Applied Mathematics E-Notes, 21(2021), 533-541 © ISSN 1607-2510
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Non-Existence Of Global Solutions Of Semi-Linear Moore-Gibson-Thompson Equation On The Heisenberg Group*

Ali Hakem[†]

Received 8 July 2020

Abstract

This paper is concerned with the following semi-linear Moore-Gibson-Thompson equation, namely

$$\begin{cases} u_{ttt} + u_{tt} - \Delta_{\mathbb{H}} u - \Delta_{\mathbb{H}} u_t = |u|^p, \quad \eta \in \mathbb{H}, \quad t > 0, \\ u(0,\eta) = u_0(\eta) \quad , \quad u_t(0,\eta) = u_1(\eta), \quad u_{tt}(0,\eta) = u_2(\eta), \quad \eta \in \mathbb{H}, \end{cases}$$

where $p > 1$, and $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}$. We intend to apply the method of test function to establish the non-existence of global weak solutions. Then, this result is extended to the case of 2×2-system of the same type.

1 Introduction

The main goal of this paper is to discuss the non-existence of global weak solutions to the following semi-linear Moore-Gibson-Thompson equation

$$\begin{cases} u_{ttt} + u_{tt} - \Delta_{\mathbb{H}} u - \Delta_{\mathbb{H}} u_t = |u|^p, \quad \eta \in \mathbb{H}, \quad t > 0, \\ u(0,\eta) = u_0(\eta), \quad u_t(0,\eta) = u_1(\eta), \quad u_{tt}(0,\eta) = u_2(\eta), \quad \eta \in \mathbb{H}, \end{cases} \tag{1}$$

where $p > 1$, and $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}$. Then we extend our analysis to the following 2×2-system of the same type,

$$\begin{cases} u_{ttt} + u_{tt} - \Delta_{\mathbb{H}} u - \Delta_{\mathbb{H}} u_t = |v|^q, \quad \eta \in \mathbb{H}, \quad t > 0, \\ v_{ttt} + v_{tt} - \Delta_{\mathbb{H}} v - \Delta_{\mathbb{H}} v_t = |u|^p, \quad \eta \in \mathbb{H}, \quad t > 0, \\ u(0,\eta) = u_0(\eta), \quad u_t(0,\eta)) = u_1(\eta), \quad u_{tt}(0,\eta) = u_2(\eta), \quad \eta \in \mathbb{H}, \\ v(0,\eta) = v_0(\eta), v_t(0,\eta) = v_1(\eta), v_{tt}(0,\eta) = v_2(\eta), \quad \eta \in \mathbb{H}. \end{cases} \tag{2}$$

Our article is motivated by the paper of W. Chen and A. Palmieri [8] which deals with the blow-up of solutions for the following semi-linear Cauchy problem for MGT equation in the conservative case with nonlinearity of derivative type, namely

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = |u_t|^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\ \\ (u, u_t, u_{tt})(0,x) = \varepsilon(u_0, u_1, u_2)(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where $p > 1$ and ε is a positive parameter describing the size of initial data. More precisely, they proved that there exists a positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution u blows up in finite time.

*Mathematics Subject Classifications: 35A01, 35R03.

[†]Laboratory ACEDP, Djillali Liabes university, 22000 Sidi Bel Abbes, Algeria

Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-(\frac{1}{p-1}-\frac{n-1}{2})^{-1}} & \text{if} \quad 1 < p < p_{Gl_a}(n), \\ e^{C\varepsilon^{-(p-1)}}, & \text{if} \quad p = p_{Gl_a}(n), \end{cases}$$

holds, where $C > 0$ is a constant independent of ε and $p_{Gl_a}(n) = \frac{n+1}{n-1}$ is the so called Glassey exponent. The MGT was previously analyzed by several authors from a different point of view. We can mention, among others, the works ([1], [8], [7]) for a variety of problems related to this equations. Recently, the critical exponent to the following structurally damped wave equation with the power nonlinearity $|u_t|^p$:

