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Existence, Uniqueness And Stability Results Of Impulsive Stochastic Semilinear Neutral Functional Partial Integrodifferential Equations With Infinite Delay And Poisson Jumps *

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Abstract

In this manuscript, we present results on existence, uniqueness and stability of mild solution of impulsive stochastic semilinear neutral functional partial integrodifferential equations with Poisson jumps under non-Lipschitz condition and Lipschitz condition. The theory of resolvent operator is utilized to exhibit the existence of these mild solutions. The results are obtained by using the method of successive approximation and Bihari's inequality.

1 Introduction

Stochastic differential equations have been investigated as mathematical models to describe the dynamical behavior of real life phenomena. It is essential to take into account the environmental disturbances as well as the time delay while constructing realistic models in the field of mechanical, electrical, engineering, biology, etc. Until now, various studies have been carried out on stochastic functional differential equations involving existence and stability results see references therein [1, 2, 3, 4, 5, 7, 9, 19]. Impulsive differential equations thrive to be a promising area and have gained much attention among the researchers due to their potential application in various fields such as orbital transfer of satellite, dosage supply in pharmacokinetics, etc. It is worth mentioning that many real world systems are subjected to stochastic abrupt changes, and therefore it is necessary to investigate them using impulsive stochastic functional differential equations. Recently, few works have been reported in the study of stochastic functional differential equations with impulsive effects, [10, 11, 12].

Moreover, many practical systems (such as sudden price variations [jumps] due to market crashes, earthquakes, hurricanes, epidemics, and so on) may undergo some jump type stochastic perturbations. The sample paths of such systems are not continuous. Therefore, it is more appropriate to consider stochastic processes with jumps to describe such models. In general, these jump models are derived from Poisson random measure. The sample paths of such systems are right continuous and possess left limits. Recently, many researchers are focusing the theory and applications of impulsive stochastic functional differential equations with Poisson jumps. To be more precise, existence and stability results on impulsive stochastic functional differential equations with Poisson jumps can be found in [12, 13, 14] and the references therein. Subsequently, few works have been reported in the study of stochastic differential equations with Poisson jumps,

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refer to [6, 8, 14, 15]. Moreover, we study the stability through the continuous dependence on the initial values by means of a corollary of Bihari's inequality. Further, we refer [16, 17].

To the best of our knowledge, there are only very few articles in the existing literature that report the study of impulsive stochastic functional differential equation with jumps. The aim of this paper is to close this gap and investigate the existence, uniqueness and stability of impulsive stochastic semilinear neutral functional integrodifferential equations and Poisson jumps under non-Lipschitz condition with Lipschitz condition being considered as a special case by means of the successive approximation. Furthermore, we give the continuous dependence of solutions on the initial data by means of a corollary of the Bihari's inequality.

In this article, we will examine impulsive stochastic semilinear neutral functional integrodifferential equations and Poisson jumps form:

$$\begin{aligned} d\big[x(t)+g(t,x_t)\big] &= A\big[x(t)+g(t,x_t)\big]dt + \left[\int_0^t B(t-s)\,[x(s)+g(s,x_s)]\,ds + f(t,x_t)\right] \\ &\quad + \sigma(t,x_t)dw(t) + \int_{\mathcal{U}} h(t,x_t,u)\tilde{N}(dt,du), \quad t\in J=[0,T], \qquad (1) \\ \Delta x(t_k) &= x(t_k^+)-x(t_k^-) = I_k(x(t_k)), \quad t=t_k, \quad k=1,2,...m, \qquad (2) \\ x_0 &= \varphi\in\mathcal{D}^b_{\mathcal{B}_0}((-\infty,0],\mathbb{X}). \qquad (3) \end{aligned}$$

