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Denumerably Many Positive Solutions For Fractional Differential Equation With Riemann-Stieltjes Integral Boundary Conditions*

Kapula Rajendra Prasad†, Mahammad Khuddush , Pogadadanda Veeraiah

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Abstract

In this paper, we consider singular two point fractional order boundary value problems satisfying Riemann-Stieltjes integral boundary conditions with increasing homeomorphism and positive homomorphism operator(IHPHO). By applying Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of denumerably many positive solutions. Finally, we provide an example to check the validity of our obtained results.

1 Introduction

In the qualitative theory of classical and fractional differential equations, various theorems have been extensively deployed by researchers in establishing the existence and uniqueness of solutions to both the initial and boundary value problems, for more detailes we refer [1, 6, 9, 13, 15, 16] and the references therein. The fractional derivatives are used for a better description of considered material properties, mathematical modelling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations [18]. It should be noted that most papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions. Recently, there have been some papers dealing with the existence, multiplicity and positive solutions of nonlinear initial fractional differential equations by the use of techniques of nonlinear analysis, see [3, 7, 19] and reference therein. It is also noted that the differential equations with boundary conditions have the same requirements, see [10, 2, 11] and references therein.

The turbulent flow through porous media is important for a wide range of scientific and engineering applications such as fluidized bed combustion, compact heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors [4] and drying of different products such as iron ore [14]. In studying such type of problems, Leibenson [12] introduced the p-Laplacian operator into the following equation,

$$\big(\varphi_p(\omega'(t))\big)' = f\big(t, \omega(t), \omega'(t)\big),$$

where $\varphi_p(\omega) = |\omega|^{p-2}\omega,\ p > 1$, is the p-Laplacian operator whose inverse function is denoted by $\varphi_q(\tau)$ with $\varphi_q(\tau) = |\tau|^{q-2}\tau$, and p, q satisfy $1/p + 1/q = 1$. It is a well known fact that the fractional p-Laplacian operator arises in many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science and hence is is worth studying the fractional differential equations with p-Laplacian operator.

In [5], Ege and Topal considered the fractional boundary value problem with IHPHO,

$$^{\mathcal{C}}\mathscr{D}^q\big(\varphi(^{\mathcal{C}}\mathscr{D}^r\omega(t))\big) + f\big(t, \omega(t)\big) = 0,\ 0 < t < 1$$

$$\alpha_1\omega(0) - \beta_1\omega'(0) = -\gamma_1\omega(\xi_1), \alpha_2\omega(1) + \beta_2\omega'(1) = -\gamma_2\omega(\xi_2),\ ^{\mathcal{C}}\mathscr{D}^r\omega(0) = 0,$$

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†§¶Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, 530003, Andhra Pradesh, India

where ${}^{C}\mathscr{D}^{q}$ and ${}^{C}\mathscr{D}^{r}$ are Caputo fractional derivatives of orders $0 < q \le 1 < r \le 2$ and established existence of positive solutions by utilizing Krasnoselskii's and Legget–Williams cone fixed point theorems on a Banach space.

In [17], Song and Cui studied the following fractional differential equation with the Riemann-Stieltjes integral boundary conditions

$$ {}^{C}\mathscr{D}^{\alpha}_{1^-}\mathscr{D}^{\beta}_{0^+}\omega(t) = f(t,\omega(t),\mathscr{D}^{\beta+1}_{0^+}\omega(t),\mathscr{D}^{\beta}_{0^+}\omega(t)), \quad 0<t<1, $$

$$ \omega(0)=\omega'(0)=0, \quad \omega(1)=\int_0^1 \omega(t)dA(t), $$

where ${}^{C}\mathscr{D}^{\alpha}_{1^-}$ is the left Caputo fractional derivative of order $1<\alpha\le 2$ and $\mathscr{D}^{\beta}_{0^+}$ is the right Riemann-Liouville fractional derivative of order $0<\beta\le 1$, $\int_0^1\omega(t)dA(t)$ is the Riemann-Stieltjes integral of ω with respect to A, $A:[0,1]\to\mathbb{R}$ is a function of bounded variation and dA is a signed measure and established existence of solutions by the method of coincidence degree theory.

Inspired by the works mentioned above, we focus in establishing the existence of denumerably many positive solutions for IHPHO Riemann-Liouville fractional order boundary value problem,

$$ \left.\begin{aligned} \varphi\big(\mathscr{D}^{\sigma}_{0^+}\omega(t)\big)+\psi(t)f\big(\omega(t)\big)=0,\ 0<t<1,\\ \omega(0)=\omega'(0)=0,\ \mathscr{D}^{\delta}_{0^+}\omega(1)=\int_0^1\Upsilon(\omega(s))\,d\mathcal{G}(s), \end{aligned}\right\} \tag{1} $$

