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Existence And Stability Results For Integrodifferential Evolution System With Random Impulse*

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Abstract

In this manuscript, the existence results of random impulsive integrodifferential evolution system are studied. To obtain the results, Leray-Schauder alternative fixed point theorem and Banach contraction principle are used. Also the stability results for the same evolution system has been studied by using continuous dependence of solutions on initial condition.

1 Introduction

In the real life scenario, mathematical modelling of systems in the scientific and engineering fields usually results in ordinary or partial differential equations, integral or integrodifferential equations or stochastic equations. In the fields of mechanics, fluid dynamics and chemical kinetics the equations will be of integrodifferential type. For details see [1, 2, 3, 4, 5, 6]. Several evolution processes are characterised by the fact that at a determined time the change of state is experienced which are subject to short term perturbations. Comparatively, the duration of this short term perturbation is negligible to the duration of the process. Naturally, the instantaneous act of perturbations are in the form of impulses. The differential equations involving impulsive effects appears as a natural description of evolution phenomena of real world problems. The impulsive effects are exhibited in the fields of science and technology. Randomness is one among the mathematical formulation of economical and biological phenomena.

The impulses may occur at deterministic or random points. The researchers have investigated the properties of deterministic impulses see [7, 8, 23] and the references therein. On the other hand, if impulses exist at random then the solution would behave as stochastic process. It is different from deterministic impulsive differential equations and stochastic differential equations. The main aspect of mathematical theory of impulsive systems falls under qualitative properties. Iwankievicz et al. [9] studied the dynamic response of nonlinear systems to poisson distributed random impulse. Wu and Meng [10] brought forward random impulse ordinary differential equations and studied solutions to the models using Liapunov's direct method. Wu and Duan [11] investigated the oscillation, stability and boundedness of solutions by comparing with the corresponding non-impulsive differential systems. Wu et.at [12] discussed existence and uniqueness in mean square of solution to certain random impulsive differential systems availing Cauchy-Schwartz inequality, Lipschitz condition and techniques in stochastic Analysis. Also Wu [13] initiated the random impulsive functional differential equation and considered p-moment stability of solutions to the models using Liapunov's function coupled with Razumikhin technique. Anguraj et al. [14] investigated existence and exponential stability of semilinear functional differential equation with random impulses under non-uniqueness. In [15] the authors have investigated the existence, uniqueness and stability results of random impulsive semilinear differential systems. Furthermore, the researchers in [16] treated the existence and uniqueness of neutral functional differential equations with random impulses. Vinodkumar et al. [17] studied the existence and stability results on nonlinear delay integrodifferential equations with random impulse.

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Radhakrishnan and Balachandran [18] studied the impulsive neutral functional evolution integrodifferential systems with infinite delay. Radhakrishnan [19] organised the study of existence, uniqueness and stability results for semilinear integrodifferential non-local evolution equations with random impulse. The researchers in [20] generalized the distribution of random impulse with Erlang distribution. Also using Erlang distribution, the authors have investigated the qualitative behaviour of random impulsive semilinear differential equations and neutral functional differential equations see [21, 22] and the references therein. Thus it is clear that the existence, uniqueness and stability of nonlinear delay integrodifferential equations with random impulse involving evolution operator remains untreated. Thus motivated by the this fact here the existence, uniqueness and stability of nonlinear delay integrodifferential equations with random impulse involving evolution operator is studied.

The manuscript is organised as follows: In section 2, the notations and preliminary definitions used throughout the paper are recalled. In section 3, the existence of solutions of nonlinear delay integrodifferential equations with random impulses using Leray-Schauder alternative fixed point theory which concurrently yield the existence and maximal interval of the existence are investigated. Furthermore the existence and uniqueness of solutions of random impulsive nonlinear delay integrodifferential equations by relaxing the linear growth conditions are studied. Section 4 depicts the stability through continuous dependence on initial conditions of random impulsive nonlinear delay integrodifferential equations. In Section 5 the conclusion is derived.

2 Preliminaries

Let $\mathcal{R}^n$ be the n-dimensional Euclidean space and Ω a nonempty set. Assume that τ_k is a random variable defined from Ω to $\mathcal{D}_k \overset{def}{=} (0, d_k)$ for $k = 1, 2, ..$ where $0 < d_k + \infty$. Furthermore, assume that τ_k follows Erlang distribution where $k = 1, 2, ...$ and let τ_i and τ_j be independent with each other as $i \neq j$ for $i, j = 1, 2, ...$ For simplification, let us denote $\mathcal{R}_T = [\tau, +\infty)$, $\mathcal{R}^+ = [0, +\infty)$.

The considered nonlinear delay integrodifferntial equation with random impulses is of the form

$$\begin{cases} x'(t) = \mathfrak{A}(t)x(t) + \int_0^t \mathcal{G}(t, s, x(\sigma(\mu)))ds, & t \neq \xi_k,\ t \geq \tau, \\ x(\xi_k) = \mathfrak{b}_k(\tau_k)x(\xi_k^-) & k = 1, 2, ..., \\ x_{t_0} = \phi, \end{cases} \tag{1}$$

where $\mathfrak{A}(t)$ is a family of linear operators which generates an evolution operator $\{\mathfrak{u}(t, s), 0 \leq s \leq t \leq T\}$, the functional $\mathcal{G} : \Delta \times \mathcal{C} \to \mathcal{R}^n$, $\mathcal{C} = \mathcal{C}\left([-r, 0], \mathcal{R}^n\right)$ is the set of piecewise continuous functions mapping $[-r, 0]$ into $\mathcal{R}^n$ with some given $r > 0$; $\sigma : \mathcal{R}^+ \to \mathcal{R}^+$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, ...$ Here $t_0 \in R_\tau$ is an arbitrary real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < ... < \lim_{k\to\infty} \xi_k = \infty$; $t_0 = \xi_0 < \xi_1 < \cdots < \lim_{k\to\infty} \xi_k$ $b_k : \mathcal{D}_k \to R^{n\times n}$ is a matrix-valued function for each $k = 1, 2, ...$; $x(\xi_k^-) = \lim_{t\uparrow\xi_k} x(t)$ according to their paths with their norm $\|x\|_t = \sup_{t-r\leq s\leq t} |x(s)|$ for each t satisfying $\tau \leq t \leq T$ $\|\cdot\|$ is any given norm in $\mathbb{X}$, here Δ denotes the set $\{(t, s) : 0 \leq s \leq t \leq \infty\}$.

