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An Application Of Generalized Bessel Functions On General Class Of Analytic Functions With Negative Coefficients*

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Abstract

A new general class $\mathbb{S}^k(t_3,t_2,t_1,t_0,\mu)$ of analytic functions with negative coefficients is introduced. The main object of this paper is to find necessary and sufficient conditions for generalized Bessel functions of first kind $z(2-u_p(z))$ to be in the class $\mathbb{S}^0(t_3,t_2,t_1,t_0,\mu)$. Furthermore, we give necessary and sufficient conditions for $\mathcal{I}(m,c)f$ to be in $\mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$ provided that the function f is in the class $\mathcal{R}^\tau(A,B)$. Finally, we give conditions for the integral operator $\mathcal{G}(m,c,z)=\int_0^z(2-u_p(t))dt$ to be in the class $\mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$. A number of known or new results are shown to follow upon specializing the parameters involved in our main results.

1 Introduction

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|\,z^n,\qquad z\in\mathbb{U}. \tag{2}$$

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A,B)$,$\tau\in\mathbb{C}\backslash\{0\}$, $-1\le B<A\le 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1,\quad z\in\mathbb{U}.$$

This class was introduced by Dixit and Pal [15].

Let $p(n)=t_3n^3+t_2n^2+t_1n+t_0$ be a polynomial of degree the most three, with real coefficients t_3,t_2,t_1 and t_0. Then a function f of the form (2) is in $\mathbb{S}^k(t_3,t_2,t_1,t_0,\mu)$, if and only if it satisfies

$$\sum_{n=2}^{\infty}n^kp(n)\,|a_n|\le\mu\qquad(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \mu>0). \tag{3}$$

Remark 1 *By suitably specializing the real constants t_3,t_2,t_1,t_0,k and μ, the class $\mathbb{S}^k(t_3,t_2,t_1,t_0,\mu)$ includes as its special cases various classes of analytic functions with negative coefficients that were considered in several works. As for illustrations, we present the following examples.*

1. $\mathbb{S}^k(0,\lambda^2,1-\alpha\lambda-\lambda,\alpha(\lambda-1),1-\alpha)\equiv\mathcal{P}(\lambda,\alpha,k)$ (Aouf and Srivastava [5]);