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{\alpha}{2}} u_t = |u_t|^p, & x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

has been studied by Tuan Anh Dao and Ahmad Z. Fino [3]. It was shown in [3] that if

$$1 < p \leq 1 + \frac{\tilde{\alpha}}{n} \quad \text{where} \quad \tilde{\alpha} = \min\{1, \alpha\},$$

then, there is no global (in time) weak solution to (3). Very recently, Vladimir Georgiev and Alessandro Palmieri [9] investigated the non-existence of global (in time) solutions to the following semi-linear Cauchy problem

$$\begin{cases} u_t - \Delta_{\mathbb{H}} u = |u|^p, & \eta \in \mathbb{H}, \quad t > 0, \\ u(0,\eta) = \varepsilon u_0(\eta), & \eta \in \mathbb{H}, \end{cases} \tag{4}$$

where $p > 1$ and $\varepsilon > 0$ is a parameter describing the smallness of the data. It was shown that if

$$1 < p \leq p_{F_{uj}}(Q) = 1 + \frac{2}{Q}, \quad \text{where} \quad Q = 2n + 2,$$

and

$$u_0 \in L^1(\mathbb{H}) \quad \text{satisfies} \quad \lim\inf_{R\to\infty} \int_{D_R} u_0(\eta) d\eta > 0,$$

where $D_R = B^n(R) \times B^n(R) \times [-R^2, R^2]$, and $B^n(R)$ denotes the ball in $\mathbb{R}^n$ around the origin with radius R, then, there exists no global in time weak solution to (4). Let us underline that, to our knowledge, the MGT equation has not been widely investigated on Heisenberg group. For this reason, motivated by the above contributions, in particular by [9], our goal in this paper is to obtain sufficient conditions for the non-existence of global solutions to problems (1) and (2). For more details on Heisenberg groups and partial differential equations in Heisenberg groups, we refer the reader to ([2], [4], [5], [6]) and the references therein. First, for the sake of the reader, we give some known facts about the Heisenberg group $\mathbb{H}$ and the operator $\Delta_{\mathbb{H}}$.

The Heisenberg group $\mathbb{H}$ whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2n+1}; \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \eta' = (x + x', y + y', \tau + \tau' + 2(x.y' - x'.y)),$$

for all $\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, where . denotes the standard scalar product in $\mathbb{R}^n$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields $X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau}$ and $Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau}$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n} \left(X_i^2 + Y_i^2 \right).$$

Observe that the vector field $T = \frac{\partial}{\partial \tau}$ does not appear in the equality above. This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \quad i, j \in 1, 2, 3, ..., n.$$

The relation above proves that $\mathbb{H}$ is a nilpotent Lie group of order 2. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).$$

A natural group of dilatations on $\mathbb{H}$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \qquad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}$. The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilations δ_λ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta \circ \eta')) = \Delta_{\mathbb{H}} u(\eta \circ \eta'), \quad \Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda, \quad \text{for all} \quad \eta, \eta' \in \mathbb{H}.$$

The natural distance from η to the origin is introduced by Folland and Stein, see [4]

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{n} (x_i^2 + y_i^2) \right)^2 \right)^{\frac{1}{4}}.$$

Before stating our main results, we collect some preliminary knowledge needed in our proof. Let us set $\mathcal{H}_T = (0, T) \times \mathbb{H}$, and $\mathcal{H} = (0, \infty) \times \mathbb{H}$.

Definition 1 (Weak solution for (1)) *Let $T > 0, p > 1$, and $(u_0, u_1, u_2) \in \left(L^1(\mathbb{H})\right)^3$. We say that $u \in \mathbb{L}^p_{loc}(\mathcal{H}_T)$ is a local weak solution to (1) on $\mathcal{H}_T$ if*