Denote by $\{B_t, t \geq 0\}$ the simple counting process generated by ξ_n, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote by $\mathfrak{F}_t$ the σ-algebra generated by $\{B, t \geq 0\}$. Then $(\Omega, \mathbb{P}, \{\mathfrak{F}_t\})$ is a probability space. Let $\mathcal{L}_p = \mathcal{L}_p(\Omega, \mathfrak{F}_t, \mathcal{R}^n)$ denote the Banach space of all $\mathfrak{F}_t$-measurable square integrable random variables in $\mathcal{R}^n$.

Assume that $T > t_0$ is any fixed time and $\mathfrak{B}$ denotes the Banach space $\mathfrak{B}\left([t_0 - r, T], \mathcal{L}_2\right)$, the family of all $\mathfrak{F}_t$-measurable, $\mathcal{C}$-valued random variables ϕ with the norm

$$\|\phi\|_{\mathfrak{B}}^2 = \sup_{t_0\leq t\leq T} \mathbb{E}\,\|\phi\|_t^2\,.$$

Let $\mathcal{L}_p^0(\Omega, \mathfrak{B})$ denote the family of all $\mathfrak{F}_0$-measurable, $\mathfrak{B}$-valued random variables φ.

Definition 1 *A map $\mathcal{G}(t, s, x) : \Delta \times \mathcal{C} \to \mathbb{X}$, for every $t \in [\tau, T]$, $\mathcal{G}(t, ., .)$ is said to be $\mathcal{L}^2$-Caratheodary if*

(i) $s \to \mathcal{G}(t, s, x)$ is measurable for each $x \in \mathcal{C}$;

(ii) $x \to \mathcal{G}(t,s,x)$ *is continuous for almost all* $t \in [\tau, T]$;

(iii) *For each positive integer* $m > 0$, *there exists* $\gamma_m \in \mathcal{L}^1([\tau,T],\mathcal{R}^+)$ *such that*

$$\sup_{\mathbb{E}\|x\|^p \le m} \mathbb{E}\|\mathcal{G}(t,s,x)\|^p \le \gamma_m(t) \quad \text{for } t \in [\tau,T], a.e.$$

For the family $\{\mathfrak{A}(t) : 0 \le t \le T\}$ of linear operators, we assume the following hypotheses:

(A1) $\mathfrak{A}(t)$ is a closed linear operator and the domain $D(\mathfrak{A})$ of $\{\mathfrak{A}(t) : 0 \le t \le T\}$ is dense in the Banach space $\mathbb{X}$ and independent of t.

(A2) For each $t \in [0,T]$, the resolvent $R(\lambda, \mathfrak{A}(t)) = (\lambda I - \mathfrak{A}(t))^{-1}$ of $\mathfrak{A}(t)$ exists for all λ with $Re\lambda \le 0$ and $\|R(\lambda,\mathfrak{A}(t))\| \le \mathcal{C}(|\lambda|+1)^{-1}$.

(A3) For any $t, s, \tau \in [0,T]$, there exists a $0 < \delta < 1$ and $\mathcal{L} > 0$ so that

$$\left\|(\mathfrak{A}(t) - \mathfrak{A}(\tau))\mathfrak{A}^{-1}(s)\right\| \le \mathcal{L}|t-\tau|^{\delta}.$$

The hypotheses (A1), (A2) imply that there exists a family of evolution operator $\mathfrak{u}(t,s)$.

The family of two parameter linear evolution system $\mathfrak{u}(t,s) : 0 \le s \le t \le T$ satisfies the following properties:

(i) $\mathfrak{u}(t,s) \in \mathcal{L}(\mathbb{X})$ the space of bounded linear transformation on $\mathbb{X}$, whenever $0 \le s \le t \le T$ and for each $x \in \mathbb{X}$, the mapping $(t,s) \to \mathfrak{u}(t,s)x$ is continuous.

(ii) $\mathfrak{u}(t,s)\mathfrak{u}(s,\tau) = \mathfrak{u}(t,\tau)$ for $0 \le \tau \le s \le t \le T$.

(iii) $\mathfrak{u}(t,t) = I$.

Definition 2 *For a given* $T \in (t_0, +\infty)$, *a stochastic process* $\{ x(t) \in \mathfrak{B}, t_0 - r \le t \le T\}$ *is called a solution of the equation (1) in* $(\Omega, \mathbb{P}, \{\mathfrak{F}_t\})$ *if*

(i) $x(t) \in \mathcal{R}^n$ *is* $\mathfrak{F}_t$ *adapted for* $t \ge t_0$;

(ii) $x(t_0 + s) = \varphi(s) \in \mathcal{L}_2^0(\Omega, \mathfrak{F})$ *when* $s \in [-r, 0]$

$$\begin{aligned} x(t) \quad = \quad & \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\tau_i)\mathfrak{u}(t,t_0)\varphi(0) \\ & + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{u}(t,s)\left[\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds \\ & + \int_{\xi_k}^{t}\mathfrak{u}(t,s)\left[\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in [t_0,T], \end{aligned} \tag{2}$$

where $\prod_{j=m}^{n}(.) = 1$ *as* $m > n$, $\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j) = \mathfrak{b}_k(\tau_k)\mathfrak{b}_{k-1}(\tau_k - 1)...\mathfrak{b}_i(\tau_i)$ *and* $I_{\mathfrak{A}}(.)$ *is the index function, i.e.,*

$$I_{\mathfrak{A}}(t) = \begin{cases} 1 & \text{if } t \in \mathfrak{A}, \\ 0 & \text{if } t \notin \mathfrak{A}. \end{cases}$$

Theorem 1 *Let* $\mathbb{B}$ *be a convex subset of a Banach space* $\mathcal{E}$ *and assume that* $0 \in \mathcal{E}$. *Let* $\mathcal{G} : \mathcal{E} \to \mathcal{E}$ *be a completely continuous operator, and let*

$$\mathcal{U}(\mathcal{G}) = \{x \in \mathcal{E} = \lambda\mathcal{G}x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\mathcal{U}(\mathcal{G})$ *is unbounded or* $\mathcal{G}$ *has a fixed point.*

3 Existence Results

The hypotheses framed below are used for solving the existence theorem.

