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Existence Of Multiple Solutions For A Kirchhoff Type Equation Involving Polyharmonic Operator With Exponential Growth*

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Abstract

In this article, we establish the existence of three weak solutions for a nonlinear Kirchhoff type elliptic equation involving polyharmonic operator by using variational methods. We assume that the nonlinearity satisfies subcritical exponential growth condition. We use a critical point theorem by B. Ricceri to prove our result.

1 Introduction

In this paper, we establish the existence of solutions to the problem:

$$\begin{array}{ll} M\left(\int_\Omega |\nabla^m u|^2 dx\right)(-\Delta)^m u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \nabla u = \ldots = \nabla^{m-1} u = 0 & \text{on } \partial\Omega, \end{array} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^{2m}$, $m \geq 1$ is a smooth and bounded domain, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions having subcritical exponential growth, μ, λ are parameters. We assume that $M : [0,\infty) \to \mathbb{R}$ is a continuous, non-decreasing function satisfying the following hypothesis:

(M1) There exist $m_0 > 0$, $\alpha > 1$ and $M(t) \geq m_0 t^{\alpha-1}$ for all $t \in [0,\infty)$.

Moser-Trudinger inequality is an important tool for the study of second order elliptic equations with exponential nonlinearity. The classical Moser-Trudinger inequality [16, 18] reads as follows:

Theorem 1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $u \in W_0^{1,n}(\Omega)$, $n \geq 2$ and*

$$\int_\Omega |\nabla u(x)|^n dx \leq 1.$$

Then there exists a constant C, which depends on n only such that

$$\int_\Omega \exp(\alpha u^p) dx \leq C|\Omega|,$$

where

$$p = \frac{n}{n-1}, \quad \alpha \leq \alpha_n = n\omega_n^{\frac{1}{n-1}},$$

and ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere.

The integral on the left actually is finite for any positive α, but if $\alpha > \alpha_n$ it can be made arbitrarily large by an appropriate choice of u.

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Moser-Trudinger inequality was extended to higher order Sobolev spaces by D. R. Adams [1]. The Adams' inequality is as follows:

Theorem 2 *Let Ω be a bounded and open subset of $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C(n,m)$ such that for all $u \in C^m(\mathbb{R}^n)$ with support contained in Ω and $\|\nabla^m u\|_p \leq 1$, $p = \frac{n}{m}$, we have*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta |u(x)|^{\frac{n}{n-m}})dx \leq C(n,m) \tag{2}$$

for all $\beta \leq \beta(n,m)$ where

$$\beta(n,m) = \begin{cases} \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{p'} & \text{when } m \text{ is odd,} \\ \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{p'} & \text{when } m \text{ is even} \end{cases}$$

and $p' = \frac{p}{p-1}$. Furthermore, for any $\beta > \beta(n,m)$, the integral can be made as large as desired, where

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u & \text{when } m \text{ is even,} \\ \nabla \triangle^{\frac{m-1}{2}} u & \text{when } m \text{ is odd.} \end{cases}$$

In case of $n = 2m$, $\beta(2m,m) = 2^{2m}\pi^m\Gamma(m+1)$ for all m. Throughout this paper, we denote the constant $C(2m,m)$ by C_0.

For some applications of the Adams' inequality to polyharmonic equations with exponential nonlinearities, we refer to [11, 4]. N. Lam and G. Lu [12] established the existence of a nontrivial solution to the following polyharmonic problem:

$$\begin{aligned} &(-\Delta)^m u = f(x,u) && \text{in } \Omega, \\ &u = \nabla u = \ldots = \nabla^{m-1} u = 0 && \text{on } \partial\Omega, \end{aligned}$$

They assume that f satisfies subcritical and critical growth condition and employed mountain pass theorem to establish their result. S. Goyal and K. Sreenadh [8] used Nehari manifold and fibering maps to obtain existence of multiple solutions to the problem:

$$\begin{aligned} &\Delta_{\frac{n}{m}}^m u = \lambda h(x)|u|^{q-1}u + u|u|^p e^{|u|^\beta} && \text{in } \Omega, \\ &u = \nabla u = \ldots = \nabla^{m-1} u = 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^n$, $n \geq 2m$, $0 < q < \frac{n}{m} - 1 < p+1$ and $\beta \in (1, \frac{n}{n-m}]$.

