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Nonlocal Initial Value Problems For First-Order Dynamic Equations On Time Scales*

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Abstract

This article is concerned with a class of first-order dynamic equations on time scales with nonlocal initial conditions. Qualitative and quantitative results are discussed. Through an application of a fixed point theorem due to O'Regan, the existence of solutions is investigated. Under suitable assumptions, we deduce the existence result for nonlocal dynamic Cauchy problem. We also examine the continuous dependency of solutions on initial conditions. We illustrate our main result through examples.

1 Introduction

The theory of time scales calculus, introduced by S. Hilger [15], is an efficient tool to unify both continuous and discrete problems in one theory. The advantage of working on dynamic equations on time scales is that, under one framework, we can describe continuous–discrete hybrid processes. The results obtained, in the context of time scales, are more general and includes various other results as a particular case. Thus, the study of dynamic equations on time scales has potential applications to any field involving both continuous and discrete dynamical processes. As a consequence, in recent decades, the field of dynamic equations on time scales has become very popular to be used in mathematical modelling of several situations where simultaneous modelling is needed.

The first-order dynamic equations on time scales have been studied extensively, covering a variety of different problems; see for instance [8, 16, 21, 24] and the references therein. Moreover, there has been significant growth in the study of initial and boundary value dynamic problems, see for instance [9, 10, 11, 23], but none of them considers nonlocal conditions. The study of nonlocal initial value problems constitutes a very interesting and important class of problems; because, in many physical systems, the measurements by a nonlocal condition may be more precise than the measurement given by a local initial condition. In the literature, a great deal of attention has been given to nonlocal problems for differential equations rather than for difference or dynamic equations, see [6, 7, 13, 14] and references therein. Thus, it is worthwhile to study dynamic equations on time scales with nonlocal initial conditions.

In this paper, we are interested in establishing the existence of solutions to the first-order dynamic equation of the form

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = f\big(t, x(t)\big), \quad t \in [0,T]_{\mathbb{T}} = [0,T] \cap \mathbb{T}, \tag{1}$$

subject to the initial condition

$$x(0) = \Phi(x), \tag{2}$$

where $T \in \mathbb{R}^{+}$, $f : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous, $\Phi : C\big([0,T]_{\mathbb{T}}, \mathbb{R}\big) \to \mathbb{R}$, and $p : [0,T]_{\mathbb{T}} \to \mathbb{R}$ is a function which is regressive and rd-continuous on $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then (1) becomes the well known ordinary differential equation [18]. If $\mathbb{T} = \mathbb{Z}$, then (1) becomes familiar difference equation [17].

Also, we discuss herein, the existence of solutions to the adjoint equation of (1)

$$x^{\Delta}(t) + q(t)x(t) = g\big(t, x(t)\big), \quad t \in [0,T]_{\mathbb{T}} = [0,T] \cap \mathbb{T}, \tag{3}$$

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subject to (2), where $q : [0,T]_{\mathbb{T}} \to \mathbb{R}$ is regressive and rd-continuous function, $g : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous function.

Among various techniques available in the analysis that deals with the existence of solutions of dynamic problems, the method of fixed points is more elegant and powerful. Many authors have been employed several fixed point theorems in their investigation of the existence of solutions, see for instance [6, 7, 10, 13, 14, 22, 24]. For more details of fixed point theory, see [2, 12, 25].

In the present paper, we have applied a fixed point theorem of O'Regan, which is an extension of Krasnoselskii's fixed point theorem [25], to obtain the existence of solutions of dynamic problem (1)–(2).

The paper is organized as follows. In Section 2, we review some basic definitions and results needed in the paper. In Section 3, we present the results of the existence and continuous dependence of solutions to a nonlocal dynamic problem (1)–(2). Examples are provided in Section 4 to illustrate the main result. Finally, in Section 5 we make our concluding remarks.

2 Preliminaries

In this section, we recall some preliminary definitions and results which helps the reader to follow the paper easily. A nonempty closed subset of $\mathbb{R}$, which inherits the standard topology of $\mathbb{R}$ is a time scale. It is denoted by $\mathbb{T}$. We assume that the reader of this paper is familiar with basic concepts of time scales calculus; these preliminary materials are very common in the literature and we refer the reader to [3, 4] for an excellent review of the topic. We begin by recalling briefly some necessary material from time scales calculus which are needed to follow the paper.