(H1) The function $\mathcal{G} : [t_0, T] \times [t_0, T] \times \mathcal{C} \to \mathcal{R}^n$ is continuous. $\mathcal{G}(t, s, 0) = 0$ which satisfies the Lipschitz condition with respect to x,

$$\mathbb{E} \left\| \mathcal{G}(t, s, x_1) - \mathcal{G}(t, s, x_2) \right\|^p \leq L\left(t, s, \mathbb{E}\left\| x_1 \right\|^p, \mathbb{E}\left\| x_2 \right\|^p\right) \mathbb{E}\left\| x_1 - x_2 \right\|_s^p \quad \text{for } (t, s) \in \Delta \text{ and } x_1, x_2 \in \mathcal{R}^n,$$

where $\mathcal{L} : [t_0, T] \times [t_0, T] \times \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}^+$ and is monotonically increasing with respect to second and third arguments.

(H2) $\mathfrak{A}(t)$ generates a family of evolution operators $\mathfrak{u}(t, s)$ in $\mathbb{X}$ and there exists $\mathcal{N} > 0$ such that

$$\left\| \mathfrak{u}(t, s) \right\| \leq \mathcal{N} \quad \text{for } 0 \leq s \leq t \leq T.$$

(H3) There exists a continuous function $\mathfrak{r} : [t_0, T] \times [t_0, T] \to (0, \infty)$ such that

$$\mathbb{E} \left\| \mathcal{G}(t, s, x) \right\|^p \leq \mathfrak{r}(t, s) H\left(\mathbb{E}\left\| x \right\|_s^p\right) \quad \text{for } (t, s) \in \Delta \text{ and } x \in \mathcal{R}^n$$

where $H : \mathcal{R}^+ \to (0, \infty)$ is a continuous nondecreasing function.

(H4) $\sigma : [t_0, T] \to [t_0, T]$, is a continuous functions such that $\sigma(t) \leq t$.

(H5) $\mathbb{E}\left\{\max_{i,k} \prod_{j=i}^{k} \left\| \mathfrak{b}(\tau_j) \right\| \right\}$is uniformly bounded that there is $\mathbf{C} > 0$ such that

$$\mathbb{E}\left\{ \max_{i,k} \prod_{j=i}^{k} \left\| \mathfrak{b}(\tau_j) \right\| \right\} \leq \mathbf{C} \text{ for all } \tau_j \ \in \ \mathcal{D}, \ j = 1, 2, ...$$

Theorem 2 *If the hypotheses (H2)–(H5) holds, then the system (1) has a solution $x(t)$, defined on $[t_0, T]$ provided that the following inequality is satisfied*

$$\mathbf{M}_1 \int_{t_0}^{T} p(\mathfrak{r}, \mathfrak{r}) ds < \int_{C_1}^{\infty} \frac{ds}{H(s)}, \tag{3}$$

where $\mathbf{M}_1 = 2^{p-1} \max\left\{1, \mathbf{C}^p\right\} (T - t_0)^2$, $C_1 = 2^{p-1} \mathbf{C}^p \mathbb{E} \left\| \phi \right\|^p$ *and* $\mathbf{C}^p \geq \frac{1}{2^{p-1}}$.

Proof. Let T be an arbitrary number $t_0 < T < +\infty$ satisfying (3). To transform the problem (1) into a fixed point problem, we consider the operator $\phi : \mathfrak{B} \to \mathfrak{B}$ defined by

$$\phi x(t) = \begin{cases} \varphi(t - t_0) & \text{if } t \in [t_0 - r, t_0], \\ \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \mathfrak{b}_i(\tau_i) \mathfrak{u}(t, t_0) \varphi(0) & \\ + \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \mathfrak{u}(t, s) \left[\int_0^s \mathcal{G}(s, \mu, x(\sigma(\mu))) d\mu \right] ds & \text{if } t \in [t_0, T]. \\ + \int_{\xi_k}^{t} \mathfrak{u}(t, s) \left[\int_0^s \mathcal{G}(s, \mu, x(\sigma(\mu))) d\mu \right] ds \Big] I_{[\xi_k, \xi_{k+1})}(t), & \end{cases}$$

The priori estimates for the solution of the Integral equation and $\lambda \in (0, 1)$ are established for the transversality theorem.

$$x(t) = \begin{cases} \lambda \varphi(t - t_0), & \text{if } t \in [t_0 - r, t_0], \\ \lambda \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \mathfrak{b}_i(\tau_i) \mathfrak{u}(t, t_0) \varphi(0) & \\ + \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \mathfrak{u}(t, s) \left[\int_0^s \mathcal{G}(s, \mu, x(\sigma(\mu))) d\mu \right] ds & \text{if } t \in [t_0, T], \\ + \int_{\xi_k}^{t} \mathfrak{u}(t, s) \left[\int_0^s \mathcal{G}(s, \mu, x(\sigma(\mu))) d\mu \right] ds \Big] I_{[\xi_k, \xi_{k+1})}(t), & \end{cases}$$