where $\psi(t)=\prod_{i=1}^{n}\psi_i(t)$, $\mathscr{D}^{\sigma}_{0^+}$, $\mathscr{D}^{\delta}_{0^+}$ denote fractional derivatives of Riemann-Liouville type with $2<\sigma\le 3$, $0<\delta\le\sigma-1$, $\int_0^1\Upsilon(\omega(s))d\mathcal{G}(s)$ denotes Riemann–Stieltjes integral of $\Upsilon(\omega(s))$ with respect to $\mathcal{G}$, $\mathcal{G}:[0,1]\to\mathbb{R}$ is a function of bounded variation and $d\mathcal{G}$ is a signed measure, $\varphi:\mathbb{R}\to\mathbb{R}$ is an IHPHO with $\varphi(0)=0$ and each $\psi_i(t)\in L^{p_i}[0,1](p_i\ge 1)$ has a singularity in the interval $(0,1/2)$.

We assume that the following conditions hold throughout the paper:

(H_1) $f,\Upsilon:[0,+\infty)\to[0,+\infty)$ are continuous,

(H_2) there exists a sequence $\{t_j\}_{j=1}^{\infty}$ such that $0<t_{j+1}<t_j<\frac{1}{2}$,

$$ \lim_{j\to\infty} t_j=\ell<\frac{1}{2},\ \lim_{t\to t_j}\psi_i(t)=+\infty,\ j=1,2,3,\cdots,\ i=1,2,3,\cdots,n $$

and each $\psi_i(t)$ does not vanish identically on any subinterval of $[0,1]$. Moreover, there exists $\eta_i>0$ such that

$$ \eta_i<\Omega_i(t)<\infty \qquad a.e. \quad \text{on} \quad [0,1], $$

where $\Omega_i(t)=\varphi^{-1}\left(\psi_i(t)\right),\ i=1,2,\cdots,n.$

(H_3) $\mathcal{G}$ is a bounded variation and nondecreasing function such that $0<\Delta<\infty$, where

$$ \Delta=\frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_0^1 d\mathcal{G}(s). $$

2 Kernel and Its Bounds

In this section, we construct kernel for the boundary value problem (1) and estimate bounds for it, which are useful for our later discussions.

Lemma 1 *Let $\mathfrak{X}\in C[0,1]$. Then the boundary value problem*

$$ \varphi\big(\mathscr{D}^{\sigma}_{0^+}\omega(t)\big)+\mathfrak{X}(t)=0,\ t\in(0,1), \tag{2} $$

$$\omega(0) = \omega'(0) = 0, \ \mathscr{D}^{\delta}_{0^+}\omega(1) = \int_0^1 A(s)\, d\mathcal{G}(s), \tag{3}$$

has a unique solution

$$\omega(t) = \int_0^1 \chi(t,\tau)\varphi^{-1}\big(\mathfrak{X}(\tau)\big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} t^{\sigma-1} \int_0^1 A(\tau)\, d\mathcal{G}(\tau), \tag{4}$$

where

$$\chi(t,\tau) = \frac{1}{\Gamma(\sigma)} \begin{cases} t^{\sigma-1}(1-\tau)^{\sigma-\delta-1} - (t-\tau)^{\sigma-1}, & \tau \le t, \\ t^{\sigma-1}(1-\tau)^{\sigma-\delta-1}, & t \le \tau. \end{cases} \tag{5}$$

Proof. By simple algebraic calculations it can be proved, so we omit the details here. ■

Lemma 2 *The kernel* $\chi(t,\tau)$ *has the following properties:*

(i) $\chi(t,\tau) \ge 0$ *and continuous on* $[0,1]\times[0,1]$,

(ii) $\chi(t,\tau) \le \chi(1,\tau)$ *for* $t,\tau \in [0,1]$,

(iii) $\max_{\tau\in[0,1]} \chi(1,\tau) = \frac{1}{4}$,

(iv) there exists $\kappa \in (0,\frac{1}{2})$ *such that* $\kappa^{\sigma-1}\chi(1,\tau) \le \chi(t,\tau)$ *for* $t \in [\kappa, 1-\kappa], \tau \in [0,1]$.

Proof. $(i), (ii)$ and (iii) are not difficult to establish so we omit the details here. We prove (iv). For $0 \le \tau \le t \le 1$, $(t-\tau)^{\sigma-1} \le t^{\sigma-1}(1-\tau)^{\sigma-1}$ and for $t \in [\kappa, 1-\kappa]$, we have

$$\chi(t,\tau) \ge t^{\sigma-1}\left[\frac{(1-\tau)^{\sigma-\delta-1}}{\Gamma(\sigma)} - \frac{(1-\tau)^{\sigma-1}}{\Gamma(\sigma)}\right] \ge \kappa^{\sigma-1}\left[\frac{(1-\tau)^{\sigma-\delta-1}}{\Gamma(\sigma)} - \frac{(1-\tau)^{\sigma-1}}{\Gamma(\sigma)}\right] \ge \kappa^{\sigma-1}\chi(1,\tau).$$