Problem (1) is related to the higher order analogue of Kirchoff equation [10],

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_\Omega \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0.$$

Mishra et al. [15] used mountain pass theorem to establish the existence of a nontrivial solution to the following Kirchhoff type problem:

$$\begin{aligned} &-M\left(|\nabla^m u|^{\frac{n}{m}}\right)\Delta_{\frac{n}{m}}^m u = \frac{f(x,u)}{|x|^\alpha} && \text{in } \Omega, \\ &u = \nabla u = \ldots = \nabla^{m-1} u = 0 && \text{on } \partial\Omega. \end{aligned}$$

They assumed that f grows like $e^{\frac{n}{n-m}}$ and $0 < \alpha < n \geq 2m$. Mishra et al. [15] also established the existence result for the following Kirchhoff type problem:

$$\begin{aligned} &-M\left(|\nabla^m u|^{\frac{n}{m}}\right)\Delta_{\frac{n}{m}}^m u = \lambda h(x)|u|^{q-1}u + u|u|^p e^{|u|^\beta} && \text{in } \Omega, \\ &u = \nabla u = \ldots = \nabla^{m-1} u = 0 && \text{on } \partial\Omega. \end{aligned}$$

We also refer to [3, 7, 9, 20] and references cited therein for some more existence results for higher order Kirchoff type equations.

Several authors have used Ricceri's critical point theorem [17] to establish the existence and multiplicity results for elliptic boundary value problems. For instance, see [2, 5, 6, 13, 14] and references therein. In this article, we use Ricceri's critical point theorem [17] to prove the existence of three weak solutions to (1). The main result of the paper is as follows;

Theorem 3 *Let $f \in \mathcal{F}$ be such that*

(F1) $\sup_{u \in H_0^m(\Omega)} \int_\Omega F(x,u)dx > 0$;

(F2) $\limsup_{t\to 0} \frac{F(x,t)}{|t|^{2\alpha}} \leq 0$;

(F3) $\limsup_{|t|\to\infty} \frac{F(x,t)}{|t|^{2\alpha}} \leq 0$.

Set

$$a = \frac{1}{2} \inf \left\{ \frac{\hat{M}(\|u\|^2)}{\int_\Omega F(x,u)dx} : u \in H_0^m(\Omega),\ \int_\Omega F(x,u)dx > 0 \right\}.$$

Then for each compact interval $K \subseteq (a, +\infty)$, there exists a number $\eta > 0$ with the following property: for every $\lambda \in K$ and $g \in \mathcal{F}$ there exists $\mu^* > 0$ such that for each $\mu \in [0, \mu^*]$, (1) has at least three weak solutions having norms less than η.

The plan of the paper is as follows: In Section 2, we state some definitions and preliminary results which would be used to prove the main theorem. In Section 3, we prove Theorem 3.

2 Preliminaries

In this section, we describe some notations, state some definitions and preliminary results. We say that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has subcritical exponential growth if

$$\lim_{|u|\to\infty} \frac{|f(x,u)|}{\exp(\alpha u^2)} = 0,\ \forall \alpha > 0 \text{ and a.e. in } \Omega. \tag{3}$$

The growth is called critical if there exists $\alpha^* > 0$ such that

$$\lim_{|u|\to\infty} \frac{|f(x,u)|}{\exp(\alpha u^2)} = 0 \text{ for all } \alpha > \alpha^* \text{ and a.e. in } \Omega,$$

$$\lim_{|u|\to\infty} \frac{|f(x,u)|}{\exp(\alpha u^2)} = \infty \text{ for all } \alpha < \alpha^* \text{ and a.e. in } \Omega.$$

Definition 1 *We denote by $\mathcal{F}$ a class of functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ each of which satisfies the following properties:*