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2. $\mathbb{S}^k(0, 1-\beta\lambda, \beta\lambda-1, 0, 1-\beta) \equiv \mathcal{S}^*_{s,k}\mathcal{T}(\alpha,\beta)$ (Aouf et al. [7]);
3. $\mathbb{S}^1(0, 0, 1+\alpha, -(\alpha+\beta), 1-\beta) \equiv \mathcal{UCT}(\alpha,\beta)$ (Bharati [12]);
4. $\mathbb{S}^0(0, 0, 1, -(1+\alpha), \alpha) \equiv \mathcal{PT}(\alpha)$ (Bharati [12]);
5. $\mathbb{S}^1(0, 0, 1, -(1+\alpha), \alpha) \equiv \mathcal{CPT}(\alpha)$ (Bharati [12]);
6. $\mathbb{S}^0(0, 0, 2, -(\cos\alpha+\beta), \cos\alpha-\beta) \equiv \mathcal{SP}_P\mathcal{T}(\alpha,\beta)$ (Selvaraj and Geetha [33]);
7. $\mathbb{S}^1(0, 0, 2, -(\cos\alpha+\beta), \cos\alpha-\beta) \equiv \mathcal{UCSPT}(\alpha,\beta)$ (Selvaraj and Geetha [33]);
8. $\mathbb{S}^0(0, 0, 1-\lambda\alpha, \alpha(\lambda-1), 1-\alpha) \equiv \mathcal{T}(\lambda,\alpha)$ (Altintaş and Owa [3]);
9. $\mathbb{S}^1(0, 0, 1-\lambda\alpha, \alpha(\lambda-1), 1-\alpha) \equiv \mathcal{C}(\lambda,\alpha)$ (Altintaş and Owa [3]);
10. $\mathbb{S}^0(0, 0, (1+\beta)-\lambda(\alpha+\beta), (\alpha+\beta)(\lambda-1), 1-\alpha) \equiv \mathcal{TS}_p(\lambda,\alpha,\beta)$ (Aouf et al. [7]);
11. $\mathbb{S}^1(0, 0, (1+\beta)-\lambda(\alpha+\beta), (\alpha+\beta)(\lambda-1), 1-\alpha) \equiv \mathcal{UST}(\lambda,\alpha,\beta)$ (Murugusundaramoorthy and Magesh [23]).
12. $\mathbb{S}^0(0, \lambda, 1-\lambda-\alpha\lambda, \alpha(\lambda-1), 1-\alpha) \equiv \mathcal{P}^*_\lambda(\alpha)$(Altintaş et al. [4]);
13. $\mathbb{S}^1(0, \lambda, 1-\lambda-\alpha\lambda, \alpha(\lambda-1) \equiv \mathcal{Q}^*_\lambda(\alpha)$(Altintaş et al. [4]);
14. $\mathbb{S}^0(1, -\alpha, 0, 0, 1-\alpha) \equiv \mathcal{M}^*(\alpha)$ (Murugusundaramoorthy et al. [26]);
15. $\mathbb{S}^0(0, 0, 1+\beta, -\lambda(\gamma+\beta), 1-\gamma) \equiv \mathcal{P}^*_\lambda(\gamma,\beta)$ (Murugusundaramoorthy et al. [27]);
16. $\mathbb{S}^1(0, 0, 1+\beta, -\lambda(\gamma+\beta), 1-\gamma) \equiv \mathcal{Q}^*_\lambda(\gamma,\beta)$(Murugusundaramoorthy et al. [27]);
17. $\mathbb{S}^0(0, \lambda, 1-\lambda, -\alpha, 1-\alpha) \equiv \mathcal{G}^*(\lambda,\alpha)$ (Murugusundaramoorthy et al. [25]);
18. $\mathbb{S}^1(0, \lambda, 1-\lambda, -\alpha, 1-\alpha) \equiv \mathcal{K}^*(\lambda,\alpha)$ (Murugusundaramoorthy et al. [25]);
19. $\mathbb{S}^0(0, \lambda(1+\beta), 1+\beta-\lambda(2\beta+\alpha+1), (\alpha+\beta)(\lambda-1), 1-\alpha) \equiv \mathcal{TS}(\lambda,\alpha,\beta)$ (Aouf et al. [6]).
20. $\mathbb{S}^0(0, 0, 1+\beta, -1+\beta(1-2\alpha), 2\alpha(1-\beta)) \equiv \mathcal{S}^*(\alpha,\beta)$ (Gupta and Jain [21]);
21. $\mathbb{S}^1(0, 0, 1+\beta, -1+\beta(1-2\alpha), 2\alpha(1-\beta)) \equiv \mathcal{C}^*(\alpha,\beta)$ (Gupta and Jain [21]);
22. $\mathbb{S}^0(0, 0, \alpha, 1-\alpha, 1-\beta) \equiv \mathcal{T}(\alpha,\beta)$ (Altintaş [1]).

Further, the class $\mathbb{S}^k(t_3, t_2, t_1, t_0, \mu)$ leads to various classes of analytic functions with negative coefficients introduced and studied by several authors (see, for example, [2, 19, 30, 32, 35, 38, 39, 40]).

Let $\mathcal{P}(C, D)$ denote the class of analytic function in $\mathbb{U}$ which are of the form $\frac{1+Cw(z)}{1+Dw(z)}$, where $-1 < C < D \leq 1$ and $w(z)$ is analytic function with $w(0) = 0$, $|w(z)| < 1$ in $\mathbb{U}$. Define

$$\mathcal{S}^*(C, D) = \{f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \in \mathcal{P}(C, D)\}$$

and

$$\mathcal{K}(C, D) = \{f \in \mathcal{A}: zf'(z) \in \mathcal{S}^*(C, D)\}.$$

Goel and Sohi [20](see also, [34]) gave the following necessary and sufficient conditions for functions f of the form (2) to be in the classes $\mathcal{T}^*(C, D) = \mathcal{S}^*(C, D) \cap \mathcal{T}$ and $\mathcal{C}(C, D) = \mathcal{K}(C, D) \cap \mathcal{T}$

$$\sum_{n=2}^{\infty} (n(1+D) - (1+C)) |a_n| \leq D - C,$$

and

$$\sum_{n=2}^{\infty} n(n(1+D)-(1+C))\,|a_n| \leq D-C,$$

respectively.