Here, A is the infinitesimal generator of strongly continuous semigroup of bounded linear operator $\{R(t), t\geq 0\}$ on $\mathbb{X}$, with $\mathcal{D}(A)\subset\mathbb{X}$ and $B(t)$, $t\in J$ is a closed linear operator on $\mathbb{X}$. Let $\mathbb{R}^+=[0,\infty)$ and let the functions $f:\mathbb{R}^+\times\hat{\mathcal{D}}\to\mathbb{X}$, $\sigma:\mathbb{R}^+\times\hat{\mathcal{D}}\to\mathcal{L}_2^0(\mathbb{Y},\mathbb{X})$, $h:\mathbb{R}^+\times\hat{\mathcal{D}}\times\mathcal{U}\to\mathbb{X}$ and $I_k:\hat{\mathcal{D}}\to\mathbb{X}$ be Borel measurable and $g:\mathbb{R}^+\times\hat{\mathcal{D}}\to\mathbb{X}$ be continuous. Here $\hat{\mathcal{D}}=\mathcal{D}((-\infty,0],\mathbb{X})$ denotes the family of all right piecewise continuous functions with left-hand limit φ from $(-\infty,0]$ to $\mathbb{X}$. In $\tilde{N}(dt,du)=N(dt,du)-dt(vdu)$ the Poisson measure $\tilde{N}(dt,du)$ denotes the Poisson counting measure. The phase space $\mathcal{D}((-\infty,0],\mathbb{X})$ is assumed to be equipped with the norm $\|\varphi\|_t=\sup_{-\infty<\theta\leq 0}\|\varphi(\theta)\|$. We also assume $\mathcal{D}^b_{\mathcal{B}_0}((-\infty,0],\mathbb{X})$ to denote the family of all almost surely bounded, $\mathcal{B}_0$-measurable, $\hat{\mathcal{D}}$-valued random variables. Further, let $\mathcal{B}_T$ be a Banach space $\mathcal{B}_T((-\infty,T],\mathcal{L}_2)$, the family of all $\mathcal{B}_t$-adapted processes $\varphi(t,w)$ with almost surely continuous in t for fixed $w\in\Omega$ with norm defined for any $\varphi\in\mathcal{B}_T$ by

$$\|\varphi\|_{\mathcal{B}_T}=\left(\sup_{0\leq t\leq T}\mathbf{E}\,\|\varphi\|_t^2\right)^{\frac{1}{2}}.$$

Furthermore, the fixed moments of time t_k satisfiy $0<t_1<...<t_m<T$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively. And $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, represent the jump in the state x at time t_k with I_k determining the size of the jump.

This manuscript is organized as follows: In section 2, we recall briefly the notation, definitions, lemmas and preliminaries which are used throughout this manuscript. In section 3, we study the existence and uniqueness of impulsive stochastic semilinear neutral functional integrodifferential equations and Poisson jumps. In section 4, we study stability through the continuous dependence on the initial values.

2 Preliminaries

Let $\mathbb{X}$, $\mathbb{Y}$ be real separable Hilbert spaces and $\mathcal{L}(\mathbb{Y},\mathbb{X})$ be the space of bounded linear operators mapping $\mathbb{Y}$ into $\mathbb{X}$. Let $(\Omega,\Im,\mathbb{P})$ be a complete probability space with an increasing right continuous family $\{\Im_t\}_{t\geq 0}$ of complete sub σ algebra of $\Im$. Let $\{w(t):t\geq 0\}$ denote a $\mathbb{Y}$-valued Wiener process defined on the probability space $(\Omega,\Im,\mathbb{P})$ with covariance operator Q, that is

$$\mathbf{E}<w(t),x>_{\mathbb{Y}}<w(s),y>_{\mathbb{Y}} \;=\; (t\wedge s)<Qx,y>_{\mathbb{Y}}, \quad \text{for all } x,y\in\mathbb{Y},$$

where Q is a positive, self-adjoint, trace class operator on $\mathbb{Y}$. In particular, we denote by $w(t), t\geq 0$, a $\mathbb{Y}$-valued Q-Wiener process with respect to $\{\Im_t\}_{t\geq 0}$.

In order to define stochastic integrals with respect to the Q-Wiener process $w(t)$, we introduce the subspace $\mathbb{Y}_0 = Q^{\frac{1}{2}}(\mathbb{Y})$ to $\mathbb{Y}$ which, endowed with the inner product $< u, v >_{\mathbb{Y}_0} = < Q^{\frac{-1}{2}}u, Q^{\frac{-1}{2}}v >_{\mathbb{Y}}$ is a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}_{i\geq 1}$ in $\mathbb{Y}$, a bounded sequence of non-negative real numbers λ_i such that $Qe_i = \lambda_i e_i$, $i = 1, 2, ...,$ and a sequence $\{\beta_i\}_{i\geq 1}$ of independent Brownian motions such that

$$< w(t), e > \;=\; \sum_{n=1}^{\infty} \sqrt{\lambda_i} < e_i, e > \beta_i(t), \quad e \in \mathbb{Y},$$