Other case is trivial. This completes the proof. ■

We denote the Banach space $C([0,1],\mathbb{R})$ by $\mathcal{B}$ with the norm $\|\omega\| = \max_{t\in[0,1]} |\omega(t)|$. For $\kappa \in (0,1/2)$, the cone $\mathcal{P}_\kappa \subset \mathcal{B}$ is defined by

$$\mathcal{P}_\kappa = \Big\{\omega \in \mathcal{B} : \omega(t) \ge 0, \min_{t\in[\kappa,\, 1-\kappa]} \omega(t) \ge \kappa^{\sigma-1}\|\omega(t)\|\Big\},$$

For any $\omega \in \mathcal{P}_\kappa$, define an operator $\mathcal{L} : \mathcal{P}_\kappa \to \mathcal{B}$ by

$$(\mathcal{L}\omega)(t) = \int_0^1 \chi(t,\tau)\varphi^{-1}\big[\psi(\tau) f\big(\omega(\tau)\big)\big]d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} t^{\sigma-1} \int_0^1 \Upsilon(\omega(\tau))\, d\mathcal{G}(\tau).$$

Lemma 3 *Assume that* (H_1) *holds. Then for each* $\kappa \in (0,1/2)$, $\mathcal{L}(\mathcal{P}_\kappa) \subset \mathcal{P}_\kappa$ *and* $\mathcal{L} : \mathcal{P}_\kappa \to \mathcal{P}_\kappa$ *is completely continuous.*

Proof. Let $\kappa \in (0,1/2)$. Since $f(\omega(\tau))$ is nonnegative for $\tau \in [0,1]$, $\omega \in \mathcal{P}_\kappa$. Since $\chi(t,\tau)$ is nonnegative for all $t,\tau \in [0,1]$, it follows that $\mathcal{L}(\omega(t)) \ge 0$ for all $t \in [0,1]$, $\omega \in \mathcal{P}_\kappa$. Now, by Lemma 2, we have

$$\begin{aligned}
\min_{t\in[\kappa,1-\kappa]} (\mathcal{L}\omega)(t) &= \min_{t\in[\kappa,1-\kappa]} \left\{\int_0^1 \chi(t,\tau)\varphi^{-1}\Big(\psi(\tau) f(\omega(\tau))\Big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} t^{\sigma-1} \int_0^1 \Upsilon(\omega(\tau))\, d\mathcal{G}(\tau)\right\} \\
&\ge \kappa^{\sigma-1} \int_0^1 \chi(1,\tau)\varphi^{-1}\Big(\psi(\tau) f(\omega(\tau))\Big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} \kappa^{\sigma-1} \int_0^1 \Upsilon(\omega(\tau))\, d\mathcal{G}(\tau) \\
&\ge \kappa^{\sigma-1} \max_{t\in[0,1]} \left\{\int_0^1 \chi(t,\tau)\varphi^{-1}\Big(\psi(\tau) f(\omega(\tau))\Big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} \int_0^1 \Upsilon(\omega(\tau))\, d\mathcal{G}(\tau)\right\} \\
&\ge \kappa^{\sigma-1} \max_{t\in[0,1]} |\mathcal{L}\omega(t)|.
\end{aligned}$$

Thus $\mathcal{L}(\mathcal{P}_\kappa) \subset \mathcal{P}_\kappa$. Therefore, the operator $\mathcal{L}$ is completely continuous by standard methods and by the Arzela-Ascoli theorem. ■

3 Denumerably Many Positive Solutions

For the existence of denumerably many positive solutions to the boundary value problem (1), we utilize the following theorems.

Theorem 1 ([8]) *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and Λ_1, Λ_2 are open sets with $0 \in \Lambda_1, \overline{\Lambda}_1 \subset \Lambda_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1) \to \mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{A}z\| \leq \|z\|$, $z \in \mathcal{E} \cap \partial\Lambda_1$, *and* $\|\mathcal{A}z\| \geq \|z\|$, $z \in \mathcal{E} \cap \partial\Lambda_2$, *or*

(b) $\|\mathcal{A}z\| \geq \|z\|$, $z \in \mathcal{E} \cap \partial\Lambda_1$, *and* $\|\mathcal{A}z\| \leq \|z\|$, $z \in \mathcal{E} \cap \partial\Lambda_2$.

Then $\mathcal{A}$ has a fixed point in $\mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1)$.