We observe that

$$\mathbb{S}^0(0,0,1+D,-(1+C),D-C) \equiv \mathcal{T}^*(C,D)$$

and

$$\mathbb{S}^1(0,0,1+D,-(1+C),D-C) \equiv \mathcal{C}(C,D).$$

For $0 \leq \alpha < 1$ and $\gamma, \beta \geq 0$, let $\mathcal{W}(\alpha,\gamma,\beta)$ denote the class of functions f of the form (2) such that

$$\mathrm{Re}\{(1-\gamma+2\beta)\frac{f(z)}{z} + (\gamma-2\beta)f'(z) + \beta z f''(z)\} > \alpha, \quad (z \in \mathbb{U}).$$

For more details about this class, see [31]. We can easily prove that a function f of the form (1) is in the class $\mathcal{W}(\alpha,\gamma,\beta)$ if

$$\sum_{n=2}^{\infty}[n(n-1)\beta + (\gamma-2\beta)n + (1-\gamma+2\beta)]\,|a_n| \leq 1-\alpha, \tag{4}$$

and a function f of the form (2) is in the class $\mathcal{WT}(\alpha,\gamma,\beta) = \mathcal{W}(\alpha,\gamma,\beta) \cap \mathcal{T}$ if and only if the conditions (4) is satisfied. We note that

$$\mathbb{S}^0(0,\beta,\gamma-3\beta,1-\gamma+2\beta,1-\alpha) = \mathcal{WT}(\alpha,\gamma,\beta).$$

The generalized Bessel function w_p (see, [8])is defined as a particular solution of the linear differential equation

$$zw''(z) + bzw'(z) + [cz^2 - p^2 + (1-b)p]w(z) = 0,$$

where $b, p, c \in \mathbb{C}$. The analytic function w_p has the form

$$w_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (c)^n}{n!\Gamma(p+n+\frac{b+1}{2})} \cdot \left(\frac{z}{2}\right)^{2n+p}, \qquad z \in \mathbb{C}.$$

Now, the generalized and normalized Bessel function u_p is defined with the transformation

$$u_p(z) = 2^p\Gamma(p+n+\frac{b+1}{2})z^{-p/2}w_p(z^{1/2}) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m)_n n!} z^n,$$

where $m = p+(b+1)/2 \neq 0,-1,-2,\ldots$ and $(a)_n$ is the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for $a \neq 0,-1,-2,\ldots$ by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n=0, \\ a(a+1)(a+2)\ldots(a+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

The function u_p is analytic on $\mathbb{C}$ and satisfies the second-order linear differential equation

$$4z^2u''(z) + 2(2p+b+1)zu'(z) + czu(z) = 0.$$

Using the Hadamard product, we now considered a linear operator $\mathcal{I}(m,c) : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{I}(m,c)f = zu_p(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!} a_n z^n,$$

where $*$ denote the convolution or Hadamard product of two series.

The study of the generalized Bessel function is a recent interesting topic in geometric function theory. We refer, in this connection, to the works of [8, 9, 10, 11, 17] and others.

Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see, for example, [14, 22, 36, 37])) and the generalized Bessel functions (see, for example, [13, 16, 24, 26, 29])) in this paper we determine necessary and sufficient conditions for $z(2-u_p(z))$ to be in the class class $\mathbb{S}^0(t_3,t_2,t_1,t_0,\mu)$. Furthermore, we give necessary and sufficient conditions for $\mathcal{I}(m,c)f$ to be in $\mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$ provided that the function f is in the class $\mathcal{R}^\tau(A,B)$. Finally, we give conditions for the integral operator $\mathcal{G}(m,c,z)=\int_0^z(2-u_p(t))dt$ to be in the class $\mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$.

2 The Necessary and Sufficient Conditions

To establish our main results, we shall require the following lemmas.

Lemma 1 ([10]) *If $b,p,c\in\mathbb{C}$ and $m\neq 0,-1,-2,\dots,$ then the function u_p satisfies the recursive relation*

$$\begin{aligned} u_p'(z) &= \frac{(-c/4)}{m}u_{p+1}(z),\ u_p''(z)=\frac{(-c/4)^2}{m(m+1)}u_{p+2}(z),\ u_p'''(z)=\frac{(-c/4)^3}{m(m+1)(m+2)}u_{p+3}(z),\\ u_p^{(4)}(z) &= \frac{(-c/4)^4}{m(m+1)(m+2)(m+3)}u_{p+4}(z),\end{aligned}$$

for all $z\in\mathbb{C}$.

Lemma 2 ([15]) *If $f\in\mathcal{R}^\tau(A,B)$ is of the form, then*

$$|a_n|\le(A-B)\frac{|\tau|}{n},\qquad n\in\mathbb{N}-\{1\}.$$

The result is sharp.

Unless otherwise mentioned, we shall assume in this paper that $c<0$, $m>0(m\neq 0,-1,-2,\ldots)$, *and* $\mu>0$. First we obtain the necessary and sufficient condition for $z(2-u_p(z))$ to be in the class $\mathbb{S}^0(t_3,t_2,t_1,t_0,\mu)$.