$$\begin{aligned} & \int_{\mathcal{H}_T} |u(t,\eta)|^p \varphi(t,\eta) d\eta dt + \int_{\mathbb{H}} \left(u_1(\eta) + u_2(\eta) \right) \varphi(0,\eta) d\eta - \int_{\mathbb{H}} u_0(\eta) \Delta_{\mathbb{H}} \varphi(0,\eta) d\eta \\ = \; & - \int_{\mathcal{H}_T} u(t,\eta) \varphi_{ttt}(t,\eta) d\eta dt + \int_{\mathcal{H}_T} u(t,\eta) \varphi_{tt}(t,\eta) d\eta dt - \int_{\mathcal{H}_T} u(t,\eta) \Delta_{\mathbb{H}} \varphi(t,\eta) d\eta dt \\ & + \int_{\mathcal{H}_T} u(t,\eta) \Delta_{\mathbb{H}} \varphi_t(t,\eta) d\eta dt, \end{aligned} \tag{5}$$

for any test function $\varphi \in \mathcal{C}_0^\infty([0,\infty) \times \mathbb{H})$ such that $\varphi(T,\eta) = \varphi_t(T,\eta) = \varphi_{tt}(T,\eta) = 0$ for all $\eta \in \mathbb{H}$. If $T = \infty$, we say that u is a global weak solution to (1).

Definition 2 (Weak solution for (2)) *Let $p, q > 1$ and $T > 0$. We say that (u, v) is a weak solution to the problem (2) if $(u, v) \in \mathbb{L}^p_{loc}(\mathcal{H}_T) \times \mathbb{L}^q_{loc}(\mathcal{H}_T)$ and satisfies the equations*

$$\begin{aligned} & \int_{\mathcal{H}_T} |v(t,\eta)|^q \varphi(t,\eta) d\eta dt + \int_{\mathbb{H}} \left(u_1(\eta) + u_2(\eta) \right) \varphi(0,\eta) d\eta - \int_{\mathbb{H}} u_0(\eta) \Delta_{\mathbb{H}} \varphi(0,\eta) d\eta \\ = \; & - \int_{\mathcal{H}_T} u(t,\eta) \varphi_{ttt}(t,\eta) d\eta dt + \int_{\mathcal{H}_T} u(t,\eta) \varphi_{tt}(t,\eta) d\eta dt - \int_{\mathcal{H}_T} u(t,\eta) \Delta_{\mathbb{H}} \varphi(t,\eta) d\eta dt \\ & + \int_{\mathcal{H}_T} u(t,\eta) \Delta_{\mathbb{H}} \varphi_t(t,\eta) d\eta dt, \end{aligned} \tag{6}$$

and

$$\begin{aligned}
&\int_{\mathcal{H}_T} |u(t,\eta)|^p \varphi(t,\eta)d\eta dt + \int_{\mathbb{H}} (v_1(\eta)+v_2(\eta))\,\varphi(0,\eta)d\eta - \int_{\mathbb{H}} v_0(\eta)\Delta_{\mathbb{H}}\varphi(0,\eta)d\eta \\
= \;& -\int_{\mathcal{H}_T} v(t,\eta)\varphi_{ttt}(t,\eta)d\eta dt + \int_{\mathcal{H}_T} v(t,\eta)\varphi_{tt}(t,\eta)d\eta dt - \int_{\mathcal{H}_T} v(t,\eta)\Delta_{\mathbb{H}}\varphi(t,\eta)d\eta dt \\
& + \int_{\mathcal{H}_T} v(t,\eta)\Delta_{\mathbb{H}}\varphi_t(t,\eta)d\eta dt, \qquad (7)
\end{aligned}$$

for any test function $\varphi \in \mathcal{C}_0^\infty([0,\infty)\times\mathbb{H})$ *such that* $\varphi(T,\eta) = \varphi_t(T,\eta) = \varphi_{tt}(T,\eta) = 0$ *for all* $\eta \in \mathbb{H}$. *If* $T = \infty$, *we say that* (u, v) *is a global weak solution to (2).*

Now, we are ready to state the main results of this paper.

Theorem 1 *Let* $(u_0, u_1, u_2) \in L^1(\mathbb{H}) \times L^1(\mathbb{H}) \times L^1(\mathbb{H})$ *satisfying the following condition:*

$$\int_{\mathbb{H}} (u_1(\eta) + u_2(\eta))\, d\eta > 0. \qquad (8)$$

If

$$1 < p \le 1 + \frac{2}{Q-1}, \qquad (9)$$

then there is no global (in time) weak solution to problem (1).

