Applied Mathematics E-Notes, 21(2021), 437-450 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Hermite-Hadamard Type Inequalities For Conformable Integrals Via Preinvex Functions*

Yousaf Khurshid[†], Muhammad Adil Khan[‡]

Received 9 June 2020

Abstract

In recent years, many results are devoted to the well-known Hermite-Hadamard inequality. This inequality has many applications in the area of pure and applied mathematics. In this paper, our main aim is to give results for conformable integral version of Hermite-Hadamard inequality for preinvex functions. First, we prove an identity associated with the Hermite-Hadamard inequality for conformable integrals using preinvex functions. By using this identity and preinvexity of function and some well-known inequalities, we obtain several results for the inequality.

1 Introduction

Let $I \in \mathbb{R}$ be an interval and $h : I \to \mathbb{R}$ be a convex function defined on I such that $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$. Then the well known Hermite-Hadamard inequality [25] states that

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} h(x)dx \leq \frac{h(\kappa_1)+h(\kappa_2)}{2} \tag{1}$$

holds. If the function h is concave on I, then both the inequalities in (1) hold in the reverse direction.

In the last few years, many researchers have shown their extensive attention on the generalizations, extensions, variations, refinements and applications of the Herimte-Hadamard inequality (see [9, 8, 16, 17, 18, 7, 20, 21, 22, 23, 33, 36, 37, 38, 39, 41, 43, 44, 45, 46, 52]). It is well-known that the convex sets and convex functions play important roles in the nonlinear programming and optimization theory. Many generalizations and extensions have been considered for the classical convexity in the last few decades. Hanson [26] introduced the invex function which is the generalization of the convex function. The basic properties for the preinvex functions and their roles in optimization theory, variational inequalities and equilibrium problems can be found in the literature [40]. The Hermite-Hadamard inequalities for preinvex and log-preinvex functions were established by Noor [32, 31].

Now, we recall some notions and definitions in invexity analysis, which will be used throughout the paper (see [13, 42] and references therein). Let $\mathfrak{A} \in \mathbb{R}$ be a non-empty set and the functions $h : \mathfrak{A} \to \mathbb{R}$ and $\Psi : \mathfrak{A} \times \mathfrak{A} \to \mathbb{R}$ be continuous.

Definition 1 *The set $\mathfrak{A} \subseteq \mathbb{R}^n$ is said to be invex with respect to $\Psi(.\ ,\ .)$ if*

$$\mu_1 + s\Psi(\mu_2, \mu_1) \in \mathfrak{A}$$

for all $\mu_1, \mu_2 \in \mathfrak{A}$ and $s \in [0, 1]$.

The invex set $\mathfrak{A}$ is also called a Ψ-connected set. If $\Psi(\mu_2, \mu_1) = \mu_2 - \mu_1$, then the invex set is also a convex set, but some of the invex sets are not convex [13].

*Mathematics Subject Classifications: 26D15, 26A51, 26A33, 26A42.
†Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
‡Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan

Definition 2 *The function h is said to be preinvex with respect to Ψ on the invex set $\mathfrak{A}$ if*

$$h(\mu_1 + s\Psi(\mu_2, \mu_1)) \leq (1-s)h(\mu_1) + sh(\mu_2)$$

for all $\mu_1, \mu_2 \in \mathfrak{A}$ and $s \in [0,1]$. The function h is called preconcave if $-h$ is preinvex.

The following Condition C was introduced by Mohan and Neogy [30]:

Condition C: Suppose $\mathfrak{A}$ is an open invex subset of $\mathbb{R}^n$ with respect to $\Psi : \mathfrak{A} \times \mathfrak{A} \to \mathbb{R}$ and Ψ satisfies

$$\Psi(\mu_2, \mu_2 + s\Psi(\mu_1, \mu_2)) = -s\Psi(\mu_1, \mu_2),$$

$$\Psi(\mu_1, \mu_2 + s\Psi(\mu_1, \mu_2)) = (1-s)\Psi(\mu_1, \mu_2),$$

for any $\mu_1, \mu_2 \in \mathfrak{A}$ and $s \in [0,1]$.

From condition C we clearly see that

$$\Psi(\mu_2 + s_2\Psi(\mu_1, \mu_2),\ \mu_2 + s_1\Psi(\mu_1, \mu_2)) = (s_2 - s_1)\Psi(\mu_1, \mu_2) \tag{2}$$

for any $\mu_1, \mu_2 \in \mathfrak{A}$ and $s \in [0,1]$. The following Herimte-Hadamard inequality for the preinvex functions was proved by Noor [32]:

Theorem 1 *Let $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2, \kappa_1)] \to (0, \infty)$ be a preinvex function on the interval K° (the interior of K) and $\kappa_1, \kappa_2 \in K^\circ$ with $\kappa_1 < \kappa_1 + \Psi(\kappa_2, \kappa_1)$. Then the following inequality holds*

$$h\left(\frac{2\kappa_1 + \Psi(\kappa_2, \kappa_1)}{2}\right) \leq \frac{1}{\Psi(\kappa_2, \kappa_1)} \int_{\kappa_1}^{\kappa_1 + \Psi(\kappa_2, \kappa_1)} h(x)dx \leq \frac{h(\kappa_1) + h(\kappa_2)}{2}. \tag{3}$$

Several important variants of Herimte-Hadamard inequality for preinvex functions have been provided in the literature [28]. Recently, the authors in [29] defined a new well-behaved simple fractional derivative called the "conformable fractional derivative". Namely, the conformable fractional derivative of order $0 < \beta \leq 1$ at $s > 0$ for the function $h : [0, \infty) \to \mathbb{R}$ is defined by

$$D_\beta(h)(s) = \lim_{\epsilon \to 0} \frac{h(s + \epsilon s^{1-\beta}) - h(s)}{\epsilon}.$$

If the conformable fractional derivative of h of order β exists, then we say that h is β-fractional differentiable. The fractional derivative at 0 is defined as $h^\beta(0) = \lim_{s \to 0^+} h^\beta(s)$.

Next, we present some basic results related to conformable fractional derivative in the following theorem.