Therefore (H2)–(H5) gives

$$\begin{aligned}
\|x(t)\|^p &\leq \lambda^p \Big[\sum_{k=0}^{+\infty} \Big[\Big\| \prod_{i=1}^{k} \mathfrak{b}_i(\tau_i) \Big\| \|\mathfrak{u}(t,t_0)\| \|\varphi(0)\| \\
&\quad + \sum_{i=1}^{k} \Big\| \prod_{j=i}^{k} \mathfrak{b}_j(\tau_j) \Big\| \Big\{ \int_{\xi_{i-1}}^{\xi_i} \|\mathfrak{u}(t,s)\| \Big\| \int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \Big\| ds \Big\} \\
&\quad + \int_{\xi_k}^{t} \|\mathfrak{u}(t,s)\| \Big\| \int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \Big\| ds \Big] I_{[\xi_k,\xi_{k+1})}(t) \Big]^p \\
&\leq 2^{p-1} \Big[\sum_{k=0}^{+\infty} \Big[\Big\| \prod_{i=1}^{k} \mathfrak{b}_i(\tau_i) \Big\|^p \|\mathfrak{u}(t,t_0)\|^p \|\varphi(0)\|^p I_{[\xi_k,\xi_{k+1})}(t) \Big] \\
&\quad + \Big[\sum_{i=1}^{k} \Big\| \prod_{j=i}^{k} \mathfrak{b}_j(\tau_j) \Big\| \Big\{ \int_{\xi_{i-1}}^{\xi_i} \|\mathfrak{u}(t,s)\| \Big\| \int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \Big\| ds \Big\} \\
&\quad + \int_{\xi_k}^{t} \|\mathfrak{u}(t,s)\| \Big\| \int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \Big\| ds \Big] I_{[\xi_k,\xi_{k+1})}(t) \Big]^p \Big] \\
&\leq 2^{p-1} \max_k \Big\{ \prod_{i=1}^{k} \|\mathfrak{b}_i(\tau_i)\|^p \Big\} \|\mathfrak{u}(t,t_0)\|^p \|\varphi(0)\|^p \\
&\quad + 2^{p-1} \Big[\max_{i,k} \Big\{ 1, \prod_{j=i}^{k} \|\mathfrak{b}_j(\tau_j)\| \Big\} \Big]^p \|\mathfrak{u}(t,s)\|^p . \Big(\int_{t_0}^{t} \Big\| \int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \Big\| ds \Big)^p .
\end{aligned}$$

From the above inequality, the last term of the right side increases in t and let us consider, $\mathbf{C}^p \geq \frac{1}{2^{p-1}}$, we would obtain

$$\begin{aligned}
\|x\|_t^p &\leq 2^{p-1} \max_k \Big\{ \prod_{i=1}^{k} \|\mathfrak{b}_i(\tau_i)\|^p \Big\} \|\mathfrak{u}(t,t_0)\|^p \|\varphi\|^p \\
&\quad + 2^{p-1} \Big[\max_{i,k} \Big\{ 1, \prod_{j=i}^{k} \|\mathfrak{b}_j(\tau_j)\| \Big\} \Big]^p \|\mathfrak{u}(t,s)\|^p (T-t_0) \int_{t_0}^{t} \Big\| \int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \Big\|^p ds.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}\|x\|_t^p &\leq 2^{p-1}\mathcal{N}^p\mathbf{C}^p\mathbb{E}\left[\|\varphi\|^p\right] \\
&\quad + 2^{p-1}\mathcal{N}^p max\left\{1,\mathbf{C}^p\right\}(T-t_0)\int_{t_0}^{t}\left[\int_0^s \mathbb{E}\|\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\|^p\right]ds \\
&\leq 2^{p-1}\mathbf{C}^p\mathcal{N}^p\mathbb{E}\left[\|\phi\|^p\right] + 2^{p-1}\mathcal{N}^p max\left\{1,\mathbf{C}^p\right\}(T-t_0)\int_{t_0}^{t}\left[\int_0^s \mathfrak{r}(s,\mu)H\left(\mathbb{E}\|x(\sigma(\mu))\|_s^p\right)d\mu\right]ds \\
&\leq 2^{p-1}\mathbf{C}^p\mathcal{N}^p\mathbb{E}\left[\|\phi\|\right]^p + 2^{p-1}\mathcal{N}^p max\left\{1,\mathbf{C}^p\right\}(T-t_0)\int_{t_0}^{t}\left[\int_0^s \mathfrak{r}(s,\mu)H\left(\mathbb{E}\|x(\mu)\|_s^p\right)d\mu\right]ds \\
&\leq 2^{p-1}\mathbf{C}^p\mathcal{N}^p\mathbb{E}\left[\|\phi\|\right]^p + 2^{p-1}\mathcal{N}^p max\left\{1,\mathbf{C}^p\right\}(T-t_0)^2\int_{t_0}^{t}\mathfrak{r}(s,s)H\left(\mathbb{E}\|x\|_s^p\right)ds.
\end{aligned}$$

In the above inequality, the last term of the right side increases in t and thus we would get,

$$\begin{aligned}\sup_{t_0\le\upsilon\le t}\mathbb{E}\,\|x\|_{\upsilon}^{p} &\le 2^{p-1}\mathbf{C}^p\mathcal{N}^p\mathbb{E}\,[\|\phi\|]^p+2^{p-1}\mathcal{N}^p max\,\{1,\mathbf{C}^p\}\,(T-t_0)^2\int_{t_0}^{t}\mathfrak{r}(s,s)H\left(\mathbb{E}\,\|x\|_{s}^{p}\right)ds\\ &\le 2^{p-1}\mathbf{C}^p\mathcal{N}^p\mathbb{E}\,[\|\phi\|]^p+2^{p-1}\mathcal{N}^p max\,\{1,\mathbf{C}^p\}\,(T-t_0)^2\int_{t_0}^{t}\mathfrak{r}(s,s)H\left(\sup_{t_0\le\upsilon\le s}\mathbb{E}\,\|x\|_{\upsilon}^{p}\right)ds.\end{aligned}$$