Theorem 2 *We assume that* $(u_0, u_1, u_2) \in \left(L^1(\mathbb{H})\right)^3$ *and* $(v_0, v_1, v_2) \in \left(L^1(\mathbb{H})\right)^3$ *satisfying the following conditions:*

$$\int_{\mathbb{H}} (u_1(\eta) + u_2(\eta))\, d\eta > 0 \quad \text{and} \quad \int_{\mathbb{H}} (v_1(\eta) + v_2(\eta))\, d\eta > 0. \qquad (10)$$

If

$$1 < pq \le 1 + \frac{2}{Q-1} \max\{p+1, q+1\}, \qquad (11)$$

then there is no global (in time) weak solution to (2).

The proofs of our main results are given in the next section.

2 Proofs

In this section, we give the proofs of Theorems 1 and 2.

Remark 1 *Throughout, C denotes a positive constant, whose value may change from line to line.*

2.1 Proof of Theorem 1

Proof. Let u be a global weak solution to (1), then for any regular test function φ, one has

$$\begin{aligned}
&\int_{\mathcal{H}} |u(t,\eta)|^p \varphi(t,\eta)d\eta dt + \int_{\mathbb{H}} (u_1(\eta)+u_2(\eta))\,\varphi(0,\eta)d\eta - \int_{\mathbb{H}} u_0(\eta)\Delta_{\mathbb{H}}\varphi(0,\eta)d\eta \\
&\le \int_{\mathcal{H}} |u(t,\eta)||\varphi_{ttt}(t,\eta)|d\eta dt + \int_{\mathcal{H}} |u(t,\eta)||\varphi_{tt}(t,\eta)|d\eta dt + \int_{\mathcal{H}} |u(t,\eta)||\Delta_{\mathbb{H}}\varphi(t,\eta)|d\eta dt \qquad (12) \\
&+ \int_{\mathcal{H}} |u(t,\eta)||\Delta_{\mathbb{H}}\varphi_t(t,\eta)|d\eta dt = J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Applying the following ε-Young inequality

$$ab \leq \varepsilon a^p + C_\varepsilon b^{p'}, \quad a, b, \varepsilon, C_\varepsilon > 0,\ p > 1,\ \frac{1}{p} + \frac{1}{p'} = 1,$$

we get estimate for J_1 as follows:

$$\begin{aligned}|J_1| &\leq \int_{\mathcal{H}} |u(t,\eta)||\varphi_{ttt}(t,\eta)|d\eta dt = \int_{\mathcal{H}} |u(t,\eta)|\varphi^{\frac{1}{p}}(t,\eta)\varphi^{-\frac{1}{p}}(t,\eta)|\varphi_{ttt}(t,\eta)|d\eta dt \\ &\leq \varepsilon \int_{\mathcal{H}} |u(t,\eta)|^p \varphi(t,\eta) d\eta dt + C_\varepsilon \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}}(t,\eta)|\varphi_{ttt}(t,\eta)|^{\frac{p}{p-1}} d\eta dt.\end{aligned} \tag{13}$$

Similarly, we have

$$\begin{aligned}|J_2| &\leq \int_{\mathcal{H}} |u(t,\eta)||\varphi_{tt}(t,\eta)|d\eta dt = \int_{\mathcal{H}} |u(t,\eta)|\varphi^{\frac{1}{p}}(t,\eta)\varphi^{-\frac{1}{p}}(t,\eta)|\varphi_{tt}(t,\eta)|d\eta dt \\ &\leq \varepsilon \int_{\mathcal{H}} |u(t,\eta)|^p \varphi(t,\eta) d\eta dt + C_\varepsilon \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}}(t,\eta)|\varphi_{tt}(t,\eta)|^{\frac{p}{p-1}} d\eta dt,\end{aligned} \tag{14}$$