Theorem 1 *$z(2-u_p(z))\in\mathbb{S}^0(t_3,t_2,t_1,t_0,\mu)$ if and only if*

$$t_3u_p'''(1)+(t_2+6t_3)u_p''(1)+(t_1+3t_2+7t_3)u_p'(1)+(t_0+t_1+t_2+t_3)(u_p(1)-1)\le\mu. \tag{5}$$

Proof. Since

$$z(2-u_p(z))=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}z^n, \tag{6}$$

according to (3), we must show that

$$\sum_{n=2}^{\infty}(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\le\mu. \tag{7}$$

Writing

$$n=(n-1)+1, \tag{8}$$

$$n^2=(n-1)(n-2)+3(n-1)+1, \tag{9}$$

and

$$n^3=(n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1, \tag{10}$$

we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\\
=\;& t_3\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\\
&+(t_2+6t_3)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\\
&+(t_1+3t_2+7t_3)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}+(t_1+t_2+t_3+t_0)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\\
=\;& t_3\sum_{n=4}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-4)!}+(t_2+6t_3)\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-3)!}+(t_1+3t_2+7t_3)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-2)!}\\
&+(t_1+t_2+t_3+t_0)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\\
=\;& t_3\sum_{n=0}^{\infty}\frac{(-c/4)^{n+3}}{(m)_{n+3}n!}+(t_2+6t_3)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+2}}{(m)_{n+2}n!}+(t_1+3t_2+7t_3)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}n!}\\
&+(t_1+t_2+t_3+t_0)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}(n+1)!}\\
=\;& t_3\frac{(-c/4)^3}{m(m+1)(m+2)}\sum_{n=0}^{\infty}\frac{(-c/4)^{n}}{(m+3)_n n!}+(t_2+6t_3)\frac{(-c/4)^2}{m(m+1)}\sum_{n=0}^{\infty}\frac{(-c/4)^{n}}{(m+2)_n n!}\\
&+(t_1+3t_2+7t_3)\frac{(-c/4)}{m}\sum_{n=0}^{\infty}\frac{(-c/4)^{n}}{(m+1)_n n!}+(t_1+t_2+t_3+t_0)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}(n+1)!}\\
=\;& t_3\frac{(-c/4)^3}{m(m+1)(m+2)}u_{p+3}(1)+(t_2+6t_3)\frac{(-c/4)^2}{m(m+1)}u_{p+2}(1)\\
&+(t_1+3t_2+7t_3)\frac{(-c/4)}{m}u_{p+1}(1)+(t_1+t_2+t_3+t_0)(u_p(1)-1)\\
=\;& t_3u_p'''(1)+(t_2+6t_3)u_p''(1)+(t_1+3t_2+7t_3)u_p'(1)+(t_1+t_2+t_3+t_0)(u_p(1)-1). \qquad (11)
\end{aligned}$$

But this last expression is bounded above by μ if (5) holds. ■

Theorem 2 *$z(2-u_p(z))\in\mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$ if and only if*

$$\begin{aligned}
&t_3u_p^{(4)}(1)+(10t_3+t_2)u_p'''(1)+(25t_3+6t_2+t_1)u_p''(1)+(15t_3+7t_2+3t_1+t_0)u_p'(1)\\
&+(t_3+t_2+t_1+t_0)(u_p(1)-1)\le\mu. \qquad (12)
\end{aligned}$$

Proof. In view of (3), we must show that

$$\sum_{n=2}^{\infty}n(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\le\mu,$$

or, equivalently

$$\sum_{n=2}^{\infty}(t_3n^4+t_2n^3+t_1n^2+t_0n)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\le\mu.$$