Theorem 2 (Hölder's Inequality) *Let $f \in L^{p_i}[0,1]$ with $p_i > 1$, for $i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then $\prod_{i=1}^{n} f_i \in L^1[0,1]$ and $\|\prod_{i=1}^{n} f_i\|_1 \leq \prod_{i=1}^{n} \|f_i\|_{p_i}$. Further, if $f \in L^1[0,1]$ and $g \in L^\infty[0,1]$, then $fg \in L^1[0,1]$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.*

Consider three possible cases for $\psi \in L^{p_i}[0,1]$:

$$\sum_{i=1}^{n} \frac{1}{p_i} < 1, \quad \sum_{i=1}^{n} \frac{1}{p_i} = 1, \quad \sum_{i=1}^{n} \frac{1}{p_i} > 1.$$

Firstly, we seek denumerably many positive solutions for the case $\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

Theorem 3 *Suppose $(H_1) - (H_3)$ hold, let $\{\kappa_j\}_{j=1}^{\infty}$ be a sequence with $t_{j+1} < \kappa_j < t_j$. Let $\{E_j\}_{j=1}^{\infty}$ and $\{D_j\}_{j=1}^{\infty}$ be such that*

$$E_{j+1} < \kappa_j^{\sigma-1} D_j < D_j < \beta D_j < E_j, \ j \in \mathbb{N},$$

where

$$\beta = \max\left\{ \left[\prod_{i=1}^{n} \eta_i \int_{\kappa_1}^{1-\kappa_1} \chi(1,\tau) d\tau + \Delta_1 \right]^{-1}, \ 1 \right\}, \ \Delta_1 = \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} \int_{\kappa_1}^{1-\kappa_1} d\mathcal{G}(s).$$

Assume that f satisfies

(A_1) $f(\omega) \leq \varphi(K_1 E_j)$ *and* $\Upsilon(\omega) \leq K_1 E_j \ \forall \ t \in [0,1], 0 \leq \omega \leq E_j, \ j \in \mathbb{N}$, *where*

$$K_1 \leq \left[\|\chi(1,\cdot)\|_q \prod_{i=1}^{n} \|\Omega_i\|_{p_i} + \Delta \right]^{-1},$$

(A_2) $f(\omega) \geq \varphi(\beta D_j)$ *and* $\Upsilon(\omega) \geq \beta D_j \ \forall \ t \in [\kappa_j, 1-\kappa_j], \ \kappa_j^{\sigma-1} D_j \leq \omega \leq D_j$.

Then the BVP (1) has denumerably many positive solutions $\{\omega_j\}_{j=1}^{\infty}$ such that $D_j \leq \|\omega_j\| \leq E_j$ for $j = 1, 2, 3 \cdots$.

Proof. Let $\Lambda_{1,j} = \{\omega \in \mathcal{B} : \|\omega\| < E_j\}$ and $\Lambda_{2,j} = \{\omega \in \mathcal{B} : \|\omega\| < D_j\}$ be open subsets of $\mathcal{B}$. Let $\{\kappa_j\}_{j=1}^{\infty}$ be given in the hypothesis and we note that $\ell < t_{j+1} < \kappa_j < t_j < 1/2$, for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, we define the cone $\mathcal{P}_{\kappa_j}$ by

$$\mathcal{P}_{\kappa_j} = \left\{ \omega \in \mathcal{B} : \omega(t) \geq 0, \min_{t \in [\kappa_j, 1-\kappa_j]} \omega(t) \geq \kappa_j^{\sigma-1} \|\omega(t)\| \right\}.$$

Let $\omega \in \mathcal{P}_{\kappa_j} \cap \partial\Lambda_{1,j}$. Then, $\omega(s) \leq E_j = \|\omega\|$ for all $s \in [0,1]$. By $(A1)$,

$$\begin{aligned}
\|\mathcal{L}\omega\| &= \max_{t\in[0,1]} \left\{ \int_0^1 \chi(t,\tau)\varphi^{-1}\Big(\psi(\tau)f(\omega(\tau))\Big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}t^{\sigma-1}\int_0^1 \Upsilon(\omega(\tau))\,d\mathcal{G}(\tau)\right\}\\
&\leq \max_{t\in[0,1]} \left\{K_1E_j\int_0^1 \chi(t,\tau)\varphi^{-1}\left(\prod_{i=1}^n \psi_i(\tau)\right)d\tau + K_1E_j\frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}t^{\sigma-1}\int_0^1 d\mathcal{G}(\tau)\right\}\\
&\leq K_1E_j\left[\int_0^1 \chi(1,\tau)\varphi^{-1}\left(\prod_{i=1}^n \psi_i(\tau)\right)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_0^1 d\mathcal{G}(\tau)\right]\\
&\leq K_1E_j\left[\int_0^1 \chi(1,\tau)\prod_{i=1}^n \varphi^{-1}\left(\psi_i(\tau)\right)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_0^1 d\mathcal{G}(\tau)\right]\\
&\leq K_1E_j\left[\int_0^1 \chi(1,\tau)\prod_{i=1}^n \Omega_i(\tau)\,d\tau + \Delta\right].
\end{aligned}$$

There exists $q > 1$ such that $\sum_{i=1}^n \frac{1}{p_i} + \frac{1}{q} = 1$. By the first part of Theorem 2, we have

$$\|\mathcal{L}\omega\| \leq K_1E_j\left[\|\chi(1,\cdot)\|_q\prod_{i=1}^n \|\Omega_i\|_{p_i} + \Delta\right] \leq E_j.$$