1. *f is Carathéodory function.*
2. *f has subcritical exponential growth, i.e., (3) is satisfied.*
3. *For every $B > 0$, $\sup_{|t|\leq B} |f(x,t)| \in L^\infty(\Omega)$.*

Definition 2 *Suppose X is a Banach space. We denote by $\mathcal{L}_X$ the class of functionals $L : X \to \mathbb{R}$ with the property: If $u_n \rightharpoonup u$ weakly in X and $\liminf_{n\to\infty} L(u_n) \leq L(u)$, then $\{u_n\}$ has a convergent subsequence converging to u.*

Next, we recall the statement of Ricerri critical point theorem [17]:

Theorem 4 *Let X be a separable and reflexive real Banach space. Suppose $\Phi, I : X \to \mathbb{R}$ are C^1 functionals satisfying the following conditions:*

1. *Φ is coercive, sequentially weakly lower semicontinuous and is of class $\mathcal{L}_X$.*
2. *Φ is bounded on each bounded subset of X.*
3. *Φ' admits a continuous inverse on X^*.*
4. *Φ has a strict local minimum at u_0 with $\Phi(u_0) = I(u_0) = 0$.*
5. *I' is compact.*
6. $\max\left\{\limsup_{\|u\|\to\infty} \frac{I(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{I(u)}{\Phi(u)}\right\} \leq 0$ *and* $\sup_{u\in X} \min\{\Phi(u), I(u)\} > 0$.

Set

$$a := \inf\left\{\frac{\Phi(u)}{I(u)} : u \in X,\ \min\{\Phi(u), I(u)\} > 0\right\}.$$

Then for each compact interval $K \subseteq (a, +\infty)$, there exists a number $\eta > 0$ with the following property: for every $\lambda \in K$ and every C^1 functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\mu^* > 0$ such that for each $\mu \in [0, \mu^*]$,

$$\Phi'(u) = \lambda I'(u) + \mu J'(u)$$

has at least three solutions having norm less than η.

Throughout this paper, we consider the Sobolev space $H_0^m(\Omega)$ equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla^m u|^2 dx\right)^{\frac{1}{2}}.$$

By Sobolev embedding theorem, $H_0^m(\Omega)$ is continuously embedded into $L^q(\Omega)$ for every $q \geq 1$. Let S_q be the optimal constant of this embedding, then we have

$$\|u\|_q \leq S_q \|u\|,$$

where $\|\cdot\|_q$ is the standard norm in L^q space. Next, we define weak solution of (1).

Definition 3 *We say that $u \in H_0^m(\Omega)$ is a weak solution to (1) if*

$$M\left(\int_\Omega |\nabla^m u|^2 dx\right)\int_\Omega \nabla^m u \nabla^m v dx - \lambda \int_\Omega f(x,u)v dx - \mu \int_\Omega g(x,u)v dx = 0$$

for every $v \in H_0^m(\Omega)$.

For a given $f \in \mathcal{F}$, define $F(x,t) = \int_0^t f(x,s)ds$. We also define the functionals $\gamma, \Phi, I : H_0^m(\Omega) \to \mathbb{R}$ by

$$\gamma(u) = \int_\Omega |\nabla^m u|^2 dx,$$

$$\Phi(u) = \frac{1}{2}\hat{M}(\gamma(u)),\ \text{where } \hat{M}(t) = \int_0^t M(s)ds,$$

and

$$I(u) = \int_\Omega F(x,u)dx.$$

It is easy to see that Φ and I are of the class C^1 and

$$\langle I'(u), v\rangle = \int_\Omega f(x,u)v dx,$$

$$\langle \Phi'(u), v\rangle = M\left(\int_\Omega |\nabla^m u|^2 dx\right)\int_\Omega \nabla^m u \nabla^m v dx.$$

for all $u, v \in H_0^m(\Omega)$.

3 Proof of Theorem 3

To prove Theorem 3, we first prove some lemmas.

