semiconductor nano-scale quantum rings, Surface Science, 532(2003), 811–815.

[10] F. B. Adda and J. Cresson, Fractional differential equations and the Schrödinger equation, Applied Mathematics and Computation, 161(2005), 323–345.

[11] J. Banerjee, U. Ghosh, S. Sarkar, and S. Das, A study of fractional Schrödinger equation composed of Jumarie fractional derivative, Pramana, 88(2017), 70.

[12] G. M. Bisci, and V. D. Radulescu, Ground state solutions of scalar field fractional Schrödinger equations, Calc. Var. Partial Differential Equations, 54(2015), 2985-3008.

[13] T. Das, U. Ghosh, S. Sarkar and S. Das, Time independent fractional Schrödinger equation for generalized Mie-type potential in higher dimension framed with Jumarie type fractional derivative, J. Math. Phys., 59(2018), 17pp.

[14] T. Das, U. Ghosh, S. Sarkar and S. Das, Higher-dimensional fractional time-independent Schrödinger equation via fractional derivative with generalised pseudoharmonic potential, Pramana, 93(2019), 76.

[15] J. Dong and M. Xu, Some solutions to the space fractional Schrödinger equation using momentum representation method, J. Math. Phys., 48(2007), 14pp.

[16] Y. Luchko, Fractional Schrdinger equation for a particle moving in a potential well, Journal of Mathematical Physics, J. Math. Phys., 54(2013), 10pp.

[17] M. Naber, Time fractional Schrödinger equation, J. Math. Phys., 45(2004), 3339–3352.

[18] S. Wang, and M. Xu, Generalized fractional Schrödinger equation with space-time fractional derivatives, J. Math. Phys., 48(2007), 10 pp.

[19] N. Laskin, Fractional quantum mechanics, Phys. Rev. E, 62(2000), 3135–3145.

[20] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E, 66(2002), 7 pp.

[21] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of Their Applications, 198, Elsevier, 1998.

[22] J. Sabatier, O. P. Agrawal and J. A. T. Machado, Advances in Fractional Calculus, 9, Springer, 2007.

[23] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, Comput. Math. Appl., 51(2006), 1367–1376

[24] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl, 1(2015), 1–13.

[25] R. Khalil, M. Al. Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., 264(2014), 6–70.

[26] U. N. Katugampola, A new approach to generalized fractional derivatives, Bull. Math. Anal. Appl, 6(2014), 1–15.

[27] A. D. Alhaidari, Solutions of the non-relativistic wave equation with position-dependent effective mass, Phys. Rev. A, 66(2002), 042116.

[28] C.Tezcan and R. Sever, PT-symmetric Solutions of Schrödinger Equation with position-dependent mass via Point Canonical Transformation, International Journal of Theoretical Physics, 47(2008), 1471–1478.

[29] R. De, R. Dutt and U. Sukhatme, Mapping of shape invariant potentials under point canonical transformations, J. Phys. A, 25(1992), L843–L850.

[30] D. R. Anderson and D. J. Ulness, Properties of the Katugampola fractional derivative with potential application in quantum mechanics, J. Math. Phys., 56(2015), 18pp.

[31] J. J. Pena, J. Morales and J. Garcia-Ravelo, Schrödinger Equation with Position-Dependent Mass: Staggered Mass Distributions, World Academy of Science, Engineering and Technology, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 11(2017), 329–332.

8 Appendix

Deduction of equation (26): In this scenario $G(x) = [m(x)]^{-\frac{1}{4}}$ and $[P^{'}(x)]^{-1} = [m(x)]^{-\frac{1}{2}}$.
The Katugampola fractional derivative provides

$$D^{\alpha}G(x) = x^{1-\alpha}\frac{d}{dx}[m(x)]^{-\frac{1}{4}} = -\frac{1}{4}x^{1-\alpha}m^{-\frac{5}{4}}(x)m'(x),$$

$$\begin{aligned} D^{2\alpha}G(x) &= -\frac{1}{4}\Big[m'(x)m^{-\frac{5}{4}}(x)D^{\alpha}(x^{1-\alpha}) + x^{1-\alpha}m^{-\frac{5}{4}}(x)D^{\alpha}(m'(x)) + x^{1-\alpha}m'(x)D^{\alpha}m^{-\frac{5}{4}}(x)\Big] \\ &= -\frac{1}{4}\Big[m'(x)m^{-\frac{5}{4}}(x)(1-\alpha)x^{1-2\alpha} + x^{2-2\alpha}m^{-\frac{5}{4}}(x)(m''(x)) - \frac{5}{4}x^{2-2\alpha}(m')^2(x)m^{-\frac{5}{4}}(x)\Big], \end{aligned}$$

$$D^{\alpha}[P^{'}(x)]^{-1} = D^{\alpha}[m(x)]^{-\frac{1}{2}} = -\frac{1}{2}x^{1-\alpha}m^{-\frac{3}{2}}(x)m'(x),$$

$$D^{\alpha}G(x)D^{\alpha}[P^{'}(x)]^{-1} = \frac{1}{8}x^{2-2\alpha}(m')^2m^{-\frac{11}{4}},$$

$$\frac{D^{\alpha}G(x)D^{\alpha}[P^{'}(x)]^{-1}}{G(x)[P^{'}(x)]^{-1}} = \frac{1}{8}x^{2-2\alpha}\left(\frac{m'(x)}{m(x)}\right)^2.$$