Since $E_j = \|\omega\|$ for $\omega \in \mathcal{P}_{\kappa_j} \cap \partial\Lambda_{2,j}$, we get

$$\|\mathcal{L}\omega\| \leq \|\omega\|. \tag{6}$$

Let $t \in [\kappa_j, 1-\kappa_j]$. Then

$$D_j = \|\omega\| \geq \omega(t) \geq \min_{t\in[\kappa_j,\,1-\kappa_j]} \omega(t) \geq \kappa_j^{\sigma-1}\|\omega\| \geq \kappa_j^{\sigma-1}D_j.$$

Thus,

$$\begin{aligned}
\|\mathcal{L}\omega\| &= \max_{t\in[0,1]} \left\{ \int_0^1 \chi(t,\tau)\varphi^{-1}\Big(\psi(\tau)f(\omega(\tau))\Big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}t^{\sigma-1}\int_0^1 \Upsilon(\omega(\tau))\,d\mathcal{G}(\tau)\right\}\\
&\geq \max_{t\in[0,1]} \left\{ \int_{\kappa_j}^{1-\kappa_j} \chi(t,\tau)\varphi^{-1}\Big(\psi(\tau)f(\omega(\tau))\Big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}t^{\sigma-1}\int_{\kappa_j}^{1-\kappa_j} \Upsilon(\omega(\tau))\,d\mathcal{G}(\tau)\right\}.
\end{aligned}$$

By (A_2),

$$\begin{aligned}
\|\mathcal{L}\omega\| \geq& \int_{\kappa_j}^{1-\kappa_j} \chi(1,\tau)\varphi^{-1}\Big(\psi(\tau)f(\omega(\tau))\Big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_{\kappa_j}^{1-\kappa_j} \Upsilon(\omega(\tau))\,d\mathcal{G}(\tau)\\
\geq& \beta D_j\left[\int_{\kappa_j}^{1-\kappa_j} \chi(1,\tau)\varphi^{-1}\left(\psi(\tau)\right)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_{\kappa_j}^{1-\kappa_j} d\mathcal{G}(\tau)\right]\\
\geq& \beta D_j\left[\int_{\kappa_1}^{1-\kappa_1} \chi(1,\tau)\varphi^{-1}\left(\prod_{i=1}^n \psi_i(\tau)\right)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_{\kappa_1}^{1-\kappa_1} d\mathcal{G}(\tau)\right]\\
\geq& \beta D_j\left[\int_{\kappa_1}^{1-\kappa_1} \chi(1,\tau)\prod_{i=1}^n \Omega_i(\tau)d\tau + \Delta_1\right] \geq \beta D_j\left[\prod_{i=1}^n \eta_i\int_{\kappa_1}^{1-\kappa_1} \chi(1,\tau)d\tau + \Delta_1\right] \geq D_j = \|\omega\|.
\end{aligned}$$

Thus, if $\omega \in \mathcal{P}_\kappa \cap \partial\Lambda_{2,k}$, then

$$\|\mathcal{L}\omega\| \geq \|\omega\|. \tag{7}$$

It is evident that $0 \in \Lambda_{2,j} \subset \overline{\Lambda}_{2,j} \subset \Lambda_{1,j}$. From (6) and (7), it follows from Theorem 1 that the operator $\mathcal{L}$ has a fixed point $\omega_j \in \mathcal{P}_{\kappa_j} \cap \left(\overline{\Lambda}_{1,j} \backslash \Lambda_{2,j}\right)$ such that $D_j \leq \|\omega_j\| \leq E_j$. This completes the proof of the theorem. ■

For $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have the following theorem.

Theorem 4 *Suppose $(H_1) - (H_3)$ hold, let $\{\kappa_j\}_{j=1}^{\infty}$ be a sequence with $t_{j+1} < \kappa_j < t_j$. Let $\{E_j\}_{j=1}^{\infty}$ and $\{D_j\}_{j=1}^{\infty}$ be such that*

$$E_{j+1} < \kappa_j^{\sigma-1} D_j < D_j < \beta D_j < E_j,\ j \in \mathbb{N},$$

Assume that f satisfies (A_2) and

(B_1) *$f(\omega(t)) \leq \varphi(K_2 E_j)$ and $\Upsilon(\omega) \leq K_2 E_j$ for all $t \in [0,1]$, $0 \leq \omega \leq E_j$, $j \in \mathbb{N}$, where*

$$K_2 \leq \min\left\{\left[\|\chi(1,\cdot)\|_\infty \prod_{i=1}^{n} \|\Omega_i\|_{p_i} + \Delta\right]^{-1},\ \beta\right\}.$$

Then the BVP (1) has denumerably many positive solutions $\{\omega_j\}_{j=1}^{\infty}$. Furthermore, $D_j \leq \|\omega_j\| \leq E_j$ for each $j \in \mathbb{N}$.