Theorem 2 ([29]) *Let $\beta \in (0,1]$ and h_1, h_2 be β-differentiable at a point $s > 0$. Then*

(i) $\frac{d_\beta}{d_\beta s}(s^n) = ns^{n-\beta}$ *for all $n \in \mathbb{R}$.*

(ii) $\frac{d_\beta}{d_\beta s}(c) = 0$ *for any constant $c \in \mathbb{R}$.*

(iii) $\frac{d_\beta}{d_\beta s}(\kappa_1 h_1(s) + \kappa_2 h_2(s)) = \kappa_1 \frac{d_\beta}{d_\beta s}(h_1(s)) + \kappa_2 \frac{d_\beta}{d_\beta s}(h_2(s))$ *for all $\kappa_1, \kappa_2 \in \mathbb{R}$.*

(iv) $\frac{d_\beta}{d_\beta s}(h_1(s)h_2(s)) = h_1(s)\frac{d_\beta}{d_\beta s}(h_2(s)) + h_2(s)\frac{d_\beta}{d_\beta s}(h_1(s))$.

(v) $\frac{d_\beta}{d_\beta s}\left(\frac{h_1(s)}{h_2(s)}\right) = \frac{h_2(s)\frac{d_\beta}{d_\beta s}(h_1(s)) - h_1(s)\frac{d_\beta}{d_\beta s}(h_2(s))}{(h_2(s))^2}$.

(vi) $\frac{d_\beta}{d_\beta s}((h_1 \circ h_2)(s)) = h_1'(h_2(s))\frac{d_\beta}{d_\beta s}(h_2(s))$ *if h_1 differentiable at $h_2(s)$.*

If in addition h_1 is differentiable, then

$$\frac{d_\beta}{d_\beta s}(h_1(s)) = s^{1-\beta}\frac{d}{ds}(h_1(s)).$$

Definition 3 ([29]) *(**Conformable fractional integral**). Let $\beta \in (0,1]$ and $0 \leq \kappa_1 < \kappa_2$. A function $h_1 : [\kappa_1, \kappa_2] \to \mathbb{R}$ is β-fractional integrable on $[\kappa_1, \kappa_2]$ if the integral*

$$\int_{\kappa_1}^{\kappa_2} h_1(x) d_\beta x := \int_{\kappa_1}^{\kappa_2} h_1(x) x^{\beta-1} dx$$

exists and is finite. All β-fractional integrable functions on $[\kappa_1, \kappa_2]$ is indicated by $L_\beta([\kappa_1, \kappa_2])$.

Remark 1

$$I_\beta^{\kappa_1}(h_1)(s) = I_1^{\kappa_1}(s^{\beta-1}h_1) = \int_{\kappa_1}^{s} \frac{h_1(x)}{x^{1-\beta}} dx,$$

where the integral is the usual Riemann improper integral and $\beta \in (0,1]$.

Recently, the conformable integrals and derivatives have been the subject of intensive research, many remarkable properties and inequalities involving the conformable integrals and derivatives can be found in the literature [1, 2, 3, 4, 5, 6, 11, 12, 14, 15, 19, 27, 34, 35, 50, 51, 53, 54, 55].

In [10], Anderson provided the conformable integral version of Herimte-Hadamard inequality as follows:

Theorem 3 ([10]) *If $\beta \in (0,1]$ and $h_1 : [\kappa_1, \kappa_2] \to \mathbb{R}$ is an β-fractional differentiable function such that $D_\beta h$ is increasing, then we have the following inequality*

$$\frac{\beta}{\kappa_2^\beta - \kappa_1^\beta} \int_{\kappa_1}^{\kappa_2} h(x) d_\beta x \leq \frac{h(\kappa_1) + h(\kappa_2)}{2}. \tag{4}$$

Moreover if the function h is decreasing on $[\kappa_1, \kappa_2]$, then we have

$$h\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{\beta}{\kappa_2^\beta - \kappa_1^\beta} \int_{\kappa_1}^{\kappa_2} h(x) d_\beta x. \tag{5}$$

If $\beta = 1$, then this reduces to the classical Herimte-Hadamard inequality.

Theorem 4 ([49]) *Let $\kappa_1, \kappa_2 > 0$ such that $\Psi(\kappa_2, \kappa_1) > 0$ and $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2, \kappa_1)] \to (0, \infty)$ is a preinvex function and symmetric with respect to $\frac{2\kappa_1 + \Psi(\kappa_2,\kappa_1)}{2}$, then the following conformable fractional integrals inequality*

$$h\left(\frac{2\kappa_1 + \Psi(\kappa_2, \kappa_1)}{2}\right) \leq \frac{\beta}{(\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta} \int\limits_{\kappa_1}^{\kappa_1 + \Psi(\kappa_2,\kappa_1)} h(x) d_\beta x \leq \frac{h(\kappa_1) + h(\kappa_1 + \Psi(\kappa_2, \kappa_1))}{2} \tag{6}$$

holds for any $\beta \in (0,1]$.

In recent years, many results are devoted to the well-known Hermite-Hadamard inequality. This inequality has many applications in the area of pure and applied mathematics. In this paper, our main aim is to give results for conformable integral version of Hermite-Hadamard inequality for preinvex functions. First, we prove an identity associated with the Hermite-Hadamard inequality for conformable integrals using preinvex functions. By using this identity and preinvexity of function and some well-known inequalities, we obtain several results for the inequality.

2 Results Connected with Left Part of Hadamard's Type Inequality

In this section, first we prove the following lemma associated with the inequality (6), which will be used in the derivation of our main results.

Lemma 1 *Let $\kappa_1, \kappa_2 > 0$ such that $\Psi(\kappa_2,\kappa_1) > 0$ and $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)] \to (0,\infty)$ be an β-fractional differentiable function on $(\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1))$ for $\beta \in (0,1]$. If $D_\beta(h) \in L_\beta([\kappa_1, \kappa_1 + \Psi(\kappa_2,\kappa_1)])$, then the following identity holds:*