$$\begin{aligned}|J_3| &\leq \int_{\mathcal{H}} |u(t,\eta)||\Delta_{\mathbb{H}}\varphi(t,\eta)|d\eta dt = \int_{\mathcal{H}} |u(t,\eta)|\varphi^{\frac{1}{p}}(t,\eta)\varphi^{-\frac{1}{p}}(t,\eta)|\Delta_{\mathbb{H}}\varphi(t,\eta)|d\eta dt \\ &\leq \varepsilon \int_{\mathcal{H}} |u(t,\eta)|^p \varphi(t,\eta) d\eta dt + C_\varepsilon \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}}(t,\eta)|\Delta_{\mathbb{H}}\varphi(t,\eta)|^{\frac{p}{p-1}} d\eta dt,\end{aligned} \tag{15}$$

and

$$\begin{aligned}|J_4| &\leq \int_{\mathcal{H}} |u(t,\eta)||\Delta_{\mathbb{H}}\varphi_t(t,\eta)|d\eta dt = \int_{\mathcal{H}} |u(t,\eta)|\varphi^{\frac{1}{p}}(t,\eta)\varphi^{-\frac{1}{p}}(t,\eta)|\Delta_{\mathbb{H}}\varphi_t(t,\eta)|d\eta dt \\ &\leq \varepsilon \int_{\mathcal{H}} |u(t,\eta)|^p \varphi(t,\eta) d\eta dt + C_\varepsilon \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}}(t,\eta)|\Delta_{\mathbb{H}}\varphi_t(t,\eta)|^{\frac{p}{p-1}} d\eta dt.\end{aligned} \tag{16}$$

Combining the estimates from (12) to (16), one has

$$\begin{aligned}&\int_{\mathcal{H}} |u(t,\eta)|^p \varphi(t,\eta) d\eta dt + \int_{\mathbb{H}} (u_1(\eta) + u_2(\eta))\, \varphi(0,\eta) d\eta \\ &\leq \left(\mathcal{A}(p,\varphi) + \mathcal{B}(p,\varphi) + \mathcal{C}(p,\varphi) + \mathcal{D}(p,\varphi) + \int_{\mathbb{H}} |u_0(\eta)||\Delta_{\mathbb{H}}\varphi(0,\eta)|d\eta \right),\end{aligned} \tag{17}$$

where

$$\mathcal{A}(p,\varphi) = \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}}(t,\eta)|\varphi_{ttt}(t,\eta)|^{\frac{p}{p-1}} d\eta dt, \tag{18}$$

$$\mathcal{B}(p,\varphi) = \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}}(t,\eta)|\varphi_{tt}(t,\eta)|^{\frac{p}{p-1}} d\eta dt, \tag{19}$$

$$\mathcal{C}(p,\varphi) = \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}}(t,\eta)|\Delta_{\mathbb{H}}\varphi(t,\eta)|^{\frac{p}{p-1}} d\eta dt, \tag{20}$$

and

$$\mathcal{D}(p,\varphi) = \int_{\mathcal{H}} \varphi^{-\frac{1}{p-1}}(t,\eta)|\Delta_{\mathbb{H}}\varphi_t(t,\eta)|^{\frac{p}{p-1}} d\eta dt. \tag{21}$$

We introduce the following test function as defined in ([9])

$$\varphi_R(t,\eta) = \phi^\sigma\left(\frac{\tau^2 + |x|^4 + |y|^4 + t^4}{R^4}\right), \quad R > 0, \quad \sigma >> 1, \tag{22}$$

where $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ is a decreasing function satisfying

$$\phi(r) = \begin{cases} 1 & \text{if} \quad 0 \leq r \leq 1, \\ 0, & \text{if} \quad r \geq 2. \end{cases}$$

We point out that $supp(\varphi_R)$ is a subset of

$$\mathcal{T}_R = \left\{(t,x,y,\tau) \in \mathcal{H} : 0 \leq \tau^2 + |x|^4 + |y|^4 + t^4 \leq 2R^4\right\},$$

while $supp(\varphi_R)_{ttt})$, $supp(\varphi_R)_{tt})$, $supp(\Delta_{\mathbb{H}}\varphi_R)$ and $supp(\Delta_{\mathbb{H}}(\varphi_R)_t)$ are subsets of

$$\mathfrak{T}_R = \left\{(t,x,y,\tau) \in \mathcal{H} : R^4 \leq \tau^2 + |x|^4 + |y|^4 + t^4 \leq 2R^4\right\}.$$