and $\Im_t = \Im_t^w$, where $\Im_t^w$ is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{Y}_0, \mathbb{X})$ denote the space of all Hilbert-Schmidt operators from $\mathbb{Y}_0$ into $\mathbb{X}$. It turns out to be a separable Hilbert space equipped with the norm $\|\zeta\|^2_{\mathcal{L}_2^0} = \mathrm{tr}((\zeta Q^{\frac{1}{2}})(\zeta Q^{\frac{1}{2}})^*)$ for any $\zeta \in \mathcal{L}_2^0$. Clearly for any bounded operators $\zeta \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ this norm reduces to

$$\|\zeta\|^2_{\mathcal{L}_2^0} \;=\; \mathrm{tr}\left(\zeta Q \zeta^*\right).$$

The resolvent operator plays an important role in the study of the existence of solutions and to give a variation of constant formula for linear systems. However, we need to know when the linear system (1)-(3) has a resolvent operator. For more details on resolvent operator, reader may refer to [18].

To obtain our results, consider the integrodifferential abstract Cauchy problem

$$\begin{cases} dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds\right]dt, & t \geq 0, \\ x(0) = x_0 \in \mathbb{X}. \end{cases} \tag{4}$$

Definition 1 ([18]) *A family of bounded linear operator $R(t) \in \mathcal{P}(\mathbb{X})$, $t \in J$ is called a resolvent operator for*

$$\frac{dx}{dt} = A\big[x(t) + \int_0^t B(t-s)x(s)ds\big],$$

if

(i) $R(0) = I$, the identity operator on $\mathbb{X}$.

(ii) for all $x \in \mathbb{X}$, $R(t)x$ is continuous for $t > 0$.

(iii) $R(t) \in \mathcal{P}(\mathbb{X})$, $t \in J$. For $x \in \mathbb{X}$, $R(\cdot)x \in C^1(J, \mathbb{X}) \bigcap C(J, \mathbb{X})$ and

$$\begin{aligned} \frac{d}{dt}R(t)x &= A\big[R(t)x + \int_0^t B(t-s)R(s)xds\big], \\ &= R(t)Ax + \int_0^t R(t-s)AB(s)ds, \ \text{for } t \geq 0. \end{aligned}$$

In what follows we make the following assumptions:

(A1) A is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{X}$.

(A2) For all $t \geq 0$, $B(t)$ is a closed linear operator $\mathcal{D}(A)$ to $\mathbb{X}$, and $B(t) = B(\mathbb{Y}, \mathbb{X})$. For any $y \in \mathbb{Y}$, the map $t \to B(t)y$ is bounded differentiable and derivative $t \to B'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Lemma 1 ([16]) *Let $T > 0$ and $u_0 > 0$, $u(t), v(t)$ be continuous functions on $[0, T]$. Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave continuous and nondecreasing function such that $K(r) > 0$ for $r > 0$. If*

$$u(t) \leq u_0 + \int_0^t v(t)K(u(s))ds \quad \text{for } \ 0 \leq t \leq T,$$

then

$$u(t) \le G^{-1}\left(G(u_0) + \int_0^t v(s)ds\right)$$

for all $t \in [0,T]$ *such that*

$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$*, for* $r \ge 0$ *and* G^{-1} *is the inverse function of the* G*. Moreover if* $u_0 = 0$ *and* $\int_0^+ \frac{ds}{K(s)} = +\infty$*, then* $u(t) = 0$ *for all* $t \in [0,T]$*.*

In order to obtain the stability of solutions, we use the following extended Bihari's inequality.

Lemma 2 **([4])** *Let the assumptions of Lemma 1 hold. If*

$$u(t) \le u_0 + \int_0^t v(t)K(u(s))ds \quad for \;\; 0 \le t \le T,$$

then

$$u(t) \le G^{-1}\left(G(u_0) + \int_0^t v(s)ds\right), \qquad t \in [0,T]$$

such that

$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$*, for* $r \ge 0$ *and* G^{-1} *is the inverse function of the* G*.*

Corollary 1 **([4])** *Let the assumptions of Lemma 1 hold and* $v(t) \ge 0$ *for* $t \in [0,T]$*. If for all* $\epsilon > 0$*, there exists* $t_1 \ge 0$ *such that for* $0 \le u_0 < \epsilon$*,* $\int_{t_1}^T v(s)ds \le \int_{u_0}^{\epsilon} \frac{ds}{K(s)}$ *holds, then for every* $t \in [t_1, T]$*, the estimate* $u(t) \le \epsilon$ *holds.*

Definition 2 *A stochastic process* $\{x(t), t \in (-\infty, T]\}$*, is called a mild solution of the equations (1)–(3) if*