$$\begin{aligned}
&h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right) - \frac{\beta}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta - \kappa_1^\beta}\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)d_\beta x \\
= \;& \frac{\Psi(\kappa_2,\kappa_1)}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta - \kappa_1^\beta}\Bigg[\int_0^{\frac{1}{2}} \big(((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{2\beta-1} - \kappa_1^\beta(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta-1}\big) \\
&\times D_\beta(h)(\kappa_1+s\Psi(\kappa_2,\kappa_1))s^{1-\beta}d_\beta s + \int_{\frac{1}{2}}^1 \big((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{2\beta-1} - (\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta \\
&\times(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta-1}\big) D_\beta(h)(\kappa_1+s\Psi(\kappa_2,\kappa_1))s^{1-\beta}d_\beta s\Bigg]. \qquad (7)
\end{aligned}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
I \;=\;& \int_0^{\frac{1}{2}} \big(((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{2\beta-1} - \kappa_1^\beta(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta-1}\big)D_\beta(h)(\kappa_1+s\Psi(\kappa_2,\kappa_1))ds \\
&+\int_{\frac{1}{2}}^1 \big((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{2\beta-1} - (\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta-1}\big) \\
&\times D_\beta(h)(\kappa_1+s\Psi(\kappa_2,\kappa_1))ds \\
=\;& \int_0^{\frac{1}{2}} \big((\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta - \kappa_1^\beta\big)h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))ds \\
&+\int_{\frac{1}{2}}^1 \big((\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta - (\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta\big)h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))ds \\
=\;& \big((\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta - \kappa_1^\beta\big)\frac{h(\kappa_1+s\Psi(\kappa_2,\kappa_1))}{\Psi(\kappa_2,\kappa_1)}\Bigg|_0^{\frac{1}{2}} \\
&-\int_0^{\frac{1}{2}} \beta(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta-1}\Psi(\kappa_2,\kappa_1)\frac{h(\kappa_1+s\Psi(\kappa_2,\kappa_1))}{\Psi(\kappa_2,\kappa_1)}ds \\
&+\big((\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta - (\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta\big)\frac{h(\kappa_1+s\Psi(\kappa_2,\kappa_1))}{\Psi(\kappa_2,\kappa_1)}\Bigg|_{\frac{1}{2}}^1 \\
&-\int_{\frac{1}{2}}^1 \beta(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta-1}\Psi(\kappa_2,\kappa_1)\frac{h(\kappa_1+s\Psi(\kappa_2,\kappa_1))}{\Psi(\kappa_2,\kappa_1)}ds \\
=\;& \frac{1}{\Psi(\kappa_2,\kappa_1)}\left[\left(\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right)^\beta - \kappa_1^\beta\right)h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right) - \beta\int_{\kappa_1}^{\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}} h(x)d_\beta x\right] \\
&+\frac{1}{\Psi(\kappa_2,\kappa_1)}\Bigg[\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta - \left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right)^\beta\right)h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right) \\
&-\beta\int_{\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}}^{\kappa_2} h(x)d_\beta x\Bigg]
\end{aligned}$$

$$= \frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}}{\Psi(\kappa_2,\kappa_1)} h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right) - \frac{\beta}{\Psi(\kappa_2,\kappa_1)} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x) d_{\beta}x,$$

we use the change of variable $x=\kappa_1+s\Psi(\kappa_2,\kappa_1)$ and then multiplying both sides by $\frac{\Psi(\kappa_2,\kappa_1)}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}}$ to get the desired result in (7). ∎

Remark 2 *Let $\beta=1$. Then (7) leads to*

$$\begin{aligned} &\frac{1}{\Psi(\kappa_2,\kappa_1)} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(s)ds - h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right) \\ &= \Psi(\kappa_2,\kappa_1)\left[\int_0^{\frac{1}{2}} s h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))ds + \int_{\frac{1}{2}}^1 (s-1)h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))ds\right]. \end{aligned}$$

Theorem 5 *Let $\kappa_1,\kappa_2>0$ such that $\Psi(\kappa_2,\kappa_1)>0$ and $h: K=[\kappa_1,\kappa_1+\Psi(\kappa_2,\kappa_1)]\to(0,\infty)$ be an β-differentiable function on $(\kappa_1,\kappa_1+\Psi(\kappa_2,\kappa_1))$ for $\beta\in(0,1]$ such that $D_{\beta}(h)\in L_{\beta}([\kappa_1,\kappa_1+\Psi(\kappa_2,\kappa_1)])$. If $|h'|$ is preinvex, then we have the following inequality:*

$$\begin{aligned} &\left| h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right) - \frac{\beta}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)d_{\beta}x \right| \\ &\le \frac{\Psi(\kappa_2,\kappa_1)}{((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta})}\left[(\mathfrak{A}_1+\mathfrak{B}_1)|h'(\kappa_1)| + (\mathfrak{A}_2+\mathfrak{B}_2)|h'(\kappa_2)|\right], \end{aligned}$$

where

$$\begin{aligned} \mathfrak{A}_1 &= \frac{\left(2\kappa_1+\Psi(\kappa_2,\kappa_1)\right)^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)+\left(2\kappa_1+\Psi(\kappa_2,\kappa_1)\right)}{(\beta+2)\Psi(\kappa_2,\kappa_1)}\right] \\ &\quad -\frac{\kappa_1^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)+\kappa_1}{(\beta+2)\Psi(\kappa_2,\kappa_1)}\right] - \frac{3}{8}\kappa_1^{\beta}, \end{aligned}$$

$$\begin{aligned} \mathfrak{B}_1 &= \frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}}{8} + \frac{\left(2\kappa_1+\Psi(\kappa_2,\kappa_1)\right)^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)+(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)\Psi(\kappa_2,\kappa_1)}\right] \\ &\quad -\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+2}}{(\beta+1)(\Psi(\kappa_2,\kappa_1))^2(\beta+2)}, \end{aligned}$$

$$\begin{aligned} \mathfrak{A}_2 &= \frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)-(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)\Psi(\kappa_2,\kappa_1)}\right] \\ &\quad +\frac{\kappa_1^{\beta+2}}{(\beta+1)(\Psi(\kappa_2,\kappa_1))^2(\beta+2)} - \frac{\kappa_1^{\beta}}{8}, \end{aligned}$$

$$\begin{aligned} \mathfrak{B}_2 &= \frac{3}{8}(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta} - \frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)-(\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)\Psi(\kappa_2,\kappa_1)}\right] \\ &\quad +\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)-(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)\Psi(\kappa_2,\kappa_1)}\right]. \end{aligned}$$