Moreover, it is clear that

$$\begin{aligned} \Delta_{\mathbb{H}}\varphi_R(t,\eta) \quad = \quad & \frac{4\sigma\,(N+4)}{R^4}\left(|x|^2+|y|^2\right)\phi'^{\sigma-1} \\ & + \frac{16\sigma}{R^8}\left(\left(|x|^6+|y|^6\right) + 2\tau\left(|x|^2-|y|^2\right)x.y + \tau^2\left(|x|^2+|y|^2\right)\right)\phi''^{\sigma-1} \\ & + \frac{16\sigma\,(\sigma-1)}{R^8}\left(\left(|x|^6+|y|^6\right) + 2\tau\left(|x|^2-|y|^2\right)x.y + \tau^2\left(|x|^2+|y|^2\right)\right)\phi'^2\phi^{\sigma-2}. \end{aligned}$$

We can easily check that there is a positive constant $C > 0$, independent of R, such that

$$|\Delta_{\mathbb{H}}\varphi_R(t,\eta)| \leq CR^{-2}\phi^{\sigma-2}\left(|\phi'|\phi + \phi'^2 + |\phi''|\phi\right), \tag{23}$$

and

$$|(\Delta_{\mathbb{H}}\varphi_R(t,\eta))_t| \leq CR^{-3}, \tag{24}$$

$$|(\varphi_R(t,\eta))_t| \leq CR^{-1}, \tag{25}$$

$$|(\varphi_R(t,\eta))_{tt}| \leq CR^{-2}, \tag{26}$$

$$|(\varphi_R(t,\eta))_{ttt}| \leq CR^{-3}. \tag{27}$$

From (23)–(27), one obtains

$$\mathcal{A}(p,\varphi) \leq CR^{Q+1-\frac{3p}{p-1}}, \tag{28}$$

and

$$\mathcal{B}(p,\varphi) \leq CR^{Q+1-\frac{2p}{p-1}}, \tag{29}$$

$$\mathcal{C}(p,\varphi) \leq CR^{Q+1-\frac{2p}{p-1}}, \tag{30}$$

$$\mathcal{D}(p,\varphi) \leq CR^{Q+1-\frac{3p}{p-1}}. \tag{31}$$

At this stage, we pass to the scaled variables

$$(t,\eta) = (t,x,y,\tau) \mapsto \left((\tilde{t},\tilde{\eta}) = (\tilde{t} = R^{-1}t,\ \tilde{x} = R^{-1}x,\ \tilde{y} = R^{-1}y,\ \tilde{\tau} = R^{-2}\tau)\right).$$

Let

$$\mathcal{K} = \left\{(\tilde{t},\ \tilde{x},\ \tilde{y},\ \tilde{\tau}) \in \mathcal{H} \quad \text{such that} \quad 1 \leq |\tilde{t}|^4 + |\tilde{x}|^4 + |\tilde{y}|^4 + |\tilde{\tau}|^2 \leq 2\right\},$$

and

$$\mathcal{K}_R = \left\{(x, y, \tau) \in \mathbb{H} \quad \text{such that} \quad R^4 \leq |x|^4 + |y|^4 + |\tau|^2 \leq 2R^4\right\}.$$

Employing (17), (28) and (29), one has

$$\begin{aligned}\int_{\mathcal{H}} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt &+ \int_{\mathbb{H}} (u_1(\eta) + u_2(\eta))\, \varphi_R(0,\eta) d\eta \\ &\leq C \left(R^{\gamma_1} + R^{\gamma_2} + \int_{\mathcal{K}_R} |u_0(\eta)||\Delta_{\mathbb{H}} \varphi_R(0,\eta)| d\eta \right),\end{aligned} \tag{32}$$

where

$$\gamma_1 = Q + 1 - \frac{2p}{p-1} \quad \text{and} \quad \gamma_2 = Q + 1 - \frac{3p}{p-1},$$