Lemma 1 *If $f \in \mathcal{F}$, then the functional $H : H_0^m(\Omega) \to \mathbb{R}$ defined by $H(u) = \int_\Omega F(x, u(x))dx$, where $F(x,t) = \int_0^t f(x,s)ds$ is C^1 and $H' : H_0^m(\Omega) \to (H_0^m(\Omega))^*$ is compact. Here $(H_0^m(\Omega))^*$ is the dual of $H_0^m(\Omega)$.*

Proof. Since f satisfies subcritical growth condition (3), we have

$$|f(x,t)| \le C\exp(\kappa t^2).$$

Then for every $u \in H_0^m(\Omega)$, and almost every $x \in \Omega$,

$$|F(x,u)| \le C|u|\exp(\kappa u^2).$$

By Adams inequality and Holder's inequality, H is well defined on $H_0^m(\Omega)$. Next, we show that H is Gateaux differentiable with derivative

$$\langle H'(u), v\rangle = \int_\Omega f(x,u)v dx, \;\; \forall\, u, v \in H_0^m(\Omega). \tag{4}$$

For $u, v \in H_0^m(\Omega)$ and $t \in (0,1)$, we have

$$\frac{H(u+tv) - H(u)}{t} = \int_\Omega \frac{F(x, u+tv) - F(x,u)}{t} dx = \int_\Omega f(x, u + t\tau(x)v(x))v(x)dx,$$

where τ is a measurable function taking values in $[0,1]$. This gives

$$\lim_{t\to 0} \frac{H(u+tv) - H(u)}{t} = \int_\Omega f(x,u)v dx.$$

This proves (4). Next, we show that if $\{u_n\}$ is a bounded sequence in $H_0^m(\Omega)$, then

$$\sup_n \int_\Omega |f(x,u_n)|^q dx < \infty \text{ for all } q > 0.$$

Since $\{u_n\}$ is bounded, there exists $L > 0$ such that $\|u_n\| \le L$, $\forall n \ge 1$. Since f satisfies (3),

$$f(x,u_n) \le C\exp(\kappa|u|^2)$$

for some constant $C > 0$.

$$\begin{aligned}\int_\Omega |f(x,u)|^q dx &\le \int_\Omega C^q \exp(\kappa q|u_n|^2)dx \\ &= C^q \int_\Omega \exp\left(\kappa q\,\|u_n\|^2 \left(\frac{|u_n|}{\|u_n\|}\right)^2\right)dx \\ &\le C^q \int_\Omega \exp\left(\kappa q L^2 \left(\frac{|u_n|}{\|u_n\|}\right)^2\right)dx.\end{aligned}$$

By Theorem 2 if $0 < \kappa < \frac{\beta(2m,m)}{qL^2}$, then

$$\sup_n \int_\Omega |f(x,u_n)|^q dx < \infty.$$

Now, suppose $\{u_n\}$ is a bounded sequence in $H_0^m(\Omega)$, then there exists $u \in H_0^m(\Omega)$ such that, upto a subsequence, $u_n \to u$ a.e. in Ω. We show that, for every $q > 0$, $f(\cdot, u_n(\cdot)) \to f(\cdot, u(\cdot))$ in $L^q(\Omega)$. Indeed, since $f(\cdot, u_n(\cdot)) \to f(\cdot, u(\cdot))$ a.e. in Ω, for a fixed $p > 1$ there exists a constant $C_1 > 0$ such that

$$\int_\Omega |f(x, u_n(x))|^{pq} dx \le C_1.$$

Let $\epsilon > 0$ be arbitrary and $\Omega' \subset \Omega$ be a measurable subset. By Hölder's inequality

$$\int_{\Omega'} |f(x,u_n)|^q dx \le |\Omega|^{\frac{1}{p'}} \left(\int_\Omega |f(x,u_n)|^{pq} dx \right)^{\frac{1}{p}} \le C_1^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} < \epsilon$$

provided $|\Omega|$ is small. Here $\frac{1}{p} + \frac{1}{p'} = 1$. By Vitali convergence theorem, $f(\cdot, u_n(\cdot)) \to f(\cdot, u(\cdot))$ in $L^q(\Omega)$.