Proof. From Lemma 1, using the property of the modulus and preinvexity of $|h'|$, we have

$$\begin{aligned}
&\left|h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right)-\frac{\beta}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)}h(x)d_\beta x\right| \\
&= \left|\frac{\Psi(\kappa_2,\kappa_1)}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-\kappa_1^\beta}\left[\int_0^{\frac{1}{2}}\left(((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{2\beta-1}-\kappa_1^\beta(\kappa_1+t\Psi(\kappa_2,\kappa_1))^{\beta-1}\right)\right.\right. \\
&\quad \times D_\beta(h)(\kappa_1+s\Psi(\kappa_2,\kappa_1))ds+\int_{\frac{1}{2}}^1\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{2\beta-1}-(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta\right. \\
&\quad \left.\left.\left.\times(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta-1}\right)D_\beta(h)(\kappa_1+s\Psi(\kappa_2,\kappa_1))ds\right]\right| \\
&\le \frac{\Psi(\kappa_2,\kappa_1)}{((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-\kappa_1^\beta)}\left[\int_0^{\frac{1}{2}}\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta-\kappa_1^\beta\right)|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))|ds\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta\right)|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))|ds\right] \\
&\le \frac{\Psi(\kappa_2,\kappa_1)}{((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-\kappa_1^\beta)}\left[\int_0^{\frac{1}{2}}\left((\kappa_1+s\Psi(\kappa_2,\kappa_2))^\beta-\kappa_1^\beta\right)[(1-s)|h'(\kappa_1)|+s|h'(\kappa_2)|]ds\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta\right)[(1-s)|h'(\kappa_1)|+s|h'(\kappa_2)|]ds\right] \\
&= \frac{\Psi(\kappa_2,\kappa_1)}{((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-\kappa_1^\beta)}\left[(\mathfrak{A}_1+\mathfrak{B}_1)|h'(\kappa_1)|+(\mathfrak{A}_2+\mathfrak{B}_2)|h'(\kappa_2)|\right].
\end{aligned}$$

■

Corollary 1 *Let $\beta=1$. Then Theorem 5 leads to*

$$\left|h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right)-\frac{1}{\Psi(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)}h(x)dx\right|\le\left[(\mathfrak{A}_1+\mathfrak{B}_1)|h'(\kappa_1)|+(\mathfrak{A}_2+\mathfrak{B}_2)|h'(\kappa_2)|\right],$$

where

$$\mathfrak{A}_1=\frac{\left(2\kappa_1+\Psi(\kappa_2,\kappa_1)\right)^2}{16\Psi(\kappa_2,\kappa_1)}\left[\frac{3\Psi(\kappa_2,\kappa_1)+\left(2\kappa_1+\Psi(\kappa_2,\kappa_1)\right)}{3\Psi(\kappa_2,\kappa_1)}\right]-\frac{\kappa_1^2}{2\Psi(\kappa_2,\kappa_1)}\left[\frac{3\Psi(\kappa_2,\kappa_1)+\kappa_1}{3\Psi(\kappa_2,\kappa_1)}\right]-\frac{3}{8}\kappa_1,$$

$$\mathfrak{B}_1=\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))}{8}+\frac{\left(2\kappa_1+\Psi(\kappa_2,\kappa_1)\right)^2}{16\Psi(\kappa_2,\kappa_1)}\left[\frac{3\Psi(\kappa_2,\kappa_1)+(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{3\Psi(\kappa_2,\kappa_1)}\right]-\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^3}{6(\Psi(\kappa_2,\kappa_1))^2},$$

$$\mathfrak{A}_2=\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^2}{16\Psi(\kappa_2,\kappa_1)}\left[\frac{3\Psi(\kappa_2,\kappa_1)-(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{3\Psi(\kappa_2,\kappa_1)}\right]+\frac{\kappa_1^3}{6(\Psi(\kappa_2,\kappa_1))^2}-\frac{\kappa_1}{8},$$

$$\begin{aligned}
\mathfrak{B}_2 &= \frac{3}{8}(\kappa_1+\Psi(\kappa_2,\kappa_1))-\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^2}{2\Psi(\kappa_2,\kappa_1)}\left[\frac{3\Psi(\kappa_2,\kappa_1)-(\kappa_1+\Psi(\kappa_2,\kappa_1))}{3\Psi(\kappa_2,\kappa_1)}\right] \\
&\quad +\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^2}{16\Psi(\kappa_2,\kappa_1)}\left[\frac{3\Psi(\kappa_2,\kappa_1)-(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{3\Psi(\kappa_2,\kappa_1)}\right].
\end{aligned}$$

Theorem 6 *Let $\kappa_1, \kappa_2 > 0$ such that $\Psi(\kappa_2, \kappa_1) > 0$ and $h : K = [\kappa_1, \kappa_1 + \Psi(\kappa_2, \kappa_1)] \to (0, \infty)$ be an β-differentiable function on $(\kappa_1, \kappa_1 + \Psi(\kappa_2, \kappa_1))$ for $\beta \in (0, 1]$ such that $D_\beta(h) \in L_\beta([\kappa_1, \kappa_1 + \Psi(\kappa_2, \kappa_1)])$. If $|h'^q$ is preinvex for $q > 1$ and $q^{-1} + p^{-1} = 1$, then we have the following inequality:*

$$\begin{aligned}
&\left| h\left(\frac{2\kappa_1 + \Psi(\kappa_2, \kappa_1)}{2}\right) - \frac{\beta}{(\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta} \int_{\kappa_1}^{\kappa_1 + \Psi(\kappa_2, \kappa_1)} h(x) d_\beta x \right| \\
&\leq \frac{\Psi(\kappa_2, \kappa_1)}{(\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta} \Big[(\mathfrak{A}_1(\beta, p))^{\frac{1}{p}} \left(\frac{3|h'(\kappa_1)|^q + |h'(\kappa_2)|^q}{8} \right)^{\frac{1}{q}} \\
&\quad + (\mathfrak{A}_2(\beta, p))^{\frac{1}{p}} \left(\frac{|h'(\kappa_1)|^q + 3|h'(\kappa_2)|^q}{8} \right)^{\frac{1}{q}} \Big], \qquad (8)
\end{aligned}$$

where

$$\mathfrak{A}_1(\beta, p) = \int_0^{\frac{1}{2}} \left((\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta \right)^p ds,$$

$$\mathfrak{A}_2(\beta, p) = \int_{\frac{1}{2}}^1 \left((\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - (\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta \right)^p ds.$$