Proof. For a fixed k, let $\Lambda_{1,k}$ be as in the proof of Theorem 3 and let $\omega \in P_{\kappa_j} \cap \partial\Lambda_{2,k}$. Then $\omega(s) \leq E_j = \|\omega\|$, for all $s \in [0,1]$. By $(B1)$ and Theorem 3,

$$\begin{aligned}
\|\mathcal{L}\omega\| &= \max_{t\in[0,1]} \left\{ \int_0^1 \chi(t,\tau)\varphi^{-1}\Big(\psi(\tau) f(\omega(\tau))\Big) d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} t^{\sigma-1} \int_0^1 \Upsilon(\omega(\tau))\, d\mathcal{G}(\tau) \right\} \\
&\leq \max_{t\in[0,1]} \left\{ K_1 E_j \int_0^1 \chi(t,\tau)\varphi^{-1} \left(\prod_{i=1}^{n} \psi_i(\tau) \right) d\tau + K_1 E_j \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} t^{\sigma-1} \int_0^1 d\mathcal{G}(\tau) \right\} \\
&\leq K_1 E_j \left[\int_0^1 \chi(1,\tau)\varphi^{-1} \left(\prod_{i=1}^{n} \psi_i(\tau) \right) d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)} \int_0^1 d\mathcal{G}(\tau) \right] \\
&\leq K_1 E_j \left[\int_0^1 \chi(1,\tau) \prod_{i=1}^{n} \Omega_i(\tau)\, d\tau + \Delta \right] \leq K_1 E_j \left[\|\chi(1,\cdot)\|_\infty \prod_{i=1}^{n} \|\Omega_i\|_{p_i} + \Delta \right] \leq E_j.
\end{aligned}$$

Thus, $\|\mathcal{L}\omega\| \leq \|\omega\|$, for $\omega \in \mathcal{P}_{\kappa_j} \cap \partial\Lambda_{1,j}$. Now define $\Lambda_{2,j} = \{\omega \in \mathcal{B} : \|\omega\| < D_j\}$. Let $\omega \in \mathcal{P}_{\kappa_j} \cap \partial\Lambda_{2,j}$ and let $s \in [\kappa_j,\ 1-\kappa_j]$. Then, the argument leading to (7) can be done to the present case. This completes the proof of the theorem. ■

Lastly, the case $\sum_{i=1}^{n} \frac{1}{p_i} > 1$.

Theorem 5 *Assume that $(H_1) - (H_3)$ hold. Let $\{E_j\}_{j=1}^{\infty}$ and $\{D_j\}_{j=1}^{\infty}$ be such that*

$$E_{j+1} < \kappa_j^{\sigma-1} D_j < \beta D_j < E_j,\ \ j \in \mathbb{N},$$

Assume that f satisfies $(A2)$ and

$(E1)$ *$f(\omega(t)) \leq \varphi(K_3 E_j)$ for all $t \in [0,1]$, $0 \leq \omega \leq E_j$, $j \in \mathbb{N}$, where*

$$K_3 \leq \min\left\{\left[\|\chi(1,\cdot)\|_\infty \prod_{i=1}^{n} \|\Omega_i\|_1 + \Delta\right]^{-1},\ \beta\right\}.$$

Then the BVP (1) has denumerably many positive solutions $\{\omega_j\}_{k=1}^{\infty}$ such that $D_j \leq \|\omega_j\| \leq E_j$ for $k = 1, 2, 3, \cdots$.

Proof. By $(E1)$,

$$\begin{aligned}\|\mathcal{L}\omega\| &= \max_{t\in[0,1]}\left\{\int_0^1 \chi(t,\tau)\varphi^{-1}\Big(\psi(\tau)f(\omega(\tau))\Big)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}t^{\sigma-1}\int_0^1 \Upsilon(\omega(\tau))\,d\mathcal{G}(\tau)\right\}\\ &\le \max_{t\in[0,1]}\left\{K_1E_j\int_0^1 \chi(t,\tau)\varphi^{-1}\left(\prod_{i=1}^n \psi_i(\tau)\right)d\tau + K_1E_j\frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}t^{\sigma-1}\int_0^1 d\mathcal{G}(\tau)\right\}\\ &\le K_1E_j\left[\int_0^1 \chi(1,\tau)\varphi^{-1}\left(\prod_{i=1}^n \psi_i(\tau)\right)d\tau + \frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_0^1 d\mathcal{G}(\tau)\right]\\ &\le K_1E_j\left[\int_0^1 \chi(1,\tau)\prod_{i=1}^n \Omega_i(\tau)\,d\tau + \Delta\right] \le K_1E_j\left[\|\chi(1,\cdot)\|_\infty\prod_{i=1}^n\|\Omega_i\|_1 + \Delta\right] \le E_j.\end{aligned}$$

This shows that if $\omega\in\mathcal{P}_{\kappa_j}\cap\partial\Lambda_{1,k}$, where $\Lambda_{1,k}=\{\omega\in\mathcal{B}:\|\omega\|<E_j\}$, Then, $\|\mathcal{L}\omega\|\le\|\omega\|$. Define $\Lambda_{2,k}=\{\omega\in\mathcal{B}:\|\omega\|<D_j\}$ and let $z\in\mathcal{P}_{\kappa_j}\cap\partial\Lambda_{2,k}$. Then, the argument worked in the proof of Theorem 3 can be applied directly to get $\|\mathcal{L}\omega\|\ge\|\omega\|$. This completes the proof of the theorem. ■

4 Example

In this section, we present an example to check validity of our main results.