Making use of (8)–(9) and writing

$$n^4=(n-1)(n-2)(n-3)(n-4)+10(n-1)(n-2)(n-3)+25(n-1)(n-2)+15(n-1)+1,$$

we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}(t_3n^4+t_2n^3+t_1n^2+t_0n)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\\
=\;& t_3\sum_{n=5}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-5)!}+(10t_3+t_2)\sum_{n=4}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-4)!}+(25t_3+6t_2+t_1)\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-3)!}\\
&+(15t_3+7t_2+3t_1+t_0)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-2)!}+(t_3+t_2+t_1+t_0)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\\
=\;& t_3\frac{(-c/4)^4}{m(m+1)(m+2)(m+3)}\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(m+4)_n n!}+(10t_3+t_2)\frac{(-c/4)^3}{m(m+1)(m+2)}\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(m+3)_n n!}\\
&+(25t_3+6t_2+t_1)\frac{(-c/4)^2}{m(m+1)}\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(m+2)_n n!}+(15t_3+7t_2+3t_1+t_0)\frac{(-c/4)}{m}\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(m+1)_n n!}\\
&(t_3+t_2+t_1+t_0)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}(n+1)!}\\
=\;& t_3u_p^{(4)}(1)+(10t_3+t_2)u_p'''(1)+(25t_3+6t_2+t_1)u_p''(1)+(15t_3+7t_2+3t_1+t_0)u_p'(1)\\
&+(t_3+t_2+t_1+t_0)(u_p(1)-1).
\end{aligned}$$

But this last expression is bounded above by μ if (12) holds. ■

Theorem 3 *$z(2-u_p(z))\in\mathbb{S}^0(t_3,t_2,t_1,t_0,\mu)$ if and only if*

$$e^{(\frac{-c}{4m})}[t_3(\frac{-c}{4m})^3+(t_2+6t_3)(\frac{-c}{4m})^2+(t_1+3t_2+7t_3)(\frac{-c}{4m})+(t_1+t_2+t_3+t_0)(1-e^{(\frac{c}{4m})})]\leq\mu. \tag{13}$$

Proof. We note that $(m)_{n-1}=m(m+1)(m+2)\cdots(m+n-2)\geq m(m+1)^{n-2}\geq m^{n-1}, \quad (n\in\mathbb{N})$. From (11), we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\\
\leq\;& t_3\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\frac{(-c/4m)^{n-1}}{(n-1)!}+(t_2+6t_3)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4m)^{n-1}}{(n-1)!}\\
&+(t_1+3t_2+7t_3)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4m)^{n-1}}{(n-1)!}+(t_1+t_2+t_3+t_0)\sum_{n=2}^{\infty}\frac{(-c/4m)^{n-1}}{(n-1)!}\\
=\;& t_3\sum_{n=4}^{\infty}\frac{(-c/4m)^{n-1}}{(n-4)!}+(t_2+6t_3)\sum_{n=3}^{\infty}\frac{(-c/4m)^{n-1}}{(n-3)!}+(t_1+3t_2+7t_3)\sum_{n=2}^{\infty}\frac{(-c/4m)^{n-1}}{(n-2)!}\\
&+(t_1+t_2+t_3+t_0)\sum_{n=2}^{\infty}\frac{(-c/4m)^{n-1}}{(n-1)!}\\
=\;& t_3(-c/4m)^3e^{-c/4m}+(t_2+6t_3)(-c/4m)^2e^{-c/4m}+(t_1+3t_2+7t_3)(-c/4m)e^{-c/4m}\\
&+(t_1+t_2+t_3+t_0)(e^{-c/4m}-1).
\end{aligned}$$

Therefore, we see that the last expression is bounded above by μ if (13) is satisfied. ■

The proof of Theorem 4 (below) is much akin to that of Theorem 3, and so the details may be omitted.

Theorem 4 *$z(2-u_p(z)) \in \mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$ if and only if*

$$e^{(\frac{-c}{4m})}[t_3(\frac{-c}{4m})^4+(10t_3+t_2)(\frac{-c}{4m})^3+(25t_3+6t_2+t_1)(\frac{-c}{4m})^2$$
$$+(15t_3+7t_2+3t_1+t_0)(\frac{-c}{4m})+(t_1+t_2+t_3+t_0)(1-e^{(\frac{c}{4m})})] \le \mu. \quad (14)$$

3 Inclusion Properties

Making use of Lemma 2, we have.

Theorem 5 *Let $f \in \mathcal{R}^\tau(A,B)$. Then $\mathcal{I}(m,c)f \in \mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$ if*

$$(A-B)\,|\tau|\,\left[t_3u_p'''(1)+(t_2+6t_3)u_p''(1)+(t_1+3t_2+7t_3)u_p'(1)\right.$$
$$\left.+(t_1+t_2+t_3+t_0)(u_p(1)-1)\right] \le \mu. \quad (15)$$

Proof. In view of (3), it suffices to show that

$$\sum_{n=2}^{\infty} n(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\,|a_n| \le \mu.$$

Since $f \in \mathcal{R}^\tau(A,B)$, then by Lemma 2, we get

$$|a_n| \le \frac{(A-B)\,|\tau|}{n}. \quad (16)$$

Thus, we must show that

$$\sum_{n=2}^{\infty} n(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\,|a_n|$$
$$\le \quad (A-B)\,|\tau|\left[\sum_{n=2}^{\infty}(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\right] \le \mu.$$

The remaining part of the proof of Theorem 5 is similar to that of Theorem 1, and so we omit the details. ∎

4 An Integral Operator

In this section, we obtain the necessary and sufficient conditions for the integral operator $\mathcal{G}(m,c,z)$ defined by

$$\mathcal{G}(m,c,z) = \int_0^z (2-u_p(t))dt \quad (17)$$

to be in $\mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$.