Proof. From Lemma 1, using the property of the modulus and preinvexity of $|h'|$, we have

$$\begin{aligned}
&\left| h\left(\frac{2\kappa_1 + \Psi(\kappa_2, \kappa_1)}{2}\right) - \frac{\beta}{(\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta} \int_{\kappa_1}^{\kappa_1 + \Psi(\kappa_2, \kappa_1)} h(x) d_\beta x \right| \\
&\leq \frac{\Psi(\kappa_2, \kappa_1)}{((\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta)} \Bigg[\int_0^{\frac{1}{2}} \left((\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta \right) |h'(\kappa_1 + s\Psi(\kappa_2, \kappa_1))| ds \\
&\quad + \int_{\frac{1}{2}}^1 \left((\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - (\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta \right) |h'(\kappa_1 + s\Psi(\kappa_2, \kappa_1))| ds \Bigg].
\end{aligned}$$

Now by the Holder's inequality

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left((\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta \right) |h'(\kappa_1 + s\Psi(\kappa_2, \kappa_1))| ds \\
&\leq \left(\int_0^{\frac{1}{2}} \left((\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta \right)^p ds \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |h'(\kappa_1 + s\Psi(\kappa_2, \kappa_1))|^q ds \right)^{\frac{1}{q}} \\
&\leq \left(\int_0^{\frac{1}{2}} \left((\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta - \kappa_1^\beta \right)^p ds \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} (1 - s)|h'(\kappa_1)|^q + s|h'(\kappa_2)|^q ds \right)^{\frac{1}{q}} \\
&= (\mathfrak{A}_1(\beta, p))^{\frac{1}{p}} \left(\frac{3|h'(\kappa_1)|^q + |h'(\kappa_2)|^q}{8} \right)^{\frac{1}{q}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left((\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - (\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta \right) |h'(\kappa_1 + s\Psi(\kappa_2, \kappa_1))| ds \\
&\leq \left(\int_{\frac{1}{2}}^1 \left((\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - (\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta \right)^p ds \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 |h'(\kappa_1 + s\Psi(\kappa_2, \kappa_1))|^q ds \right)^{\frac{1}{q}} \\
&\leq \left(\int_{\frac{1}{2}}^1 \left((\kappa_1 + \Psi(\kappa_2, \kappa_1))^\beta - (\kappa_1 + s\Psi(\kappa_2, \kappa_1))^\beta \right)^p ds \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 (1 - s)|h'(\kappa_1)|^q + s|h'(\kappa_2)|^q ds \right)^{\frac{1}{q}}
\end{aligned}$$

$$= \quad (\mathfrak{A}_2(\beta,p))^{\frac{1}{p}} \left(\frac{|h'(\kappa_1)|^q + 3|h'(\kappa_2)|^q}{8} \right)^{\frac{1}{q}}.$$

Hence, we have the result in (8). ■

Corollary 2 *Let $\beta = 1$. Then Theorem 6 leads to*

$$\left| h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right) - \frac{1}{\Psi(\kappa_2,\kappa_1)} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)dx \right|$$
$$\leq \frac{\Psi(\kappa_2,\kappa_1)}{2^{p+1}(p+1)^{\frac{1}{p}}} \Big[\left(\frac{3|h'(\kappa_1)|^q + |h'(\kappa_2)|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{|h'(\kappa_1)|^q + 3|h'(\kappa_2)|^q}{8} \right)^{\frac{1}{q}} \Big].$$

Theorem 7 *Let $\kappa_1, \kappa_2 > 0$ such that $\Psi(\kappa_2,\kappa_1) > 0$ and $h : K = [\kappa_1, \kappa_1+\Psi(\kappa_2,\kappa_1)] \to (0,\infty)$ be an β-differentiable function on $(\kappa_1, \kappa_1+\Psi(\kappa_2,\kappa_1))$ for $\beta \in (0,1]$ such that $D_\beta(h) \in L_\beta([\kappa_1, \kappa_1+\Psi(\kappa_2,\kappa_1)])$. If $|h'^q$ is preinvex for $q > 1$ and $q^{-1}+p^{-1} = 1$, then we have the following inequality:*

$$\left| h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right) - \frac{\beta}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta - \kappa_1^\beta} \int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)} h(x)d_\beta x \right|$$
$$\leq \frac{\Psi(\kappa_2,\kappa_1)}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta - \kappa_1^\beta} \Big[(\mathfrak{A}_1(\beta))^{1-\frac{1}{q}} \left(\mathfrak{A}_2(\beta)|h'(\kappa_1)|^q + \mathfrak{A}_3(\beta)|h'(\kappa_2)|^q\right)^{\frac{1}{q}}$$
$$+ (\mathfrak{B}_1(\beta))^{1-\frac{1}{q}} \left(\mathfrak{B}_2(\beta)|h'(\kappa_1)|^q + \mathfrak{B}_3(\beta)|h'(\kappa_2)|^q\right)^{\frac{1}{q}}, \tag{9}$$

where

$$\mathfrak{A}_1(\beta) = \frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+1}(\beta+1)\Psi(\kappa_2,\kappa_1)} - \frac{\kappa_1^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)} - \frac{1}{2}\kappa_1^\beta,$$

$$\mathfrak{B}_1(\beta) = \frac{1}{2}(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta - \left[\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)} - \frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+1}(\beta+1)\Psi(\kappa_2,\kappa_1)} \right],$$

$$\mathfrak{A}_2(\beta) = \frac{\big(2\kappa_1+\Psi(\kappa_2,\kappa_1)\big)^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)} \left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1) + \big(2\kappa_1+\Psi(\kappa_2,\kappa_1)\big)}{(\beta+2)\Psi(\kappa_2,\kappa_1)} \right]$$
$$- \frac{\kappa_1^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)} \left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)+\kappa_1}{(\beta+2)\Psi(\kappa_2,\kappa_1)} \right] - \frac{3}{8}\kappa_1^\beta,$$

$$\mathfrak{B}_2(\beta) = \frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta}{8} + \frac{\big(2\kappa_1+\Psi(\kappa_2,\kappa_1)\big)^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)} \left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1) + (2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)\Psi(\kappa_2,\kappa_1)} \right]$$
$$- \frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+2}}{(\beta+1)(\Psi(\kappa_2,\kappa_1))^2(\beta+2)},$$

$$\mathfrak{A}_3(\beta) = \frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)} \left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1) - (2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)\Psi(\kappa_2,\kappa_1)} \right]$$
$$+ \frac{\kappa_1^{\beta+2}}{(\beta+1)(\Psi(\kappa_2,\kappa_1))^2(\beta+2)} - \frac{\kappa_1^\beta}{8},$$

$$\mathfrak{B}_3(\beta) = \frac{3}{8}(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta - \frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)} \left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1) - (\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)\Psi(\kappa_2,\kappa_1)} \right]$$

$$+\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)-(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)\Psi(\kappa_2,\kappa_1)}\right].$$