Example 1 *Consider the following fractional order boundary value problem*

$$\left.\begin{aligned}\varphi(\mathscr{D}_{0^+}^{5/2}\omega(t))+\psi(t)f(\omega(t))=0,\ t\in(0,1),\\ \omega(0)=\omega'(0)=0,\ \mathscr{D}_{0^+}^{3/4}\omega(1)=\int_0^1\Upsilon(\omega(\tau))\,d\mathcal{G}(\tau),\end{aligned}\right\}\tag{8}$$

where

$$\varphi(\omega)=\begin{cases}\frac{\omega^3}{1+\omega^2}, & \omega\le 0,\\ \omega^2, & \omega>0,\end{cases}$$

$\psi(t)=\psi_1(t)\psi_2(t)$ *in which* $\psi_1(t)=\frac{1}{|t-\frac{1}{4}|^{\frac{1}{2}}}$ *and* $\psi_2(t)=\frac{1}{|t-\frac{1}{3}|^{\frac{1}{2}}}$,

$$f(\omega)=\begin{cases}\frac{1}{2}\times10^{-8}, & \omega\in(10^{-4},+\infty),\\ \frac{1260\times10^{-(8k+4)}-\frac{1}{2}\times10^{-8k}}{10^{-(4k+2)}-10^{-4k}}(\omega-10^{-4k}) +\frac{1}{2}\times10^{-8k}, & \omega\in\left[10^{-(4k+2)},10^{-4k}\right],\\ 1260\times10^{-(8k+4)}, & \omega\in\left(\frac{1}{5^{3/2}}\times10^{-(4k+2)},10^{-(4k+2)}\right),\\ \frac{1260\times10^{-(8k+4)}-\frac{1}{2}\times10^{-(8k+8)}}{\frac{1}{5^{3/2}}\times10^{-(4k+2)}-10^{-(4k+4)}}(\omega-10^{-(4k+4)}) +\frac{1}{2}\times10^{-(8k+8)}, & \omega\in\left(10^{-(4k+4)},\frac{1}{5^{3/2}}\times10^{-(4k+2)}\right],\end{cases}$$

$$\Upsilon(\omega)=\begin{cases}\frac{3}{4}\times 10^{-4}, & \omega\in(10^{-4},+\infty),\\ \frac{36\times10^{-(4k+2)}-\frac{3}{4}\times10^{-4k}}{10^{-(4k+2)}-10^{-4k}}(\omega-10^{-4k}) \\ +\frac{3}{4}\times 10^{-4k}, & \omega\in\left[10^{-(4k+2)},10^{-4k}\right],\\ 36\times10^{-(4k+2)}, & \omega\in\left(\frac{1}{5^{3/2}}\times10^{-(4k+2)},10^{-(4k+2)}\right),\\ \frac{36\times10^{-(4k+2)}-\frac{3}{4}\times10^{-(4k+4)}}{\frac{1}{5^{3/2}}\times10^{-(4k+2)}-10^{-(4k+4)}}(\omega-10^{-(4k+4)}) \\ +\frac{3}{4}\times10^{-(4k+4)}, & \omega\in\left(10^{-(4k+4)},\frac{1}{5^{3/2}}\times10^{-(4k+2)}\right],\end{cases}$$

$$\mathcal{G}(t)=\begin{cases}t, & t\in[0,1/2)\cup[2/3,5/6),\\ \frac{1}{2}, & t\in[1/2,2/3),\\ \frac{5}{6}, & t\in[5/6,1],\end{cases}$$

Let

$$t_j=\frac{31}{64}-\sum_{r=1}^{j}\frac{1}{4(r+1)^4},\ \kappa_j=\frac{1}{2}(t_j+t_{j+1}),\ j=1,2,3,\cdots.$$

Then $\kappa_1=\frac{15}{32}-\frac{1}{648}<\frac{15}{32}$ *and* $t_{j+1}<\kappa_j<t_j$, $\kappa_j>\frac{1}{5}$. *Therefore,* $\kappa_j^{\sigma-1}>\frac{1}{5^{3/2}}$, $j=1,2,3,\cdots$. *It is easy to see*

$$t_1=\frac{15}{32}<\frac{1}{2},\ t_j-t_{j+1}=\frac{1}{4(j+2)^4},\ j=1,2,3,\cdots.$$