Theorem 6 *The integral operator $\mathcal{G}(m,c,z) \in \mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$if and only if the condition (5) is satisfied.*

Proof. Since

$$\mathcal{G}(m,c,z) = z - \sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}}\frac{z^n}{n!},$$

in view of (3), we need only to show that

$$\sum_{n=2}^{\infty} n(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}n!} \le \mu$$

or, equivalently

$$\sum_{n=2}^{\infty}(t_3n^3+t_2n^2+t_1n+t_0)\frac{(-c/4)^{n-1}}{(m)_{n-1}(n-1)!}\leq\mu.$$

The remaining part of the proof is similar to that of Theorem 1, and so we omit the details. ■

The proof of Theorem 7 and Theorem 8 (below) are much akin to that of Theorem 3, and so the details may be omitted.

Theorem 7 *Let $f\in\mathcal{R}^{\tau}(A,B)$. Then $\mathcal{I}(m,c)f\in\mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$ if*

$$(A-B)\left|\tau\right|e^{(\frac{-c}{4m})}[t_3(\frac{-c}{4m})^3+(t_2+6t_3)(\frac{-c}{4m})^2+(t_1+3t_2+7t_3)(\frac{-c}{4m})+(t_1+t_2+t_3+t_0)(1-e^{(\frac{c}{4m})})]\leq\mu.$$

Theorem 8 *The integral operator $\mathcal{G}(m,c,z)\in\mathbb{S}^1(t_3,t_2,t_1,t_0,\mu)$ if and only if the condition (14) is satisfied.*

Remark 2 *By setting $t_1=2$, $t_0=-(\cos\alpha+\beta)$ and $\mu=\cos\alpha-\beta$ in the above theorems, we obtain the corresponding results for the class $\mathcal{SP}_P\mathcal{T}(\alpha,\beta)$(for $k=0$) and for the class $\mathcal{UCSPT}(\alpha,\beta)$ (for $k=1$) obtained by Frasin and Aldawish [18].*

5 Corollaries and Consequences

In this section, we apply our main results in order to deduce each of the following new corollaries and consequences for the classes $\mathcal{T}^*(C,D)$, $\mathcal{C}(C,D)$, $-1<C<D\leq1$ and $\mathcal{WT}(\alpha,\gamma,\beta)$, $0\leq\alpha<1$, $\gamma,\beta\geq0$.

Corollary 1 *$z(2-u_p(z))\in\mathcal{T}^*(C,D)$ if and only if*

$$(1+D)u_p'(1)+(D-C)u_p(1)\leq2(D-C). \tag{18}$$

Corollary 2 *$z(2-u_p(z))\in\mathcal{C}(C,D)$ if and only if*

$$(1+D)u_p''(1)+(2+3D-C)u_p'(1)+(D-C)u_p(1)\leq2(D-C).$$

Corollary 3 *$z(2-u_p(z))\in\mathcal{T}^*(C,D)$ if and only if*

$$e^{(\frac{-c}{4m})}[(1+D)(\frac{-c}{4m})+(D-C)(1-e^{(\frac{c}{4m})})]\leq D-C.$$

Corollary 4 *$z(2-u_p(z))\in\mathcal{C}(C,D)$ if and only if*

$$e^{(\frac{-c}{4m})}[(1+D)(\frac{-c}{4m})^2+(2+3D-C)(\frac{-c}{4m})+(D-C)(1-e^{(\frac{c}{4m})})]\leq(D-C). \tag{19}$$

Corollary 5 *Let $f\in\mathcal{R}^{\tau}(A,B)$. Then $\mathcal{I}(m,c)f\in\mathcal{C}(C,D)$ if*

$$(A-B)\left|\tau\right|[(1+D)u_p'(1)+(D-C)(u_p(1)-1)]\leq D-C.$$

Corollary 6 *Let $f\in\mathcal{R}^{\tau}(A,B)$. Then $\mathcal{I}(m,c)f$ is in $\mathcal{C}(C,D)$ if*

$$(A-B)\left|\tau\right|e^{(\frac{-c}{4m})}[(1+D)(\frac{-c}{4m})+(D-C)(1-e^{(\frac{c}{4m})})]\leq D-C.$$