Let us consider the function $l(t)$ defined by

$$l(t)=\sup_{t_0\le\upsilon\le t}\mathbb{E}\,\|x\|_{\upsilon}^{p},\qquad t\in[t_0,T].$$

Then for any $t\in[t_0,T]$ it follows that

$$l(t)\le 2^{p-1}\mathbf{C}^p\mathcal{N}^p\mathbb{E}\,[\|\phi\|]^p+2^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}\,(T-t_0)^2\int_{t_0}^{t}\mathfrak{r}(s,s)H(l(s))ds. \tag{4}$$

The right side of the inequality (4) is denoted by $u(t)$, given that

$$\begin{cases} l(t)\le\mathfrak{u}(t), & t\in[t_0,T],\\ \mathfrak{u}(t_0)=2^{p-1}\mathbf{C}^p\mathcal{N}^p\mathbb{E}\,[\|\phi\|]^p=\mathcal{C}_1\end{cases}$$

and

$$\begin{aligned}&\mathfrak{u}'^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}\,(T-t_0)^2\mathfrak{r}(t,t)H(l(t))\\ \le\;& 2^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}\,(T-t_0)^2\mathfrak{r}(t,t)H(\mathfrak{u}(t)),\qquad t\in[t_0,T].\end{aligned}$$

Then

$$\frac{\mathfrak{u}'(t)}{H(\mathfrak{u}(t))}\le 2^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}\,(T-t_0)^2\mathfrak{r}(t,t),\qquad t\in[t_0,T]. \tag{5}$$

Integrating the inequality (5) from t_0 to t and also using the change of variable, we would obtain

$$\begin{aligned}\int_{\mathfrak{u}(t_0)}^{\mathfrak{u}(t)}\frac{ds}{H(s)} &\le 2^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}\,(T-t_0)^2\int_{t_0}^{t}\mathfrak{r}(s,s)ds\\ &\le 2^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}\,(T-t_0)^2\int_{t_0}^{T}\mathfrak{r}(s,s)ds\\ &< \int_{\mathfrak{u}(t_0)}^{\infty}\frac{ds}{H(s)},\qquad t\in[t_0,T],\end{aligned} \tag{6}$$

where the last inequality is obtained by (3). From (6) and by mean value theorem, there is a constant β_2 such that $\mathfrak{u}(t)\le\beta_2$ and hence $l(t)\le\beta_2$. Since $\sup_{t_0\le\upsilon\le t}\mathbb{E}\,\|x\|_{\upsilon}^{p}=l(t)$ holds for every $t\in[t_0,T]$. Thus we have $\sup_{t_0\le\upsilon\le t}\mathbb{E}\,\|x\|_{\upsilon}^{p}\le\beta_2$ where β_2 depends only on T, the functions of $\mathfrak{r}$ and H and subsequently,

$$\mathbb{E}\,\|x\|_{\mathfrak{B}}^{p}=\sup_{t_0\le\upsilon\le T}\mathbb{E}\,\|x\|_{\upsilon}^{p}\le\beta_2.$$

Furthermore, we shall prove that ϕ is continuous and completely continuous.

Step 1: To prove ϕ is continuous. Let $\{x_n\}$ be a convergent sequence of elements in $\mathfrak{B}$. then for every $t\in[t_0,T]$, we have

$$\begin{aligned}\phi x_n(t) = \sum_{k=0}^{+\infty}\Bigg[&\prod_{i=1}^{k}\mathfrak{b}_i(\tau_i)\mathfrak{u}(t,t_0)\varphi(0)\\ &+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{u}(t,s)\left[\int_0^s\mathcal{G}(s,\mu,x_n(\sigma(\mu)))d\mu\right]ds\\ &+\int_{\xi_k}^{t}\mathfrak{u}(t,s)\left[\int_0^s\mathcal{G}(s,\mu,x_n(\sigma(\mu)))d\mu\right]ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t).\end{aligned}$$

Thus,

$$\begin{aligned}\phi x_n(t)-\phi x(t) &= \sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{u}(t,s)\left\{\int_0^s\mathcal{G}(s,\mu,x_n(\sigma(\mu)))d\mu-\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right\}ds\\ &\quad+\int_{\xi_k}^{t}u(t,s)\left\{\int_0^s\mathcal{G}(s,\mu,x_n(\sigma(\mu)))d\mu-\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right\}ds\Bigg]I_{[\xi_k,\xi_{k+1})(t)}.\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left\|\phi x_n(t)-\phi x(t)\right\|_t^p &\le \mathcal{N}^p max\{1,\mathbf{C}^p\}(T-t_0)\int_{t_0}^{t}\mathbb{E}\Bigg\|\Bigg[\int_0^s\mathcal{G}(s,\mu,x_n(\sigma(\mu)))d\mu\\ &\quad-\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\Bigg]\Bigg\|^p ds\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Thus ϕ is clearly continuous.

Step 2: To prove ϕ is completely continuous operator. Let us denote

$$B_m=\{x\in\mathfrak{B}\mid\|x\|_{\mathfrak{B}}^p\le m\},\ m\ge 0.$$

Step 2.1: To prove that ϕ maps B_m into an equicontinuous family.
Let $y\in\mathfrak{B}_m$ and $t_1,t_2\in[t_0,T]$. If $t_0<t_1<t_2<T$. Then by hypotheses (H2)–(H5) and condition (3), we have

$$\begin{aligned}\phi x(t_1)-\phi x(t_2) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\tau_i)\mathfrak{u}(t,t_0)\varphi(0)\\ &\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{u}(t,s)\left[\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds\\ &\quad+\int_{\xi_k}^{t_1}\mathfrak{u}(t,s)\left[\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t_1)\\ &\quad-\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\tau_i)\mathfrak{u}(t,t_0)\varphi(0)\\ &\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{u}(t,s)\left[\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds\\ &\quad+\int_{\xi_k}^{t_2}\mathfrak{u}(t,s)\left[\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t_2).\end{aligned}$$