Proof. From Lemma 1, using the property of the modulus and preinvexity of $|h'|$, we have

$$\begin{aligned}
&\left|h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right)-\frac{\beta}{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}}\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)}h(x)d_{\beta}x\right|\\
\leq\ &\frac{\Psi(\kappa_2,\kappa_1)}{((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta})}\left[\int_0^{\frac{1}{2}}\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}\right)|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))|ds\right.\\
&\left.+\int_{\frac{1}{2}}^{1}\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}\right)|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))|ds\right].
\end{aligned}$$

Now by the power-mean inequality

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}\right)|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))|ds\\
\leq\ &\left(\int_0^{1}\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}\right)ds\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^{1}\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}\right)|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))|^q ds\right)^{\frac{1}{q}}
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}\right)|h\prime(\kappa_1+s\Psi(\kappa_2,\kappa_1))|ds\\
\leq\ &\left(\int_0^{\frac{1}{2}}\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}\right)ds\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^{\frac{1}{2}}\left((\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta}-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}\right)|h'(\kappa_1+s\Psi(\kappa_2,\kappa_1))|^q ds\right)^{\frac{1}{q}}.
\end{aligned}$$

Now by the preinvexity of $|h'^q$ from above, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}\right)[(1-s)|h'(\kappa_1)|^q+s|h'(\kappa_2)|^q]ds\\
=&|h'(\kappa_1)|^q\int_0^{\frac{1}{2}}\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}\right)(1-s)ds\\
&+|h'(\kappa_2)|^q\int_0^{\frac{1}{2}}\left((\kappa_1+s\Psi(\kappa_2,\kappa_1))^{\beta}-\kappa_1^{\beta}\right)sds\\
=&|h'(\kappa_1)|^q\left(\frac{\left(2\kappa_1+\Psi(\kappa_2,\kappa_1)\right)^{\beta+1}}{2^{\beta+2}(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)+\left(2\kappa_1+\Psi(\kappa_2,\kappa_1)\right)}{(\beta+2)(\Psi(\kappa_2,\kappa_1))}\right]\right.\\
&\left.-\frac{\kappa_1^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)+\kappa_1}{(\beta+2)(\Psi(\kappa_2,\kappa_1))}\right]-\frac{3}{8}\kappa_1^{\beta}\right)\\
&+|h'(\kappa_2)|^q\left(\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+2}(\beta+1)(\Psi(\kappa_2,\kappa_1))}\left[\frac{(\beta+2)\Psi(\kappa_2,\kappa_1)-(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)(\Psi(\kappa_2,\kappa_1))}\right]\right.
\end{aligned}$$

$$+\frac{\kappa_1^{\beta+2}}{(\beta+1)(\Psi(\kappa_2,\kappa_1))^2(\beta+2)}-\frac{\kappa_1^{\beta}}{8}\Bigg)$$

and

$$\int_{\frac{1}{2}}^{1}\big((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta\big)[(1-s)|h'(\kappa_1)|^q+s|h'(\kappa_2)|^q]ds$$
$$=|h'(\kappa_1)|^q\int_{\frac{1}{2}}^{1}\big((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta\big)(1-s)ds$$
$$+|h'(\kappa_2)|^q\int_{\frac{1}{2}}^{1}\big((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta\big)sds$$
$$=|h'(\kappa_1)|^q\Bigg(\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta}{8}+\frac{\big(2\kappa_1+\Psi(\kappa_2,\kappa_1)\big)^{\beta+1}}{2^{\beta+2}(\beta+1)(\Psi(\kappa_2,\kappa_1))}$$
$$\times\left[\frac{(\beta+2)((\Psi(\kappa_2,\kappa_1)))+(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)(\Psi(\kappa_2,\kappa_1))}\right]-\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+2}}{(\beta+1)((\Psi(\kappa_2,\kappa_1)))^2(\beta+2)}\Bigg)$$
$$+|h'(\kappa_2|^q\Bigg(\frac{3}{8}(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{(\beta+1)((\Psi(\kappa_2,\kappa_1)))}$$
$$\times\left[\frac{(\beta+2)((\Psi(\kappa_2,\kappa_1)))-(\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)((\Psi(\kappa_2,\kappa_1)))}\right]+\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+2}(\beta+1)((\Psi(\kappa_2,\kappa_1)))}$$
$$\times\left[\frac{(\beta+2)((\Psi(\kappa_2,\kappa_1)))-(2\kappa_1+\Psi(\kappa_2,\kappa_1))}{(\beta+2)((\Psi(\kappa_2,\kappa_1)))}\right]\Bigg),$$

where

$$\mathfrak{A}_1(\beta)=\int_0^{\frac{1}{2}}\big((\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta-\kappa_1^\beta\big)ds=\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+1}(\beta+1)\Psi(\kappa_2,\kappa_1)}-\frac{\kappa_1^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)}-\frac{1}{2}\kappa_1^\beta,$$

$$\mathfrak{B}_2(\beta)=\int_{\frac{1}{2}}^{1}\big((\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta-(\kappa_1+s\Psi(\kappa_2,\kappa_1))^\beta\big)ds=\frac{1}{2}(\kappa_1+\Psi(\kappa_2,\kappa_1))^\beta$$
$$-\left[\frac{(\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{(\beta+1)\Psi(\kappa_2,\kappa_1)}-\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^{\beta+1}}{2^{\beta+1}(\beta+1)\Psi(\kappa_2,\kappa_1)}\right].$$