Corollary 7 *The integral operator $\mathcal{G}(m,c,z)$ is in $\mathcal{C}(C,D)$ if and only if the condition (18) is satisfied.*

Corollary 8 *The integral operator $\mathcal{G}(m,c,z)$ is in $\mathcal{C}(C,D)$if and only if the condition (19) is satisfied.*

Corollary 9 $z(2-u_p(z)) \in \mathcal{WT}(\alpha,\gamma,\beta)$ *if and only if*

$$\beta u_p''(1)+\gamma u_p'(1)+(u_p(1)-1)\leq 1-\alpha.$$

Corollary 10 $z(2-u_p(z)) \in \mathcal{WT}(\alpha,\gamma,\beta)$ *if and only if*

$$e^{(\frac{-c}{4m})}[\beta(\frac{-c}{4m})^2+\gamma(\frac{-c}{4m})+(1-e^{(\frac{c}{4m})})]\leq 1-\alpha.$$

Concluding Remark. By suitably specializing the real constants t_3, t_2, t_1, t_0, k and μ in Theorems 1, 2, 5 and 6, as stated in Remark 1, we determined necessary and sufficient conditions for $z(2-u_p(z))$ to be in the classes $\mathcal{S_P T}(\alpha,\beta)$, $\mathcal{UCT}(\alpha,\beta)$, $\mathcal{PT}(\alpha)$, $\mathcal{CPT}(\alpha)$ (see, [13]), $\mathcal{T}(\lambda,\alpha)$, $\mathcal{C}(\lambda,\alpha)$ (see, [29]), $\mathcal{P}^*_\lambda(\alpha)$, $\mathcal{Q}^*_\lambda(\alpha)$, $\mathcal{M}^*(\alpha)$ (see, [26]), $\mathcal{TS}(\lambda,\alpha,\beta)$ (see, [16]), $\mathcal{TS}_p(\lambda,\alpha,\beta)$ and $\mathcal{UCT}(\lambda,\alpha,\beta)$(see, [24]). Further, our main results can lead to several additional new results by suitably specializing the real constants t_3, t_2, t_1, t_0, k and μ in other subclasses of analytic functions with negative coefficients introduced and studied by several authors as stated in Remark 1.

References

[1] O. Altintaş, A subclass of analytic functions with negative coefficients, Bull. Sci. Eng. Hacet. Univ., 19(1990), 15–24.

[2] O. Altintaş and Y. Ertekin, A new subclass of analytic functions with negative coefficients, Curr. Top. Anal. Funct. Theory, (1992), 36–47.

[3] O. Altintas and S. Owa, On subclasses of univalent functions with negative coefficients, Pusan Kyongnam Mathematical Journal, 4(1988), 41–56.

[4] O. Altintaş, Ö. Özkan and H. M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, Appl. Math. Lett., 13(2000), 63–67.

[5] M. K. Aouf and H. M. Srivastava, Some families of starlike functions with negative coefficients, J. Math. Aanl. Appl., 203(1996), 762–790.

[6] M. K. Aouf, R. M. El-Ashwah and S. M. El-Deeb, Certain subclasses of uniformly starlike and convex functions defined by convolution, Acta Math. Acad. Paedeg. Nyr., 26(2010), 55–70.

[7] M. K. Aouf, R. M. El-Ashwah and S. M. El-Deeb, Certain classes of univalent functions with negative coefficients and n-starlike with respect to certain points, Mat. Vesnik, 62(2010), 215–226.

[8] A . Baricz, Geometric properties of generalized Bessel functions, Publ. Math. Debrecen, 73(2008), 155–178.

[9] A. Baricz, Geometric properties of generalized Bessel functions of complex order, Mathematica, 48(2006), 13–18.

[10] A. Baricz, Generalized Bessel Functions of the First Kind, Lecture Notes in Math., Vol. 1994 , Springer-Verlag (2010).

[11] A. Baricz and B. A. Frasin, Univalence of integral operators involving Bessel functions, Appl. Math. Lett., 23(2010), 371–376.

[12] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, Tamkang J. Math., 26(1997), 17–32.

[13] N. E. Cho, H. J. Lee and R. Srivastava, Characterizations for certain subclasses of starlike and convex functions associated with bessel functions, Filomat, 30(2016), 1911–1917.

[14] N. E. Cho, S. Y. Woo and S. Owa, Uniform convexity properties for hypergeometric functions, Fract. Cal. Appl. Anal., 5(2002), 303–313.