(i) $x(t)$ *is* $\Im_t$*-adapted,*

(ii) $x(t)$ *satisfies the integral equation*

$$\begin{cases} x(t) = \varphi(t) \quad for \;\; t \in]-\infty, 0], \\ x(t) = R(t)\Big[\varphi(0) + g(0,\varphi)\Big] - g(t,x_t) + \int_0^t R(t-s)f(s,x_s)ds + \int_0^t R(t-s)\sigma(t,x_s)dw(s) \\ \qquad + \int_0^t \int_{\mathcal{U}} R(t-s)h(s,x_s,u)\tilde{N}(ds,du) + \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k)) \quad a.s \;\; t \in [0,T]. \end{cases} \tag{5}$$

3 Existence and Uniqueness

In this section, we discuss the existence and uniqueness of mild solution of the system (1)–(3). We use the following hypotheses to prove our results.

(H1) A is the infinitesimal generator of a strongly continuous semigroup $R(t)$, whose domain $\mathcal{D}(A)$ is dense in $\mathbb{X}$ such that

$$\|R(t,s)\| \le M, \text{ for all, } t \in J.$$

(H2) For each $x, y \in \hat{\mathcal{D}}$ and for all $t \in [0,T]$ such that

(i)

$$\|f(t,x_t)-f(t,y_t)\|^2 \vee \|\sigma(t,x_t)-\sigma(t,y_t)\|^2 \leq K\left(\|x_t-y_t\|_t^2\right),$$

(ii)

$$\int_{\mathcal{U}} \|h(t,x,u)-h(t,y,u)\|^2 v(du)ds \vee \left(\int_{\mathcal{U}} \|h(t,x,u)-h(t,y,u)\|^4 v(du)ds\right)^{\frac{1}{2}} \leq K\left(\|x_t-y_t\|_t^2\right),$$

(iii)

$$\left(\int_{\mathcal{U}} \|h(t,x,u)\|^4 v(du)ds\right)^{\frac{1}{2}} \leq K\|x\|_t^2.$$

where $K(\cdot)$ is a concave non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, such that $K(0)=0$, $K(u)>0$, for $u>0$ and $\int_{0^+} \frac{du}{K(u)} = \infty$.

(H3) Assuming that there exists a positive number L_g such that $L_g < \frac{1}{12}$, for any $x, y \in \hat{\mathcal{D}}$ and for $t \in [0,T]$, we have

$$\|g(t,x_t)-g(t,y_t)\|^2 \leq L_g \|x-y\|_t^2.$$

(H4) The function $I_k \in C(\mathbb{X},\mathbb{X})$ and there exists some constant h_k such that

$$\|I_k(x(t_k))-I_k(y(t_k))\|^2 \leq h_k\|x-y\|^2, \quad \text{for } x,y \in \hat{\mathcal{D}} \text{ and } k=1,2,...m.$$

(H5) For all $t \in [0,T]$, it follows that $f(t,0), g(t,0), \sigma(t,0), h(t,0)$ and $I_k(0) \in \mathcal{L}^2$, for $k=1,2,...,m$ such that

$$\|f(t,0)\|^2 \vee \|\sigma(t,0)\|^2 \vee \|h(t,0,u)\|^2 \vee \|I_k(0)\|^2 \leq k_0,$$

where $k_0 > 0$ is a constant.

Let us now introduce the successive approximation to equation (4) as follows

$$\begin{cases} x^n(t) = \varphi(t), \quad \text{for } t \in (-\infty, 0], \quad \text{for } n=0,1,2..., \\ x^n(t) = R(t)\big[\varphi(0)+g(0,\varphi)\big] - g(t,x_t^n) + \int_0^t R(t-s)f(s,x_s^{n-1})ds + \int_0^t R(t-s)\sigma(s,x_s^{n-1})dw(s) \\ \qquad + \int_0^t \int_{\mathcal{U}} R(t-s)h(s,x_s^{n-1},u)\tilde{N}(ds,du) \\ \qquad + \sum_{0<t_k<t} R(t-t_k)I_k(x^{n-1}(t_k)), \quad \text{a.s } t \in [0,T], \quad \text{for } n=1,2,..., \\ x^0(t) = R(t)\varphi(0), \quad t \in [0,T], \quad \text{for } n=0, \end{cases} \tag{6}$$

with an arbitrary non-negative initial approximation $x^0 \in \mathcal{B}_T$.