Hence, we have the result in (9). ■

Corollary 3 *Let $\beta=1$. Then Theorem 7 leads to*

$$\left|h\left(\frac{2\kappa_1+\Psi(\kappa_2,\kappa_1)}{2}\right)-\frac{1}{\Psi(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\Psi(\kappa_2,\kappa_1)}h(x)dx\right|$$
$$\leq\Big[(\mathfrak{A}_1(1))^{1-\frac{1}{q}}\left(\mathfrak{A}_2(1)|h'(\kappa_1)|^q+\mathfrak{A}_3(1)|h'(\kappa_2)|^q\right)^{\frac{1}{q}}$$
$$+(\mathfrak{B}_1(\beta))^{1-\frac{1}{q}}\left(\mathfrak{B}_2(1)|h'(\kappa_1)|^q+\mathfrak{B}_3(1)|h'(\kappa_2)|^q\right)^{\frac{1}{q}},$$

where

$$\mathfrak{A}_1(1)=\frac{(2\kappa_1+\Psi(\kappa_2,\kappa_1))^2}{8\Psi(\kappa_2,\kappa_1)}-\frac{\kappa_1^2}{2\Psi(\kappa_2,\kappa_1)}-\frac{1}{2}\kappa_1^\beta,$$

$$\mathfrak{B}_1(1) = \frac{1}{2}(\kappa_1 + \Psi(\kappa_2, \kappa_1)) - \left[\frac{(\kappa_1 + \Psi(\kappa_2, \kappa_1))^2}{2\Psi(\kappa_2, \kappa_1)} - \frac{(2\kappa_1 + \Psi(\kappa_2, \kappa_1))^2}{8\Psi(\kappa_2, \kappa_1)}\right],$$

$$\mathfrak{A}_2(1) = \frac{\left(2\kappa_1 + \Psi(\kappa_2, \kappa_1)\right)^2}{16\Psi(\kappa_2, \kappa_1)}\left[\frac{3\Psi(\kappa_2, \kappa_1) + \left(2\kappa_1 + \Psi(\kappa_2, \kappa_1)\right)}{3\Psi(\kappa_2, \kappa_1)}\right] - \frac{\kappa_1^2}{2\Psi(\kappa_2, \kappa_1)}\left[\frac{3\Psi(\kappa_2, \kappa_1) + \kappa_1}{3\Psi(\kappa_2, \kappa_1)}\right] - \frac{3}{8}\kappa_1,$$

$$\mathfrak{B}_2(1) = \frac{(\kappa_1 + \Psi(\kappa_2, \kappa_1))}{8} + \frac{\left(2\kappa_1 + \Psi(\kappa_2, \kappa_1)\right)^2}{16\Psi(\kappa_2, \kappa_1)}\left[\frac{3\Psi(\kappa_2, \kappa_1) + (2\kappa_1 + \Psi(\kappa_2, \kappa_1))}{3\Psi(\kappa_2, \kappa_1)}\right] - \frac{(\kappa_1 + \Psi(\kappa_2, \kappa_1))^3}{6(\Psi(\kappa_2, \kappa_1))^2},$$

$$\mathfrak{A}_3(1) = \frac{(2\kappa_1 + \Psi(\kappa_2, \kappa_1))^2}{16\Psi(\kappa_2, \kappa_1)}\left[\frac{3\Psi(\kappa_2, \kappa_1) - (2\kappa_1 + \Psi(\kappa_2, \kappa_1))}{3\Psi(\kappa_2, \kappa_1)}\right] + \frac{\kappa_1^3}{6(\Psi(\kappa_2, \kappa_1))^2} - \frac{\kappa_1}{8},$$

$$\begin{aligned}\mathfrak{B}_3(1) &= \frac{3}{8}(\kappa_1 + \Psi(\kappa_2, \kappa_1)) - \frac{(\kappa_1 + \Psi(\kappa_2, \kappa_1))^2}{2\Psi(\kappa_2, \kappa_1)}\left[\frac{3\Psi(\kappa_2, \kappa_1) - (\kappa_1 + \Psi(\kappa_2, \kappa_1))}{3\Psi(\kappa_2, \kappa_1)}\right] \\ &\quad + \frac{(2\kappa_1 + \Psi(\kappa_2, \kappa_1))^2}{16\Psi(\kappa_2, \kappa_1)}\left[\frac{3\Psi(\kappa_2, \kappa_1) - (2\kappa_1 + \Psi(\kappa_2, \kappa_1))}{3\Psi(\kappa_2, \kappa_1)}\right].\end{aligned}$$

Acknowledgment. The authors would like to thank the referee for his/her suggestions that improved the paper.

References

[1] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math., 279(2015), 57–66.

[2] T. Abdeljawad, R. P. Agarwal, J. Alzabut, F. Jarad and A. Özbekler, Lyapunov-type inequalties for mixed non-linear forced differential equations within conformalbe derivatives, J. Inequal. Appl., 2018(2018), 1–17.

[3] T. Abdeljawad, J. Alzabut and F. Jarad, A generalized Lyapunov-type inequality in the frame of conformable derivatives, Adv. Difference Equ., 2017(2017), 1–10.

[4] M. Adil Khan, S. Begum, Y. Khurshid and Y.-M. Chu, Ostrowski type inequalities involving conformable fractional integrals, J. Inequal. Appl., 2018(2018), 1–14.

[5] A. Iqbal, M. Adil Khan, M. Suleman and Y.-M. Chu, The right Riemann-Liouville fractional Hermite-Hadamard type inequalities derived from Green's function, AIP Advances, 10(2020), 1–9.

[6] M. Adil Khan, Y.-M. Chu, A. Kashuri, R. Liko and G. Ali, Conformable fractional integrals version of Hermite-Hadamard inequalities and their generalizations, J. Funct. Spaces, 2018(2018), 1–9.

[7] M. Adil Khan, Y.-M. Chu, T. U. Khan and J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 15(2017), 1414–1430.

[8] M. Adil Khan, Y. Khurshid and T. Ali, Hermite-Hadamard inequality for fractional integrals Via η-convex functions, Acta Math. Univ. Comenian., 86(2017), 153–164.

[9] M. Adil Khan, Y. Khurshid, T. Ali and N. Rehman, Inequalities for three times differentiable functions, Punjab Univ. J. Math., 48(2016), 35–48.

[10] D. R. Anderson, Taylor's Formula and Integral Inequalities for Conformable Fractional Derivatives, Contributions in Mathematics and Engineering, Springer, Cham, 2016.