Substituting all these in equation (22)

$$Q_I(m(x), m'(x), x) = -\frac{1}{4}\left[x^{1-2\alpha}(1-\alpha)\frac{m'(x)}{m(x)} + x^{2-2\alpha}\frac{m''(x)}{m(x)} - \frac{7}{4}x^{2-2\alpha}\left(\frac{m'(x)}{m(x)}\right)^2\right].$$

Proof of equation (33). Here $G(x) = [\sigma^2 m(x)U(P(x))]^{-\frac{1}{4}}$ and $[P^{'}(x)]^{-1} = \sigma\{m(x)\}^{-\frac{1}{2}}\{U(P(x))\}^{\frac{1}{2}}$.

$$\begin{aligned} D^{\alpha}G(x) &= x^{1-\alpha}\frac{d}{dx}[\sigma^2 m(x)U(P(x))]^{-\frac{1}{4}} \\ &= \frac{1}{\sqrt{\sigma}}x^{1-\alpha}\big[-\frac{1}{4}\{m(x)\}^{-\frac{5}{4}}m'(x)\{U(P(x))\} - \frac{1}{4\sigma}\{m(x)\}^{\frac{1}{4}}U^*(P(x))\{U(P(x))\}^{-\frac{7}{4}}\big], \end{aligned}$$

$$\begin{aligned} D^{2\alpha}G(x) &= x^{1-\alpha}\frac{d}{dx}\left\{\frac{1}{\sqrt{\sigma}}x^{1-\alpha}\big[-\frac{1}{4}\{m(x)\}^{-\frac{5}{4}}m'(x)\{U(P(x))\} - \frac{1}{4\sigma}\{m(x)\}^{\frac{1}{4}}U^*(P(x))\{U(P(x))\}^{-\frac{7}{4}}\big]\right\} \\ &= -\frac{x^{1-\alpha}}{4\sqrt{\sigma}}[f_1+f_2+f_3+f_4+f_5+f_6+f_7+f_8], \end{aligned}$$

where

$$\begin{aligned} f_1 &= (1-\alpha)\{m(x)\}^{-\frac{5}{4}}m'(x)\{U(P(x))\}^{-\frac{1}{4}}x^{-\alpha}, \\ f_2 &= -\frac{5}{4}x^{1-\alpha}\{m'(x)\}^2\{U(P(x))\}^{-\frac{1}{4}}\{m(x)\}^{-\frac{9}{4}}, \\ f_3 &= x^{1-\alpha}\{m(x)\}^{-\frac{5}{4}}m''(x)\{U(P(x))\}^{-\frac{1}{4}}, \\ f_4 &= -\frac{1}{4\sigma}x^{1-\alpha}\{U(P(x))\}^{-\frac{7}{4}}U^*(P(x))m'(x)\{m(x)\}^{-\frac{3}{4}}, \\ f_5 &= \frac{1}{\sigma}(1-\alpha)x^{-\alpha}\{U(P(x))\}^{-\frac{7}{4}}U^*(P(x))\{m(x)\}^{\frac{1}{4}}, \\ f_6 &= -\frac{7}{4\sigma^2}x^{1-\alpha}\{U(P(x))\}^{-\frac{13}{4}}(U^*(P(x)))^2\{m(x)\}^{\frac{3}{4}}, \end{aligned}$$

$$
\begin{aligned}
f_7 &= \frac{1}{\sigma^2} x^{1-\alpha} \left\{U(P(x))\right\}^{-\frac{9}{4}} U^{**}(P(x)) \left\{m(x)\right\}^{\frac{3}{4}}, \\
f_8 &= \frac{1}{4\sigma} x^{1-\alpha} \left\{U(P(x))\right\}^{-\frac{7}{4}} U^{*}(P(x)) \left\{m(x)\right\}^{-\frac{3}{4}} m'(x).
\end{aligned}
$$

$$
D^\alpha [P(x)]^{-1} = -\frac{\sigma}{2} x^{1-\alpha} \left\{m(x)\right\}^{-\frac{3}{2}} m'(x) \left\{U(P(x))\right\}^{\frac{1}{2}} + \frac{1}{2} x^{1-\alpha} \frac{U^{*}(P(x))}{U(P(x))}.
$$

Using all these into equation (22) we have the results (33) to (36). ■