Since $\sum_{j=1}^{\infty}\frac{1}{j^4}=\frac{\pi^4}{90}$ *and* $\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{\pi^2}{6}$, *we see that*

$$\ell=\lim_{j\to\infty}t_j=\frac{31}{64}-\sum_{i=1}^{\infty}\frac{1}{4(i+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}>\frac{1}{5},$$

$$\psi_1,\psi_2\in L^p[0,1]\quad \textit{for all}\quad 0<p<2,\quad \textit{so}\quad \eta_1=\eta_2=\frac{1}{\sqrt{3}},$$

$$\Delta=\frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_0^1 d\mathcal{G}(s)=\frac{\Gamma(5/2-3/4)}{\Gamma(5/2)}\times\frac{5}{6}\approx 0.5761394489,$$

$$\Delta_1=\frac{\Gamma(\sigma-\delta)}{\Gamma(\sigma)}\int_{\kappa_1}^{1-\kappa_1} d\mathcal{G}(s)=\frac{\Gamma(5/2-3/4)}{\Gamma(5/2)}[\mathcal{G}(1-\kappa_1)-\mathcal{G}(\kappa_1)]\approx 0.02053830443,$$

$$\int_{\kappa_1}^{1-\kappa_1}\chi(1,\tau)d\tau=\int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}-\frac{1}{648}}\left[(1-\tau)^{\frac{5}{2}-\frac{3}{4}-1}-(1-\tau)^{\frac{5}{2}-1}\right]d\tau\approx 0.02301858242,$$

So, we get

$$\beta=\max\left\{\left[\prod_{i=1}^{n}\eta_i\int_{\kappa_1}^{1-\kappa_1}\chi(1,\tau)d\tau+\Delta_1\right]^{-1},\ 1\right\}=\max\left\{35.44695838,\ 1\right\}=35.44695838,$$

$$\|\chi(1,\cdot)\|_q=\left[\int_0^1|\chi(1,\tau)|^q d\tau\right]^{\frac{1}{q}}<\frac{1}{4}\quad \textit{for any } q>0.$$

Next, let $0<\varepsilon<1$ *be fixed. Then* $\psi_1,\psi_2\in L^{1+\varepsilon}[0,1]$. *It follows that*

$$\|\Omega_1\|_{1+\varepsilon}=\|\varphi^{-1}(\psi_1)\|_{1+\varepsilon}=\left[\frac{1}{3-\varepsilon}\left(3^{\frac{3-\varepsilon}{4}}+1\right)2^{\frac{1+\varepsilon}{2}}\right]^{\frac{1}{1+\varepsilon}},$$

$$\|\Omega_2\|_{1+\varepsilon}=\|\varphi^{-1}(\psi_2)\|_{1+\varepsilon}=\left[\frac{4}{3-\varepsilon}\left(2^{\frac{3-\varepsilon}{4}}+1\right)(1/3)^{\frac{3-\varepsilon}{4}}\right]^{\frac{1}{1+\varepsilon}}.$$

So, for $0<\varepsilon<1$, *we have*

$$0.7898442122\le\left[\|\chi(1,\cdot)\|_q\prod_{i=1}^{n}\|\Omega_i\|_{p_i}+\Delta\right]^{-1}\le 0.8457314598.$$

Taking $K_1=0.78$. *In addition if we take* $E_j=10^{-4j}$, $D_j=10^{-(4j+2)}$, *then*

$$E_{j+1}=10^{-(4j+4)}<\frac{1}{5^{3/2}}\times10^{-(4j+2)}<\kappa_j^{\sigma-1}D_j<D_j=10^{-(4j+2)}<E_j=10^{-4j},$$

$\beta D_j=35.44695838\times10^{-(4j+2)}<0.78\times10^{-4j}=K_1E_j$, $j=1,2,3,\cdots$, *and* f *and* χ *satisfies the following growth conditions:*

$$f(\omega)\le\ \varphi(K_1E_j)=K_1^2E_j^2=0.6084\times10^{-8j},\quad \omega\in\left[0,10^{-4j}\right],$$

$$f(\omega)\ge\ \varphi(\beta D_j)=\beta^2D_j^2=1256.486858\times10^{-(8j+4)},\quad \omega\in\left[\frac{1}{5^{3/2}}\times10^{-(4j+2)},10^{-(4j+2)}\right],$$

$$\chi(t,\omega)\le\ M_1E_j=0.78\times10^{-4j},\quad \omega\in\left[0,10^{-4j}\right],$$

$$\chi(t,\omega)\ge\ \beta D_j=35.44695838\times10^{-(4j+2)},\quad \omega\in\left[\frac{1}{5^{3/2}}\times10^{-(4j+2)},10^{-(4j+2)}\right].$$

Then all the conditions of Theorem 3 are satisfied. Therefore, by Theorem 3, the boundary value problem (8) *has countably many positive solutions* $\{\omega^{[j]}\}_{j=1}^{\infty}$ *such that* $10^{-(4j+2)}\le\|\omega^{[j]}\|\le10^{-4j}$ *for each* $j=1,2,3,\cdots$.

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