[15] K. K. Dixit and S. K. Pal, On a class of univalent functions related to complex order, Indian J. Pure Appl. Math., 26(1995), 889–896.

[16] R. M. El-Ashwah and A. H. Hassan, Some characterizations for a certain generalized bessel function of the first kind to be in certain subclasses of analytic functions, Int. J. Open Problems Complex Analysis, 9(2017), 14–37.

[17] B. A. Frasin, Sufficient conditions for integral operators defined by Bessel functions, J. Math. Inequal., 4(2010), 301–306.

[18] B. A. Frasin and I. Aldawish, On subclasses of uniformly spiral-like functions associated with generalized Bessel functions, J. Funct. Spaces 2019, Art. ID 1329462, 6 pp.

[19] B. A. Frasin, T. Al-Hawary and F. Yousef, Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions, Afr. Mat., 30(2019), 223–230.

[20] R. M. Goel and N. S. Sohi, Multivalent functions with negative coefficients, Indian J. Pure Appl. Math., 12(1981), 844–853.

[21] V. P. Gupta and P. K. Jain, Certain classes of univalent functions with negative coefficients, Bull. Aust. Math. Soc., 14(1976), 409–416.

[22] E. Merkes and B. T. Scott, Starlike hypergeometric functions, Proc. Amer. Math. Soc., 12(1961), 885–888.

[23] G. Murugusundaramoorthy and N. Magesh, On certain subclasses of analytic functions associated with hypergeometric functions, Appl. Math. Lett., 24(2011), 494–500.

[24] G. Murugusundaramoorthy and T. Janani, An application of generalized Bessel functions on certain subclasses of analytic functions, Turkish Journal of Analysis and Number Theory, 3(2015), 1–6.

[25] G. Murugusundaramoorthy, K. Vijaya and S. Porwal, Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series, Hacet. J. Math. Stat., 45(2016), 1101–1107.

[26] G. Murugusundaramoorthy, K. Vijaya and M. Kasthuri, A note on subclasses of starlike and convex functions, associated with Bessel functions, J. Nonlinear Funct. Anal., 2014(2014), Article ID 11.

[27] G. Murugusundaramoorthy, K. Vijaya, and K. Uma, Subordination results for a class of analytic functions involving the Hurwitz-Lerch zeta function, International Journal of Nonlinear Science, 10(2010), 430–437.

[28] S. Porwal, Mapping properties of generalized Bessel functions on some subclasses of univalent functions, Anal. Univ. Oradea Fasc. Matematica, 20(2013), 51–60.

[29] S. Porwal and K. K. Dixit, An application of generalized Bessel functions on certain analytic functions, Acta Univ. M. Belii Ser. Math., 21(2013), 55–61.

[30] S. Porwal and M. Kumar, A unified study on starlike and convex functions associated with Poisson distribution series, Afr. Mat., 27(2016), 1021–1027.

[31] C. Ramachandran and L. Vanitha, Certain Aspect of Subordination for a class of analytic function, International Journal of Mathematical Analysis, 9(2015), 979–984.

[32] T. Rosy, B. A. Stephen, K. G. Subramanian and H. Silverman, Classes of convex functions, Internat. J. Math. Math. Sci., 23(2000), 819–825.

[33] C. Selvaraj and R. Geetha, On subclasses of uniformly convex spirallike functions and corresponding class of spirallike functions, International Journal of Contemporary Mathematical Sciences, 5(2010), 37–40.

[34] N. Shukla and P. Shukla, Mapping properties of analytic function defined by hypergeometric function, II, Soochow Journal of Mathematics, 25(1999), 29–36.

[35] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51(1975), 109–116.

[36] H. Silverman, Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl., 172(1993), 574–581.

[37] H. M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, Hypergeometric functions in the parabolic starlike and uniformly convex domains, Integr. Transf. Spec. Func., 18(2007), 511–520.

[38] H. M. Srivastava, T. N. Shanmugam, C. Ramachandran and S. Sivasubramanian, A new subclass of k-uniformly convex functions with negative coefficients, JIPAM. J. Inequal. Pure Appl. Math., 8(2007), 14 pp.

[39] K. G. Subramanian, T. V. Sudharsan, P. Balasubrahmanyam and H. Silverman, Classes of uniformly starlike functions, Publ. Math. Debrecen, 53(1998), 309–315.

[40] K. G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahmanyam and H. Silverman, Subclasses of uniformly convex and uniformly starlike functions, Math. Japonica, 42(1995), 517–522.