Therefore,

$$\begin{aligned}\phi x(t_1)-\phi x(t_2) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\tau_i)\mathfrak{u}(t,t_0)\varphi(0)\\ &\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{u}(t,s)\left[\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds\\ &\quad+\int_{\xi_k}^{t_1}\mathfrak{u}(t,s)\left[\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds\Bigg]\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right)\\ &\quad+\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{t_1}^{t_2}\mathfrak{u}(t,s)\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t_2).\end{aligned}$$

Thus,

$$\mathbb{E}\left\|\phi x(t_1)-\phi x(t_2)\right\|^p \leq 2^{p-1}\mathbb{E}\left\|I_1\right\|^p+2^{p-1}\mathbb{E}\left\|I_2\right\|^p, \tag{7}$$

where

$$\begin{aligned} I_1 &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\tau_i)\mathfrak{u}(t,t_0)\varphi(0) \\ &\quad +\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{u}(t,s)\left[\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds \\ &\quad +\int_{\xi_k}^{t_1}\mathfrak{u}(t,s)\left[\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right]ds\Bigg]\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right) \end{aligned}$$

and

$$I_2=\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{t_1}^{t_2}\mathfrak{u}(t,s)\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu ds\right]I_{[\xi_k,\xi_{k+1})}(t_2).$$

However,

$$\begin{aligned} \mathbb{E}\left\|I_1\right\|^p &\leq 2^{p-1}\mathcal{N}^p\mathbf{C}^p\mathbb{E}\left\|\varphi(0)\right\|^p\mathbb{E}\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right)+2^{p-1}\mathcal{N}^p max\left\{1,\mathbf{C}^p\right\}(t_1-t_0) \\ &\quad \times\mathbb{E}\int_{t_0}^{t_1}\left\|\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right\|^p ds\mathbb{E}\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right) \\ &\leq 2^{p-1}\mathcal{N}^p\mathbf{C}^p\mathbb{E}\left\|\varphi(0)\right\|^p\mathbb{E}\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right)+2^{p-1}\mathcal{N}^p max\left\{1,\mathbf{C}^p\right\}(t_1-t_0) \\ &\quad \times\int_{t_0}^{t_1}\int_0^s \mathfrak{r}(s,\mu)\mathcal{H}\left(\mathbb{E}\left\|x\right\|_s^p\right)d\mu ds\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right) \\ &\leq 2^{p-1}\mathcal{N}^p\mathbf{C}^p\mathbb{E}\left\|\varphi(0)\right\|^p\mathbb{E}\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right) \\ &\quad +2^{p-1}\mathcal{N}^p max\left\{1,\mathbf{C}^p\right\}(t_1-t_0)^2\int_{t_0}^{t_1}\mathfrak{r}(s,s)H\left(\mathbb{E}\left\|x\right\|_s^p\right)ds\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right) \\ &\leq 2^{p-1}\mathcal{N}^p\mathbf{C}^p\mathbb{E}\left\|\varphi(0)\right\|^p\mathbb{E}\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right) \\ &\quad +2^{p-1}\mathcal{N}^p max\left\{1,\mathbf{C}^p\right\}(t_1-t_0)^2\int_{t_0}^{t_1}\mathcal{N}^*H(\mathbb{E}(m))ds\left(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\right) \\ &\to 0 \quad \text{as } t_2\to t_1, \end{aligned} \tag{8}$$

where $\mathcal{N}^*=\sup\{\mathfrak{r}(t,t):t\in[t_0,T]\}$ and

$$\begin{aligned} \mathbb{E}\left\|I_2\right\|^p &\leq \mathcal{N}^p\mathbf{C}^p(t_2-t_1)\mathbb{E}\int_{t_1}^{t_2}\left\|\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\right\|^p ds \\ &\leq \mathbf{C}^p(t_2-t_1)^2\int_{t_1}^{t_2}\mathcal{N}^*H(m)ds \\ &\to 0 \quad \text{as } t_2\to t_1. \end{aligned} \tag{9}$$

The right side of the equations (8) and (9) is independent of $x\in B_m$. It shows that the right side of (7) tends to zero as $t_2\to t_1$. Thus, ϕ maps B_m into an equicontinuous family of functions.

Step 2.2: To prove $\phi\mathfrak{B}_m$ is uniformly bounded.

From (3), $\|x\|_{\mathfrak{B}}^p \leq m$ and by (H2)–(H5) it yields that

$$\begin{aligned}\|(\phi x)(t)\|^p &\leq 2^{p-1} \max_k \left\{ \prod_{i=1}^{p} \|\mathfrak{b}_i(\tau_i)\|^p \right\} \|\mathfrak{u}(t,t_0)\|^p \|\varphi(0)\|^p \\ &\quad + 2^{p-1} \|\mathfrak{u}(t,s)\|^p \left[\max_{i,k} \left\{ 1, \prod_{i=1}^{k} \|\mathfrak{b}_i(\tau_i)\| \right\} \right]^p \left(\sum_{k=0}^{+\infty} \int_{t_0}^{t} \left\| \int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \right\| ds I_{[\xi_k,\xi_{k+1})}(t) \right)^p .\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\,\|(\phi x)\|_t^p &\leq 2^{p-1}\mathcal{N}^p \mathbf{C}^p \mathbb{E}\,\|\varphi(0)\|^p \\ &\quad + 2^{p-1}\mathcal{N}^p \max\{1, \mathbf{C}^p\}(T-t_0) \int_{t_0}^{t} \mathbb{E} \left\| \int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \right\|^p ds \\ &\leq 2^{p-1}\mathcal{N}^p \mathbf{C}^p \mathbb{E}\,\|\varphi(0)\|^p + 2^{p-1}\mathcal{N}^p \max\{1, \mathbf{C}^p\}(T-t_0)^2 \|\alpha_m\|_{L^1} .\end{aligned}$$

This yields that the set $\{(\phi x)(t), \|x\|_{\mathfrak{B}}^p \leq m\}$ is uniformly bounded, so $\phi \mathfrak{B}_m$ is uniformly bounded. It is so far shown that $\phi \mathfrak{B}_m$ is an equicontinuous collection. Now by Arzela-Ascoli theorem, we may show that ϕ maps $\mathfrak{B}_m$ int a precompact set in $\mathcal{R}^n$.