Now, we show that $H' : H_0^m(\Omega) \to (H_0^m(\Omega))^*$ is continuous and compact. Let $u_n \to u$ in $H_0^m(\Omega)$. Then, $\{u_n\}$ is bounded and $u_n \to u$ a.e. in Ω. For some $v \in H_0^m(\Omega)$ with $\|v\| \le 1$, we have

$$\begin{aligned}
|\langle H'(u_n) - H'(u), v\rangle| &\le \left(\int_\Omega |f(x,u_n) - f(x,u)|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dx \right)^{\frac{1}{2}} \\
&\le C \|v\| \left(\int_\Omega |f(x,u_n) - f(x,u)|^2 dx \right)^{\frac{1}{2}} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus H' is continuous. Similarly, we can show that H' is compact. ∎

Lemma 2 *1. The functional Φ is sequentially weak lower semicontinuous.*

2. Φ belongs to the class $\mathcal{L}_X$.

Proof. (i). Let $\{u_n\}$ be a sequence in $H_0^m(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^m(\Omega)$. Then

$$\int_\Omega |\nabla^m u|^2 dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx. \tag{5}$$

Since the function $t \mapsto \hat{M}(t)$ is continuous and non-decreasing,

$$\begin{aligned}
\liminf_{n\to\infty} \Phi(u_n) &= \frac{1}{2} \liminf_{n\to\infty} \hat{M} \left(\int_\Omega |\nabla^m u_n|^2 dx \right) \\
&= \frac{1}{2} \hat{M} \left(\liminf_{n\to\infty} \int_\Omega |\nabla^m u_n|^2 dx \right) \\
&\ge \frac{1}{2} \hat{M} \left(\int_\Omega |\nabla^m u|^2 dx \right) = \Phi(u).
\end{aligned}$$

Thus Φ is sequentially weak lower semicontinuous.

(ii). It is easy to see that $\gamma(u) \in L_X$. Since $\hat{M}$ is continuous and non-decreasing, we deduce that $\Phi \in \mathcal{L}_X$.
∎

Proof of Theorem 3. By Lemma 1, I is well defined and continuously Gateaux differentiable function with compact derivative $\langle I'(u), v\rangle = \int_\Omega f(x,u) v dx$, $\forall\, u,\, v \in H_0^m(\Omega)$. By Lemma 2, Φ is sequentially weakly lower semicontinuous functional which belongs to the class $\mathcal{L}_X$. Next, we show that Φ is coercive.

$$\Phi(u) = \frac{1}{2} \hat{M} \left(\|u\|^2 \right) \ge \frac{1}{2} \|u\|^{2\alpha}.$$

Thus Φ is coercive. It is easy to see that $u_0 = 0$ is only global minimum of Φ and $\Phi(0) = 0 = I(0)$. Moreover, if $\|u\| \leq r$ then $\Phi(u) \leq \frac{1}{2}\hat{M}(r^n)$ and hence Φ is bounded on each bounded subset of $H_0^m(\Omega)$.

Next, we show that the operator $\Phi' : H_0^m(\Omega) \to (H_0^m(\Omega))^*$ is invertible on $H_0^m(\Omega)$. In view of Minty-Browder theorem [19, Theorem 26 A], it is enough to show that Φ is strictly convex, hemicontinuous and coercive. Let $u, v \in H_0^m(\Omega)$ with $u \neq v$ and $t \in [0,1]$. Since the operator $\gamma' : H_0^m(\Omega) \to (H_0^m(\Omega))^*$ given by

$$\langle \gamma'(u), v\rangle = \int_\Omega \nabla^m u \nabla^m v dx$$

is strictly monotone, γ is strictly convex, see [19, Proposition 25.10]. Furthermore, as M is non-decreasing, the function $\hat{M}$ is convex in $[0, +\infty)$. Thus

$$\hat{M}(\gamma(tu + (1-t)v) < \hat{M}(t\gamma(u) + (1-t)\gamma(v)) \leq t\hat{M}(\gamma(u)) + (1-t)\hat{M}(\gamma(v)).$$