Theorem 1 *Let the assumptions (H1)–(H5) hold. Then the system (1)–(3) has unique mild solution $x(t)$ in $\mathcal{B}_T$ and*

$$\mathbf{E}\left\{\sup_{0\leq t\leq T} \|x^n(t)-x(t)\|^2\right\} \to 0, \quad \text{as } n \to \infty$$

where $\{x^n(t)\}_{n\geq 1}$ are the successive approximations (6)

Proof. The proof will be split into the following steps:
Step 1: For all $t \in (-\infty, T]$, the sequence $x^n(t)$, $n \geq 1 \in \mathcal{B}_T$ is bounded.

Let $x^0 \in \mathcal{B}_T$ be a fixed initial approximation to (6). Let $M = \sup_{t\in[0,T]} \|R(t)\|$. Then for any $n \geq 1$, we have

$$\begin{aligned}\|x^n(t)\|^2 \quad &\leq \quad 6M^2\mathbf{E}\,\|\varphi(0)+g(0,\varphi)\|^2 + 12\mathbf{E}\Big[\,\|g(t,x_t^n)-g(t,0)\|^2 + \|g(t,0)\|^2\Big] \\ &+12M^2T\mathbf{E}\int_0^t \Big[\big\|f(s,x_s^{n-1})-f(s,0)\big\|^2 + \|f(s,0)\|^2\Big]\,ds \\ &+12M^2\mathbf{E}\int_0^t \Big[\big\|\sigma(s,x_s^{n-1})-\sigma(s,0)\big\|^2 + \|\sigma(s,0)\|^2\Big]\,ds \\ &+12M^2\mathbf{E}\int_0^t\int_{\mathcal{U}} \Big[\big\|h(s,x_s^{n-1},u)-h(s,0,u)\big\|^2 + \|h(s,0,u)\|^2\Big]\,ds \\ &+6M^2\mathbf{E}\Big(\int_0^t\int_{\mathcal{U}} \big\|h(s,x_s^{n-1},u)\big\|^4 v(du)ds\Big)^{\frac{1}{2}} \\ &+12M^2m\mathbf{E}\sum_{k=1}^m \Big[\big\|I_k(x^{n-1}(t_k))-I_k(0)\big\|^2 + \|I_k(0)\|^2\Big].\end{aligned}$$

Thus,

$$\|x^n(t)\|_{\mathcal{B}}^2 \leq \frac{Q_1}{1-12L_g} + \frac{6M^2(2T+5)}{1-12L_g}\mathbf{E}\int_0^t K\left(\|x^{n-1}\|_{\mathcal{B}}^2\right)ds + \frac{12M^2m}{1-12L_g}\sum_{k=1}^m h_k\left\{\mathbf{E}\,\|x^{n-1}\|_{\mathcal{B}}^2\right\}.$$

where,

$$Q_1 = 12M^2\left[\mathbf{E}\,\|\varphi(0)\|^2 + L_g\mathbf{E}\,\|\varphi\|_0^2\right] + 12\Big[\left(1+M^2T(T+2)+M^2m\sum_{k=1}^m h_k\right)\Big]k_0.$$

Given that $K(\cdot)$ is concave and $K(0)=0$, we can find positive constants a and b such that

$$K(u) \leq a+bu, \qquad \text{for all} \quad u \geq 0.$$

Then

$$\mathbf{E}\,\|x^n(t)\|_{\mathcal{B}}^2 \leq Q_2 + \frac{6M^2(2T+5)b}{1-12L_g}\int_0^t \mathbf{E}\,\|x^{n-1}\|_s^2\,ds + \frac{12M^2m}{1-12L_g}\sum_{k=1}^m h_k\left\{\mathbf{E}\,\|x^{n-1}\|_{\mathcal{B}}^2\right\}, \qquad n=1,2,... \tag{7}$$

where $Q_2 = \frac{Q_1}{1-12L_g} + \frac{6M^2(2T+5)Ta}{1-12L_g}$. Since

$$\mathbf{E}\,\big\|x^0(t)\big\|_{\mathcal{B}}^2 \leq M^2\mathbf{E}\,\|\varphi(0)\|^2 = Q_3 < \infty, \tag{8}$$

we see that

$$\mathbf{E}\,\|x^n(t)\|^2 \leq \infty, \quad \text{for all} \;\; n=1,2,... \;\; \text{and} \;\; t\in[0,T].$$