Step 2.3: To prove $\phi \mathfrak{B}_m$ is compact.
Let $t_0 < t \leq T$ be fixed and ϵ a real number satisfying $\epsilon \in (0, t-t_0)$, for $x \in B_m$. We define

$$\begin{aligned}(\phi_\epsilon)(t) &= \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} \mathfrak{b}_i(\tau_i)\mathfrak{u}(t,t_0)\varphi(0) \\ &\quad + \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \mathfrak{u}(t,s) \left[\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \right] ds \\ &\quad + \int_{\xi_k}^{t-\epsilon} \mathfrak{u}(t,s) \left[\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu \right] ds \Bigg] I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in (t_0, t-\epsilon).\end{aligned}$$

The set

$$\mathcal{H}_\epsilon(t) = \{(\phi_\epsilon x)(t) : x \in \mathfrak{B}_m\}$$

is precompact in $\mathcal{R}^n$ for each $\epsilon \in (0, t-t_0)$. Using (H2)–(H5), (3) and $\mathbb{E}\,\|x\|_{\mathfrak{B}}^p \leq m$ we would obtain

$$\mathbb{E}\,\|(\phi x) - (\phi_\epsilon x)\|_t^p \leq \max\{1, \mathbf{C}^p\}(T-t_0)^2 \int_{t-\epsilon}^{t} \mathcal{N}^* H(m) ds.$$

Thus there are precompact sets arbitrarily close to the set $\{(\phi x)(t) : x \in \mathfrak{B}_m\}$. Hence the set $\{(\phi x)(t) : x \in \mathfrak{B}_m\}$ is precompact in $\mathcal{R}^n$ is precompact in $\mathcal{R}^n$. Therefore, ϕ is a completely continuous operator. Furthermore, the set $\mathbf{U}(\phi) = x \in \mathfrak{B} : x = \lambda \phi x,$ for some $0 < \lambda 1$ is bounded. Subsequently, by Theorem 1 the operator ϕ has a fixed point in $\mathfrak{B}$. Therefore, the system (1) has a solution which completes the proof. ■

Now, we present another existence result for the system (1) by means of the Banach contraction principle.

Theorem 3 *If the hypotheses (H1), (H4) and (H5) holds, then the initial value system (1) has a solution on $[t_0, T]$.*

Proof. Consider the nonlinear operator $\phi : \mathfrak{B} \to \mathfrak{B}$ defined as in Theorem 2

$$
\begin{aligned}
& \mathbb{E}\left\|\phi_x - \phi_y\right\|_t^p \\
\leq\; & 2^{p-1}\mathcal{N}^p \max\{1, \mathbf{C}^p\}(T-t_0)\int_{t_0}^{t}\left\{\int_0^s \mathbb{E}\left\|\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu - \mathcal{G}(s,\mu,y(\sigma(\mu)))d\mu\right\|^p\right\}ds \\
\leq\; & 2^{p-1}\mathcal{N}^p \max\{1, \mathbf{C}^p\}(T-t_0) \\
& \times \int_{t_0}^{t}\left\{\int_0^s \mathcal{L}\left(s,\mu,\mathbb{E}\left\|x(\sigma(\mu))\right\|^p, \mathbb{E}\left\|y(\sigma(\mu))\right\|^p\right)\mathbb{E}\left\|x(\sigma(\mu)) - y(\sigma(\mu))\right\|_s^p d\mu\right\}ds \\
\leq\; & 2^{p-1}\mathcal{N}^p \max\{1, \mathbf{C}^p\}(T-t_0)\int_{t_0}^{t}\left\{\int_0^s \mathcal{L}\left(s,\mu,\mathbb{E}\left\|x\right\|^p, \mathbb{E}\left\|y\right\|^p\right)\mathbb{E}\left\|x(\mu) - y(\mu)\right\|_s^p d\mu\right\}ds \\
\leq\; & 2^{p-1}\mathcal{N}^p \max\{1, \mathbf{C}^p\}(T-t_0)^2\int_{t_0}^{t}\mathcal{L}\left(s,s,\mathbb{E}\left\|x\right\|^p, \mathbb{E}\left\|y\right\|^p\right)\mathbb{E}\left\|x-y\right\|_s^p ds.
\end{aligned}
$$

Taking supremum over t, we get

$$
\left\|\phi(x) - \phi(y)\right\|_{\mathfrak{B}}^p \leq \Lambda(T)\left\|x-y\right\|_{\mathfrak{B}}^p,
$$

with

$$
\Lambda(T) = 2^{p-1}\mathcal{N}^p \max\{1, \mathbf{C}^p\}(T-t_0)^2\int_{t_0}^{t}\mathcal{L}\left(s,s,\mathbb{E}\left\|x\right\|^p, \mathbb{E}\left\|y\right\|^p\right)ds.
$$

Then we can take a suitable $0 \leq T_1 \leq T$ sufficiently small where $\Lambda(T) \leq 1$ and hence ϕ is a contraction on $\mathfrak{B}_{T_1}$ (where T is substituted with T_1). Thus, using Banach fixed point theorem we obtain a unique fixed point $x \in \mathfrak{B}_{T_1}$ for operator ϕ, and hence $\phi x = x$ is a solution of the system (1). The process is repeated to extend the solution to the entire interval $[-r, T]$ in finitely many similar steps, thus completing the proof of existence and uniqueness of solutions on the whole interval $[-r, T]$. ■

4 Stability

Here, we study the stability of the system (1) through the continuous dependence of solutions on initial condition.