This shows that Φ' is strictly monotone. For any $u \in H_0^m(\Omega)$, by (M1), we see that

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|} = \frac{M(\gamma(u))\,\|u\|^2}{\|u\|} \geq m_0 \|u\|^{2\alpha-1}.$$

Thus Φ' is coercive. By using standard arguments, we can conclude that Φ' is hemicontinuous. By Theorem [19, Theorem 26-A] there exists $\Phi'^{-1} : (H_0^m(\Omega))^* \to H_0^m(\Omega)$ and Φ'^{-1} is bounded. Now, we show that Φ'^{-1} is continuous. Let $\{v_n\} \subseteq (H_0^m(\Omega))^*$ be a sequence converging to $v \in (H_0^m(\Omega))^*$, $u_n = \Phi'^{-1}(v_n)$ and $u = \Phi'^{-1}(v)$. Then $\{u_n\}$ is bounded in $H_0^m(\Omega)$ and upto a subsequence $u_n \rightharpoonup u_0$ weakly in $H_0^m(\Omega)$. Since $v_n \to v$, it is easy to see that

$$\lim_{n\to\infty} \langle \Phi'(u_n), u_n - u_0\rangle = \lim_{n\to\infty} \langle v_n, u_n - u_0\rangle = 0$$

which implies

$$\lim_{n\to\infty} M\left(\int_\Omega |\nabla^m u_n|^2 dx\right) \int_\Omega \nabla^m u_n \nabla^m (u_n - u_0) dx = 0. \tag{6}$$

Since $\{u_n\}$ is bounded in $H_0^m(\Omega)$, we have

$$\int_\Omega |\nabla^m u_n|^2 dx \to b \geq 0 \text{ as } n \to \infty.$$

If $b = 0$, then $\{u_n\}$ converges to $u_0 = 0$ in $H_0^m(\Omega)$ and the proof is complete. If $b > 0$,

$$M\left(\int_\Omega |\nabla^m u_n|^2 dx\right) \to M(b) \text{ as } n \to \infty.$$

By (M1),

$$M\left(\int_\Omega |\nabla^m u_n|^2 dx\right) \geq C > 0. \tag{7}$$

From (6) and (7),

$$\lim_{n\to\infty} \int_\Omega \nabla^m u_n \nabla^m (u_n - u_0) dx = 0. \tag{8}$$

Since $u_n \rightharpoonup u_0$ weakly in $H_0^m(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega \nabla^m u_0 \nabla^m (u_n - u_0) dx = 0. \tag{9}$$

From (8) and (9), $u_n \to u_0$ in $H_0^m(\Omega)$. Since Φ' is continuous and injective, $u_n \to u$ in $H_0^m(\Omega)$ and hence Φ'^{-1} is continuous. In the following, we prove that

$$\limsup_{u\to 0} \frac{I(u)}{\Phi(u)} \leq 0. \tag{10}$$

By the hypothesis (H2), for every $\epsilon > 0$, there exists $\delta_1 > 0$ such that for all $x \in \Omega$ and $|t| \leq \eta_1$,

$$F(x,t) \leq \epsilon |t|^{2\alpha}. \tag{11}$$

Since $f \in \mathcal{F}$, for a fixed $\alpha > 0$ and $q > 2\alpha$ there exists $C > 0$ such that for every $x \in \Omega$ and $|t| \geq \delta_1$,