This proves the boundedness of $\{x^n(t), n\in\mathbb{N}\}$.
Step 2: The sequence $\{x^n(t)\}$, $n \geq 1$ is a Cauchy sequence.
Let us next show that $\{x^n(t)\}$ is Cauchy sequence in $\mathcal{B}_T$. For this consider,

$$\begin{aligned}\mathbf{E}\,\big\|x^{n+1}(t)-x^n(t)\big\|^2 \quad &\leq \quad 5L_g\mathbf{E}\,\big\|x^{n+1}-x^n\big\|_t^2 + 5M^2(T+3)\int_0^t K\left(\mathbf{E}\,\big\|x^n-x^{n-1}\big\|_s^2\right)ds \\ &+5M^2m\sum_{k=1}^m h_k\mathbf{E}\left\{\big\|x^n-x^{n-1}\big\|_t^2\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{E}\left\|x^{n+1}(t)-x^{n}(t)\right\|^{2} &\leq \frac{5M^{2}(T+3)}{1-5L_{g}}\int_{0}^{t}K\left(\mathbf{E}\left\|x^{n}-x^{n-1}\right\|_{s}^{2}\right)ds\\ &\quad+\frac{5M^{2}m\sum_{k=1}^{m}h_{k}}{1-5L_{g}}\mathbf{E}\left\{\left\|x^{n}-x^{n-1}\right\|_{t}^{2}\right\}.\end{aligned} \tag{9}$$

Set

$$\Psi_{n}(t)=\sup_{t\in[0,T]}\mathbf{E}\left\|x^{n+1}-x^{n}\right\|_{t}^{2}.$$

Then, we have in the view of (9),

$$\Psi_{n}(t)\leq\frac{5M^{2}(T+3)}{1-5L_{g}}\int_{0}^{t}K\left(\Psi_{n-1}(s)\right)ds+\frac{5M^{2}m\sum_{k=1}^{m}h_{k}}{1-5L_{g}}\Psi_{n-1}(t),\quad 0\leq t\leq T. \tag{10}$$

Choose $T_1 \in [0,T)$ such that

$$C_{1}\int_{0}^{t}K\left(\Psi_{n-1}(s)\right)ds\leq C_{1}\Psi_{n-1}(s)ds,\quad n=1,2,...\quad \text{for all}\;\; 0\leq t\leq T_{1}.$$

Moreover,

$$\begin{aligned}\left\|x^{1}(t)-x^{0}(t)\right\|^{2} &= \Big\|R(t)g(0,\varphi)-\left[g(t,x_{t}^{1})-g(t,x_{t}^{0})\right]-g(t,x_{t}^{0})\\ &\quad+\int_{0}^{t}R(t-s)f(s,x_{s}^{0})ds+\int_{0}^{t}R(t-s)\sigma(s,x_{s}^{0})dw(s)\\ &\quad+\int_{0}^{t}\int_{\mathcal{U}}R(t-s)h(s,x_{s}^{0},u)\tilde{N}(ds,du)+\sum_{0<t_{k}<t}R(t-t_{k})I_{k}(x^{0}(t_{k}))\Big\|^{2}.\end{aligned}$$

Then, we get

$$\mathbf{E}\left\|x^{1}(t)-x^{0}(t)\right\|_{t}^{2}\leq Q_{4}+\frac{14L_{g}+14M^{2}m\sum_{k=1}^{m}h_{k}}{1-7L_{g}}\mathbf{E}\left\|x^{0}\right\|_{t}^{2}+\frac{7M^{2}(2T+5)}{1-7L_{g}}\int_{0}^{t}K\left(\mathbf{E}\left\|x^{0}\right\|_{s}^{2}\right)ds.$$

If we take the supremum over t, and use (8), we get

$$\Psi_{0}(t)=\sup_{t\in[0,T]}\mathbf{E}\left\|x^{1}-x^{0}\right\|_{t}^{2}\leq Q_{5}+\frac{7M^{2}(2T+5)}{1-7L_{g}}\int_{0}^{t}K(Q_{3})ds\leq Q_{6}.$$

Now, for $n=1$ in (10) we get

$$\Psi_{1}(t)\leq C_{1}\int_{0}^{t}K\left(\Psi_{0}(s)\right)ds+C_{2}\Psi_{0}(t),\quad 0\leq t\leq T_{1}.$$

where $C_1=\frac{5M^2(T+3)}{1-5L_g}$ and $C_2=\frac{5M^2m\sum_{k=1}^{m}h_k}{1-5L_g}$. Therefore,

$$\begin{aligned}\Psi_{1}(t) &\leq C_{1}\int_{0}^{t}K\left(\Psi_{0}(s)\right)ds+C_{2}\Psi_{0}(t)\\ &\leq C_{1}\int_{0}^{t}Q_{6}ds+C_{2}Q_{6}\\ &\leq \left(C_{1}+C_{2}\right)T_{1}Q_{6}.\end{aligned}$$