which follows from (32) that

$$\int_{\mathbb{H}} (u_1(\eta) + u_2(\eta))\, \varphi_R(0,\eta) d\eta \leq C \left(R^{\gamma_1} + R^{\gamma_2} + \int_{\mathcal{K}_R} |u_0(\eta)||\Delta_{\mathbb{H}} \varphi_R(0,\eta)| d\eta \right), \tag{33}$$

and

$$\int_{\mathcal{H}} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt \leq C \left(R^{\gamma_1} + R^{\gamma_2} + \int_{\mathcal{K}_R} |u_0(\eta)||\Delta_{\mathbb{H}} \varphi_R(0,\eta)| d\eta \right). \tag{34}$$

It is clear that the assumption (9) is equivalent to $\gamma_1 = \max(\gamma_1, \gamma_2) \leq 0$. For this reason, we will split our consideration into two cases.

Case 1: In the subcritical case $1 < p < 1 + \frac{2}{Q-1}$, letting $R \to \infty$ in (33) we easily deduce

$$\int_{\mathbb{H}} (u_1(\eta) + u_2(\eta))\, d\eta < 0,$$

which contradicts the assumption (8).

Case 2: For the critical case $p = 1 + \frac{2}{Q-1}$, from (34) we can see that

$$\int_{\mathcal{H}} |u(t,\eta)|^p d\eta dt \leq C \quad \text{hence} \quad \lim_{R\to\infty} \int_{\mathfrak{T}_R} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt = 0. \tag{35}$$

From (12), one obtains

$$\int_{\mathcal{H}} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt + \int_{\mathbb{H}} (u_1(\eta) + u_2(\eta))\, \varphi_R(0,\eta) d\eta \leq C \int_{\mathfrak{T}_R} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt.$$

Letting $R \to \infty$ and invoking (35), we get easily

$$\int_{\mathcal{H}} |u(t,\eta)|^p) d\eta dt + \int_{\mathbb{H}} (u_1(\eta) + u_2(\eta))\, d\eta = 0,$$

which contradicts the assumption (8). Summarizing, the proof of the Theorem 1 is completed. ■

2.2 Proof of Theorem 2

Proof. First, we introduce the same test function as in Theorem 1. Let us assume that (u, v) is the global solution to (2). Then for any regular test function φ, we have

$$\begin{aligned}&\int_{\mathcal{H}} |u(t,\eta)|^p \varphi(t,\eta) d\eta dt + \int_{\mathbb{H}} (v_1(\eta) + v_2(\eta))\, \varphi(0,\eta) d\eta - \int_{\mathbb{H}} v_0(\eta) \Delta_{\mathbb{H}} \varphi(0,\eta) d\eta \\ &\leq \int_{\mathcal{H}} |v(t,\eta)||\varphi_{ttt}(t,\eta)| d\eta dt + \int_{\mathcal{H}} |v(t,\eta)||\varphi_{tt}(t,\eta)| d\eta dt + \int_{\mathcal{H}} |v(t,\eta)||\Delta_{\mathbb{H}} \varphi(t,\eta)| d\eta dt \\ &+ \int_{\mathcal{H}} |v(t,\eta)||\Delta_{\mathbb{H}} \varphi_t(t,\eta)| d\eta dt,\end{aligned} \tag{36}$$

and

$$\begin{aligned}
&\int_{\mathcal{H}} |v(t,\eta)|^q \varphi(t,\eta) d\eta dt + \int_{\mathbb{H}} (u_1(\eta)+u_2(\eta))\,\varphi(0,\eta) d\eta - \int_{\mathbb{H}} u_0(\eta)\Delta_{\mathbb{H}}\varphi(0,\eta) d\eta \\
&\le \int_{\mathcal{H}} |u(t,\eta)||\varphi_{ttt}(t,\eta)| d\eta dt + \int_{\mathcal{H}} |u(t,\eta)||\varphi_{tt}(t,\eta)| d\eta dt + \int_{\mathcal{H}} |u(t,\eta)||\Delta_{\mathbb{H}}\varphi(t,\eta)| d\eta dt \\
&+ \int_{\mathcal{H}} |u(t,\eta)||\Delta_{\mathbb{H}}\varphi_t(t,\eta)| d\eta dt.
\end{aligned} \tag{37}$$