$$F(x,t) \leq C|t|^q \exp(\alpha t^2). \tag{12}$$

On combining (11) and (12), we obtain

$$F(x,t) \leq \epsilon |t|^{2\alpha} + C|t|^q \exp(\alpha t^2). \tag{13}$$

On using (13), (2) and Hölder's inequality

$$\begin{aligned} I(u) &= \int_\Omega F(x,u)dx \\ &\leq \int_\Omega \left(\epsilon |u|^{2\alpha} + C|u|^q \exp(\alpha |u|^2)\right) dx \\ &\leq \epsilon \int_\Omega |u|^{2\alpha} dx + C\left(\int_\Omega \exp\left(p\alpha \|u\|^2 \left(\frac{|u|^2}{\|u\|^2}\right)\right)\right)^{\frac{1}{p}} \left(\int_\Omega |u|^{p'q}\right)^{\frac{1}{p'}} \\ &\leq \epsilon S_{2\alpha}^{2\alpha} \|u\|^{2\alpha} + C(S_{p'q})^q C_0^{\frac{1}{p}} \|u\|^q \\ &\leq \frac{2\epsilon\alpha}{m_0} S_{2\alpha}^{2\alpha} \Phi(u) + C(S_{p'q})^q C_0^{\frac{1}{p}} \left(\frac{2\alpha}{m_0}\Phi(u)\right)^{\frac{q}{2\alpha}}. \end{aligned}$$

Then

$$\frac{I(u)}{\Phi(u)} \leq \frac{2\epsilon\alpha}{m_0} S_{2\alpha}^{2\alpha} + C(S_{p'q})^q C_0^{\frac{1}{p}} \left(\frac{2\alpha}{m_0}\right)^{\frac{q}{2\alpha}} \Phi(u)^{\frac{q-2\alpha}{2\alpha}},$$

where $C_0 = C(2m, m)$ is defined in (2). Since $q > 2\alpha$ and $\Phi(u) \to 0$ as $u \to 0$, we see that

$$\lim_{\|u\| \to 0} \frac{I(u)}{\Phi(u)} \leq 0.$$

Next, we show that

$$\limsup_{\|u\| \to \infty} \frac{I(u)}{\Phi(u)} \leq 0. \tag{14}$$

By the assumptions (F3), for every $\epsilon > 0$ there exists $\delta_2 > 0$ such that

$$F(x,t) \leq \epsilon |t|^{2\alpha} \text{ for every } x \in \Omega \text{ and } |t| > \delta_2. \tag{15}$$

Since $f \in \mathcal{F}$, there exists $K > 0$ such that for every $x \in \Omega$,

$$\sup_{|t| \leq \delta_2} |f(x,t)| \leq K. \tag{16}$$

On combining (15) and (16), we get

$$F(x,t) \leq K\delta_2 + \epsilon |t|^{2\alpha} \text{ for every } x \in \Omega \text{ and } t \in \mathbb{R}.$$

Thus

$$I(u) \leq K\delta_2 |\Omega| + \epsilon \int_\Omega |u|^{2\alpha} dx.$$

Since $H_0^m(\Omega) \hookrightarrow L^{2\alpha}(\Omega)$,

$$\frac{I(u)}{\Phi(u)} \leq \frac{2\alpha}{m_0 \|u\|^{2\alpha}}(K\delta_2|\Omega| + \epsilon \int_\Omega |u|^{2\alpha} dx) \leq \frac{2\alpha K \delta_2 |\Omega|}{m_0 \|u\|^{2\alpha}} + \frac{2\alpha\epsilon S_{2\alpha}^{2\alpha}}{m_0}.$$

This proves (14). From (10) and (14), we obtain

$$\max\left\{\limsup_{\|u\|\to\infty} \frac{I(u)}{\Phi(u)}, \limsup_{u\to 0} \frac{I(u)}{\Phi(u)}\right\} \leq 0.$$

Thus all the conditions of Theorem 4 are satisfied. Moreover, the functional $\Lambda(u) = \int_\Omega G(x,u)dx$, where $G(x,t) = \int_\Omega g(x,s)dx$, is continuously Gateaux differentiable in $H_0^m(\Omega)$. It is easy to see that Λ has compact derivative given by

$$\langle \Lambda'(u), v\rangle = \int_\Omega g(x,u)v dx.$$

By Theorem 4 there exists $\eta > 0$ such that for every $\lambda \in K$ there exists $\mu^* > 0$ such that for each $\mu \in [0, \mu^*]$, the functional $\Phi - \lambda I - \mu\Lambda$ has at least three critical points whose norm is less than η. Hence, (1) has three weak solutions. This completes the proof. ■

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