Now, for $n = 2$ in (10), we get

$$\begin{aligned}\Psi_2(t) &\leq C_1 \int_0^t K\left(\Psi_1(s)\right) ds + C_2 \Psi_1(t) \\ &\leq C_1 \int_0^t \left(C_1 + C_2\right) s Q_6 ds + C_2 (C_1 + C_2) T_1 Q_6 \\ &\leq \left(C_1 + C_2\right)^2 \frac{T_1^2}{2!} Q_6.\end{aligned}$$

Thus by applying mathematical induction in (10) and using the above work we get

$$\Psi_n(t) \leq \frac{\left(C_1 + C_2\right)^n T_1^n}{n!} Q_6, \quad n \geq 0, \quad t \in [0, T].$$

Note that for any $m > n \geq 0$, we have,

$$\begin{aligned}\sup_{t\in[0,T_1]} \mathbf{E} \left\|x^m(t) - x^n(t)\right\|^2 &\leq \sum_{r=n}^{+\infty} \sup_{t\in[0,T_1]} \mathbf{E} \left\|x^{r+1} - x^r\right\|_t^2 \\ &\leq \sum_{r=n}^{+\infty} \frac{(C_1 + C_2)^r T_1^r}{r!} Q_6 \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \tag{11}$$

This shows that $\{x^n\}$ is Cauchy in $\mathcal{B}_T$.

Step 3: The existence and uniqueness of the solution (1)–(3).

The Borel-Cantelli lemma argument can be used to show that as $x^n(t) \rightarrow x(t)$ uniformly in t on $[0, T_1]$. By iteration, the existence of solution of (1)–(3) on $[0, T]$ can be obtained.

Now, we prove the uniqueness of the solution (5). Let $x_1, x_2 \in \mathcal{B}_T$ be two solutions to (5) on some interval $(-\infty, T]$. Then, for $t \in (-\infty, 0]$, the uniqueness is obvious and for $0 \leq t \leq T$, we have

$$\begin{aligned}\mathbf{E} \left\|x_1(t) - x_2(t)\right\|^2 &\leq 5\big[L_g + M^2 m \sum_{k=1}^m h_k\big] \mathbf{E} \left\|x_1 - x_2\right\|_t^2 \\ &\quad + 5M^2(T+3) \int_0^t K\left(\mathbf{E} \left\|x_1 - x_2\right\|_s^2\right) ds.\end{aligned}$$

Thus,

$$\mathbf{E} \left\|x_1(t) - x_2(t)\right\|_t^2 \leq \frac{5M^2(T+3)}{1 - Q_7} \int_0^t K\left(\mathbf{E} \left\|x_1 - x_2\right\|_s^2\right) ds.$$

where, $5\big[L_g + M^2 m \sum_{k=1}^m h_k\big]$. Thus, Bihari's inequality yields that

$$\sup_{t\in[0,T]} \mathbf{E} \left\|x_1(t) - x_2(t)\right\|_t^2 = 0, \quad 0 \leq t \leq T.$$

Thus, $x_1(t) = x_2(t)$, for all $0 \leq t \leq T$. Therefore, for all $-\infty < t \leq T$, $x_1(t) = x_2(t)$ a.s.
This completes the proof. ■

4 Stability

In this section, we study the stability through the continuous dependence on initial values.

Definition 3 . *A mild solution $x(t)$ of the system (1)–(3) with inital value ϕ is said to be stable in the mean square if for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\mathbf{E}\left\|x - \hat{x}\right\|_{\mathcal{B}}^2 \leq \epsilon, \text{ whenever } \mathbf{E}\left\|\phi - \hat{\phi}\right\|_{\mathcal{B}}^2 \leq \delta,$$

where $\hat{x}(t)$ is another mild solution of the system (1)–(3) with initial $\hat{\phi}$.

Theorem 2 *Let $x(t)$ and $y(t)$ be the mild solution of the system (1)–(3) with initial values φ_1 and φ_2 respectively. If the assumption of Theorem 1 are satisfied, then the mild solution of the system (1)–(3) is stable in the mean square.*