Taking $\varphi = \varphi_R$ and using the Holder inequality with parameters p and $\frac{p}{p-1}$, one obtains

$$\begin{aligned}
&\int_{\mathcal{H}} |v(t,\eta)|^q \varphi_R(t,\eta) d\eta dt + \int_{\mathbb{H}} (u_1(\eta)+u_2(\eta))\,\varphi_R(0,\eta) d\eta - \int_{\mathbb{H}} |u_0(\eta)||\Delta_{\mathbb{H}}\varphi_R(0,\eta)| d\eta \\
&\le \left(\mathcal{A}(p,\varphi_R)^{\frac{p-1}{p}} + \mathcal{B}(p,\varphi_R)^{\frac{p-1}{p}} + \mathcal{C}(p,\varphi_R)^{\frac{p-1}{p}} + \mathcal{D}(p,\varphi_R)^{\frac{p-1}{p}}\right)\left(\int_{\mathcal{H}} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt\right)^{\frac{1}{p}}.
\end{aligned} \tag{38}$$

An analogous treatment with parameters q and $\frac{q}{q-1}$, gives

$$\begin{aligned}
&\int_{\mathcal{H}} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt + \int_{\mathbb{H}} (v_1(\eta)+v_2(\eta))\,\varphi_R(0,\eta) d\eta - \int_{\mathbb{H}} |v_0(\eta)||\Delta_{\mathbb{H}}\varphi_R(0,\eta)| d\eta \\
&\le \left(\mathcal{A}(q,\varphi_R)^{\frac{q-1}{q}} + \mathcal{B}(q,\varphi_R)^{\frac{q-1}{q}} + \mathcal{C}(q,\varphi_R)^{\frac{q-1}{q}} + \mathcal{D}(q,\varphi_R)^{\frac{q-1}{q}}\right)\left(\int_{\mathcal{H}} |v(t,\eta)|^q \varphi_R(t,\eta) d\eta dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{39}$$

Using the same change of variables as in Theorem 1, one has

$$\int_{\mathcal{H}} |v(t,\eta)|^q \varphi_R(t,\eta) d\eta dt \le C R^{\frac{(Q+1)(p-1)}{p}-2}\left(\int_{\mathcal{H}} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt\right)^{\frac{1}{p}}, \tag{40}$$

and

$$\int_{\mathcal{H}} |u(t,\eta)|^p \varphi_R(t,\eta) d\eta dt \le C R^{\frac{(Q+1)(q-1)}{q}-2}\left(\int_{\mathcal{H}} |v(t,\eta)|^q \varphi_R(t,\eta) d\eta dt\right)^{\frac{1}{q}}. \tag{41}$$

We easily deduce from (38) and (39) by letting $R \to \infty$ that

$$\int_{\mathbb{H}} (u_1(\eta)+u_2(\eta))\, d\eta < 0 \quad \text{and} \quad \int_{\mathbb{H}} (v_1(\eta)+v_2(\eta))\, d\eta < 0,$$

which contradicts the assumption (10). In the second case $pq = 1 + \frac{2}{Q-1}\max\{p+1, q+1\}$, from (41) there exists a positive constant C such that

$$\int_{\mathcal{H}} |u(t,\eta)|^p d\eta dt \le C \quad \text{hence} \quad \lim_{R\to\infty} \int_{\mathfrak{T}_R} |u(\eta,t)|^p \varphi_R(t,\eta) d\eta dt = 0.$$

As in the case 2 in the proof of Theorem 1, we deduce a contradiction. Summarizing, the proof of the Theorem 2 is completed. ■

Acknowledgment. This project was funded by PRFU at Djillali Liabes University, Sidi Bel Abbes, Algeria, under grant no. (C00L03UN220120200002). This research work is also supported by the General Direction of Scientific Research and Technological Development (DGRSDT), Algeria.

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