[11] A. O. Akdemir, A. Ekinci and E. Set, Conformable fractional integrals and related new integral inequalities, J. Nonlinear convex Anal., 18(2017), 661–674.

[12] M. Al-Refai and T. Abdeljawad, Fundamental results of conformable Sturm-Liouville engenvalue problems, Complexity, 2017(2017), 1–7.

[13] T. Antczak, Mean value in invexity analysis, Nonlinear Anal., 60(2005), 1473–1484.

[14] M. Awan, M. A. Noor, M. V. Mihai and K. I. Noor, Inequalities via harmonic convex functions: conformable fractional calculus approach, J. Math. Inequal., 12(2018), 143–153.

[15] Y.-M. Chu, M. Adil Khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, J. Inequal. Appl., 2017(2017), 1–12.

[16] M. Adil Khan, T. Ali and T. U. Khan, Hermite-Hadamard type inequalities with applications, Fasciculi Mathematici, 59(2017), 57–74.

[17] A. Iqbal, M. Adil Khan, Sana Ullah and Y.-M. Chu, Some new Hermite-Hadamard type inequalities associated with conformable fractional integrals and their applications, J. Funct. Spaces, 2020(2020), 1–18.

[18] M. Adil Khan, N. Mohammad, Eze R. Nwaeze and Y.-M. Chu, Quantum Hermite-Hadamard inequality by means of a Green function, Adv. Difference Equ., 2020(2020), 1–20.

[19] W. S. Chung, Fractional Newton mechanics with conformable fractional derivative, J. Comput. Appl. Math., 290(2015), 150–158.

[20] S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl., 167(1992), 49–56.

[21] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 11(1998), 91–95.

[22] S. S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense, Demonstratio Math., 32(1999), 687–696.

[23] S. S. Dragomir and A. McAndrew, Refinements of the Hermite-Hadamard inequality for convex functions, J. Inequal. Pure Appl. Math., 6(2005), 1–6.

[24] L. Fejér, Uberdie Fourierreihen II, Math. Natur. Anz Ungar. Akad. Wiss., 24(1906), 369–390.

[25] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d̀une fonction considérée par Riemann, J. Math. Pures Appl., 58(1893), 171–215.

[26] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl., 80(1981), 545–550.

[27] M. S. Hashemi, Invariant subspaces admitted by fractional differential equations with conformable derivatives, Chaos Solitons Fractals, 107(2018), 161–169.

[28] İ. İşcan, Hermite-Hadamard's inequalities for Preinvex function via fractional integrals and related fractional inequalities, Amer. J. Math. Anal., 1(2013), 33–38.

[29] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., 264(2014), 65–70.

[30] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl., 189(1995), 901–908.

[31] M. A. Noor, On Hadamard integral inequalities invoving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), 1–6.

[32] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory, 2(2007), 126–131.

[33] S. Khan, M. Adil Khan and Y.-M. Chu, New converses of Jensen inequality via Green functions with applications, RACSAM, 114(2020) 1–14.

[34] M. Z. Sarikaya, H. Yaldiz and H. Budak, On weighted Iyengar-type inequalities for conformable fractional integrals, Math. Sci., 11(2017), 327–331.

[35] E. Set, İ. Mumcu and M. E. Özdemir, On the more general Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals, Topol. Algebra Appl., 5(2017), 67–73.

[36] Y.-Q. Song, M. Adil Khan, S. Zaheer Ullah and Y.-M. Chu, Integral inequalities involving strongly convex functions, J. Funct. Spaces, 2018(2018), 1–8.

[37] M. Adil Khan, S. Khan and Y.-M. Chu, A new bound for the Jensen gap with applications in information theory, IEEE Access, 20(2020), 98001–98008.

[38] M. Adil Khan, J. Pečarić and Y.-M. Chu, Refinements of Jensen's and McShane's inequalities with applications, AIMS Mathematics, 5(2020), 4931–4945.

[39] S. Khan, M. Adil Khan, S. I. Butt and Y.-M. Chu, A new bound for the Jensen gap pertaining twice differentiable functions with applications, Adv. Difference Equ., 2020(2020), 1–11.

[40] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization, J. Math. Anal. Appl., 136(1998), 29–38.

[41] Z.-H. Yang and Y.-M. Chu, A monotonicity properties involving the generalized elliptic integral of the first kind, Math. Inequal. Appl., 20(2017), 729–735.

[42] X.-M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl., 256(2001), 229–241.

[43] Z.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017(2017), 1–17.

[44] Z.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 462(2018), 1714–1726.

[45] Z.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the error function, Math. Inequal. Appl., 21(2018), 469–479.

[46] Z.-H. Yang, W. Zhang and Y.-M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, 20(2017), 1107–1120.

[47] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, J. Inequal. Appl., 2010(2010), 1–11.

[48] A. Barani, A. G. Ghazanfari and S. S Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, J. Inequal. Appl., 2012(2012), 1–9.

[49] Y. Khurshid, M. Adil Khan, Y.-M. Chu and Z. A. Khan, Hermite-Hadamard-Fejér inequalities for conformable fractional integrals via preinvex functions, J. Funct. Spaces, 2019(2019), 1–9.

[50] Y. Khurshid, M. Adil Khan and Y.-M. Chu, Conformable integral inequalities of the Hermite-Hadamard type in terms of GG- and GA-convexities, J. Funct. Spaces, 2019(2019), 1–8.

[51] Y. Khurshid, M. Adil Khan and Y.-M. Chu, Ostrowski type inequalities involving conformable integrals via preinvex functions, AIP Advances, 10(2020), 1–9.

[52] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comput., 147(2004), 137–146.

[53] M. Z. Sarikaya, A. Akkurt, H. Budak, M. E. Yildirim and H. Yildirim, Hermite-Hadamard inequalities for conformable fractional integrals, An International Journal of Optimization and Control: Theories & Applications, 9(2019), 49–59.

[54] Y. Khurshid, M. Adil Khan and Y.-M. Chu, Conformable fractional integral inequalities for GG- and GA-convex functions, AIMS Mathematics, 5(2020), 5012–5030.

[55] Y. Khurshid, M. Adil Khan and Y.-M. Chu, Conformable integral version of Hermite-Hadamard-Fejér inequalities via η-convex functions, AIMS Mathematics, 5(2020), 5106–5120.