Theorem 4 *Let $x(t)$ and $\overline{x}(t)$ be solutions of the system (1) with the initial values $\varphi(0)$ and $\overline{\varphi(0)} \in \mathfrak{B}$ respectively. If the assumptions of Theorem 3 are satisfied, then the solution of the system (1) is stable in the p^{th} mean.*

Proof. By assumptions, $x(t)$ and $\overline{x}(t)$ are the solutions of the system (1) for $t \in [t_0, T]$. Then

$$
\begin{aligned}
x(t) - \overline{x}(t) \;=\; & \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\tau_i)\mathfrak{u}(t,t_0)\left[\varphi(0) - \overline{\varphi}(0)\right] \\
& + \sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\mathfrak{u}(t,s)\left[\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu - \int_0^s \mathcal{G}(s,\mu,\overline{x}(\sigma(\mu)))d\mu\right]ds \\
& + \int_{\xi_k}^{t}\mathfrak{u}(t,s)\left[\int_0^s \mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu - \int_0^s \mathcal{G}(s,\mu,\overline{x}(\sigma(\mu)))d\mu\right]ds\Bigg] I_{[\xi_k,\xi_{k+1})}(t).
\end{aligned}
$$

By using the hypotheses (H1), (H4) and (H5) we get

$$\begin{aligned}
\mathbb{E}\left\|x-\overline{x}\right\|_t^p &\leq 2^{p-1}\mathcal{N}^p\sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k}\|\mathfrak{b}_i(\tau_i)\|^p\,\mathbb{E}\left\|\varphi(0)-\overline{\varphi(0)}\right\|^p I_{[\xi_k,\xi_{k+1})}(t)\right]\\
&+2^{p-1}\mathcal{N}^p\mathbb{E}\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\|\mathfrak{b}_j(\tau_j)\|\\
&\times\int_{\xi_{i-1}}^{\xi_i}\left\|\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu-\int_0^s\mathcal{G}(s,\mu,\overline{x}(\sigma(\mu)))d\mu\right\|ds\\
&+\int_{\xi_k}^{t}\left\|\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu-\int_0^s\mathcal{G}(s,\mu,\overline{x}(\sigma(\mu)))d\mu\right\|ds\Big]I_{[\xi_k,\xi_{k+1})(t)}\Big]^p\\
&\leq 2^{p-1}\mathcal{N}^p\mathbb{E}\left\{\max_k\{\prod_{i=1}^{k}\|\mathfrak{b}_i(\tau_i)\|^p\}\right\}\mathbb{E}\left\|\varphi(0)-\overline{\varphi}(0)\right\|^p\\
&+2^{p-1}\mathcal{N}^p\mathbb{E}\left[\max_{i,k}\{1,\prod_{j=i}^{k}\|\mathfrak{b}_j(\tau_j)\|\}\right]^p\\
&\times\mathbb{E}\left(\int_{t_0}^{t}\left\|\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu-\int_0^s\mathcal{G}(s,\mu,\overline{x}(\sigma(\mu)))d\mu\right\|dsI_{[\xi_k,\xi_{k+1})}\right)^p\\
&\leq 2^{p-1}\mathcal{N}^p\mathbf{C}^p\mathbb{E}\left\|\varphi(0)-\overline{\varphi}(0)\right\|^p\\
&+2^{p-1}\mathcal{N}^p max\{1,\mathbf{C}^p\}(t-t_0)\int_{t_0}^{t}\mathbb{E}\Big\|\int_0^s\mathcal{G}(s,\mu,x(\sigma(\mu)))d\mu\\
&-\int_0^s\mathcal{G}(s,\mu,\overline{x}(\sigma(\mu)))d\mu\Big\|^p ds
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-\overline{x}\right\|_t^p &\leq 2^{p-1}\mathcal{N}^p\mathbb{E}\left\|\varphi(0)-\overline{\varphi}(0)\right\|^p\\
&+2^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}(T-t_0)^2\int_{t_0}^{t}\mathcal{L}\left(s,s,\mathbb{E}\left\|x\right\|^p,\mathbb{E}\left\|y\right\|^p\right)\sup_{s\in[t_0,t]}\mathbb{E}\left\|x-\overline{x}\right\|_s^p ds.
\end{aligned}$$

By applying the Gronwall's inequality, we would obtain

$$\begin{aligned}
\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-\overline{x}\right\|_t^p &\leq 2^{p-1}\mathcal{N}^p\mathbb{E}\left\|\varphi(0)-\overline{\varphi}(0)\right\|^p\\
&\times exp\left(2^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}(T-t_0)^2\int_{t_0}^{t}\mathcal{L}\left(s,s,\mathbb{E}\left\|x\right\|^p,\mathbb{E}\left\|y\right\|^p\right)ds\right)\\
&\leq \varrho\mathbb{E}\left\|\varphi(0)-\overline{\varphi}(0)\right\|^2
\end{aligned}$$

where

$$\varrho=2^{p-1}\mathcal{N}^p\mathbf{C}^p\exp\left(2^{p-1}\mathcal{N}^p\max\{1,\mathbf{C}^p\}(T-t_0)^2\int_{t_0}^{t}\mathcal{L}\left(s,s,\mathbb{E}\left\|x\right\|^p,\mathbb{E}\left\|y\right\|^p\right)ds\right).$$

Now given $\epsilon>0$, choose $\delta=\frac{\epsilon}{\varrho}$ such that $\|\varphi(0)-\overline{\varphi}(0)\|^p<\delta$. Then

$$\sup_{t\in[t_0,T]}\mathbb{E}\left\|x-\overline{x}\right\|_t^p\leq\epsilon.$$

Thus the proof is completed. ■

5 Conclusion

In this manuscript, the existence results of random impulsive integrodifferential evolution system are investigated. To obtain the results, Leray-Schauder alternative fixed point theorem and Banach contraction principle are used. Also, stability results for the considered evolution system has been studied by using continuous dependence of solutions on initial condition. Hence, in the near future, we would like to extend this problem to fractional integrodifferential evolution system with inclusions.

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