Proof. Let $x(t)$ and $y(t)$ be the mild solutions of equation (1)–(3) with initial values φ_1 and φ_2 respectively. Then for $0 \leq t \leq T$,

$$\begin{aligned} x(t) - y(t) &= R(t)\big[\,[\varphi_1(0) - \varphi_2(0)] + [g(0,\varphi_1) - g(0,\varphi_2)]\,\big] - \big[g(t,x_t) - g(t,y_t)\big] \\ &\quad + \int_0^t R(t-s)\,[f(s,x_s) - f(s,y_s)]\,ds + \int_0^t R(t-s)\,[\sigma(s,x_s) - \sigma(s,y_s)]\,dw(s) \\ &\quad + \int_0^t \int_{\mathcal{U}} R(t-s)\,[h(s,x_s,u) - h(s,y_s,u)]\,\tilde{N}(ds,du) + \sum_{0<t_k<t} R(t-s)\,[I_k(x(t_k)) - I_k(y(t_k))]\,. \end{aligned}$$

So, estimating as before, we get

$$\begin{aligned} \mathbf{E}\left\|x - y\right\|^2 &\leq 7M^2\,[1 + L_g]\,\mathbf{E}\left\|\varphi_1 - \varphi_2\right\|^2 + 7M^2\,[T + 3]\int_0^t K\left(\mathbf{E}\left\|x - y\right\|_s^2\right) ds \\ &\quad + 7\big[L_g + M^2 m\sum_{k=1}^m h_k\big]\,\mathbf{E}\left\|x - y\right\|_t^2\,. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{E}\left\|x - y\right\|_t^2 &\leq \frac{7M^2(1 + L_g)}{1 - 7(L_g + M^2 m\sum_{k=1}^m h_k)}\mathbf{E}\left\|\varphi_1 - \varphi_2\right\|^2 \\ &\quad + \frac{7M^2(T + 3)}{1 - 7(L_g + M^2 m\sum_{k=1}^m h_k)}\int_0^t K\left(\mathbf{E}\left\|x - y\right\|_s^2\right) ds. \end{aligned}$$

Let

$$K_1(u) = \frac{7M^2(T + 3)}{1 - 7(L_g + M^2 m\sum_{k=1}^m h_k)}K(u),$$

where K is a concave increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K(0) = 0$, $K(u) > 0$ for $u > 0$ and $\int_{0_+} \frac{du}{K(u)} = +\infty$. Then, $K_1(u)$ is concave from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K(0) = 0$, $K_1(u) \geq K(u)$ for $0 \leq u \leq 1$ and $\int_{0_+} \frac{du}{K(u)} = +\infty$. Now for any $\epsilon > 0$, $\epsilon_1 = \frac{1}{2}\epsilon$, we have $\lim_{s \to 0} \int_s^{\epsilon_1} \frac{du}{K_1(u)} = \infty$. Then, there is a positive constant $\delta < \epsilon_1$, such that $\int_\delta^{\epsilon_1} \frac{du}{K_1(u)} \geq T$. Let

$$\begin{aligned} u_0 &= \frac{7M^2(1 + L_g)}{1 - 7(L_g + M^2 m\sum_{k=1}^m h_k)}\mathbf{E}\left\|\varphi_1 - \varphi_2\right\|^2, \\ u(t) &= \mathbf{E}\left\|x - y\right\|_{\mathcal{B}}^2, \qquad v(t) = 1, \end{aligned}$$

when $u_0 \leq \delta \leq \epsilon_1$. From Corollary 1, we have

$$\int_{u_0}^{\epsilon_1} \frac{du}{K_1(u)} \geq \int_\delta^{\epsilon_1} \frac{du}{K_1(u)} \geq T = \int_0^T v(t)ds.$$

It follows, for any $t \in [0, T]$, the estimate $u(t) \leq \epsilon_1$ hold. This completes the proof. ∎

Remark 1 *If $m = 0$ in (1)–(3), then the system behaves as stochastic partial neutral functional integrodifferential equations with infinite delays and poisson jumps of the form:*

$$d\big[x(t) + g(t, x_t)\big] = A\big[x(t) + g(t, x_t)\big]dt + \left[\int_0^t B(t-s)\left[x(s) + g(s, x_s)\right]ds + f(t, x_t)\right] + \sigma(t, x_t)dw(t) + \int_{\mathcal{U}} h(t, x_t, u)\tilde{N}(ds, du), \qquad t \in J = [0, T], \tag{12}$$

$$x_0 = \varphi \in \mathcal{D}^b_{\mathcal{B}_0}((-\infty, 0], \mathbb{X}). \tag{13}$$

By applying Theorem 1, under the hypotheses (H1)–(H3), (H5) the system (11)–(12) guarantees the existence and uniqueness of the mild solution.

Remark 2 *If the system (12)–(13) satisfies the Remark 1, then by Theorem 2, the mild olution of the system (12)–(13) is stable in mean square.*

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