Definition 1 *A function $x : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at every right-dense points in $\mathbb{T}$ and its left sided limits exist at left-dense points in $\mathbb{T}$.*

Definition 2 *A function $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is said to be rd-continuous on $\mathbb{T} \times \mathbb{R}$ if $f(\cdot, x)$ is rd-continuous on $\mathbb{T}$ for each fixed $x \in \mathbb{R}$ and $f(t, \cdot)$ is continuous on $\mathbb{R}$ for each fixed $t \in \mathbb{T}$.*

Definition 3 *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, where $\mu : \mathbb{T} \to [0, \infty)$ is the graininess function defined by $\mu(t) := \sigma(t) - t$.*

Definition 4 *For a regressive function p, the exponential function $e_p(\cdot, t_0)$ on the time scale $\mathbb{T}$ is defined as*

$$e_p(t, t_0) := \exp\left(\int_{t_0}^{t} \xi_{\mu(s)}\big(p(s)\big)\Delta s \right) \quad \text{with} \quad \xi_h(z) = \begin{cases} \dfrac{Log\big(1 + hz\big)}{h} & \text{if} \quad h > 0, \\ z & \text{if} \quad h = 0, \end{cases}$$

where Log is the principal logarithm function. In fact, $e_p(\cdot, t_0)$ is the unique solution of the dynamic initial value problem

$$x^\Delta(t) = p(t)x, \quad x(t_0) = 1, \quad t, t_0 \in \mathbb{T},$$

where p is regressive and rd-continuous function.

For regressive functions $p, q : \mathbb{T} \to \mathbb{R}$, we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := \frac{-p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

Throughout the paper, we denote

$$E_0 := \sup_{t \in [0,T]_{\mathbb{T}}} \left| e_{\ominus p}(t, 0) \right| \quad \text{and} \quad E := \sup_{s,t \in [0,T]_{\mathbb{T}}} \left| e_{\ominus p}(t, s) \right|.$$

Some fundamental properties of the exponential function are stated below.

Theorem 1 *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are regressive and rd-continuous. Then the following hold.*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t),s) = \big(1+\mu(t)p(t)\big)e_p(t,s)$*;*

(iii) $1/e_p(t,s) = e_{\ominus p}(t,s)$*;*

(iv) $e_p(t,s) = 1/e_p(s,t)$*;*

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

(vii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$*.*

We now remind of essential features from the fixed point theory. Let $C\big([0,T]_{\mathbb{T}},\mathbb{R}\big)$ be the family of all continuous functions from $[0,T]_{\mathbb{T}}$ into $\mathbb{R}$, which is Banach space coupled with the norm $\|\cdot\|$ defined as $\|x\| := \sup\limits_{t\in[0,T]_{\mathbb{T}}} \big|x(t)\big|$. We denote this Banach space by X and a closed ball of radius r centered in the null element of X by B_r; that is, $B_r := \big\{x \in X : \|x\| \le r\big\}$.

Definition 5 ([12]) *Let X, Y be two Banach spaces. A mapping $F : X \to Y$ is said to be completely continuous if the image of each bounded set B of X, $F(B)$, is relatively compact in Y.*

Definition 6 ([19]) *Let X be a Banach space. A mapping $F : X \to X$ is said to be nonlinear contraction map if there exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that*

$$\big\|F(x) - F(y)\big\| \le \phi\big(\|x-y\|\big)$$

for all $x, y \in X$.

The version of Arzelà-Ascoli theorem in the context of time scales is stated as follows, see [1, Lemma 2.6].

Theorem 2 *A bounded equicontinuous subset D of $C\big(\mathbb{T},\mathbb{R}\big)$ is relatively compact.*

The main result of this paper is based upon an application of the following fixed point theorem of D. O'Regan [19].

Theorem 3 *Let U be an open set in a closed, convex set C of a Banach space X. Assume $0 \in U$, $F(\overline{U})$ bounded and $F : \overline{U} \to C$ is given by $F := F_1 + F_2$, where $F_1 : \overline{U} \to X$ is continuous and completely continuous and $F_2 : \overline{U} \to X$ is a nonlinear contraction. Then either,*

(A1) F has a fixed point in $\overline{U}$; or

(A2) there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

3 Main Results

In this section, we establish a result concerning the existence of solutions to (1)–(2) using Theorem 3 and its applications.

First, we present an auxiliary lemma which reformulate our dynamic problem (1)–(2) as equivalent delta integral equation. The idea is same as that of Lemma 3.1 [5]. Hence the proof is omitted.

Lemma 1 *Let $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous and regressive. Suppose $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous, $t_0 \in \mathbb{T}$, and $\Phi : C\big(\mathbb{T}, \mathbb{R}\big) \to \mathbb{R}$. Then x is the unique solution of the nonlocal dynamic problem (1)–(2) on $\mathbb{T}$ if and only if*

$$x(t) = e_{\ominus p}(t, t_0)\Phi(x) + \int_{t_0}^{t} e_{\ominus p}(t, s) f\big(s, x(s)\big)\Delta s \tag{4}$$

for all $t \in \mathbb{T}$.

Theorem 4 *Assume that $f : [0, T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous function such that for all $t \in [0, T]_{\mathbb{T}}$ and for each $x \in B_r$, $\big|f(t, x)\big| \le h_r(t)$, where $h_r : [0, T]_{\mathbb{T}} \to [0, \infty)$ is such that*

$$\lim_{r\to\infty} \left(\frac{1}{r} \int_0^T h_r(t)\Delta t\right) = \beta < \frac{1}{E} < \infty. \tag{5}$$

Also, suppose that there is a continuous, nondecreasing function $m : [0, \infty) \to [0, \infty)$ satisfying $m(z) < z$ for $z > 0$ such that

$$\big|\Phi(x) - \Phi(y)\big| \le \frac{1}{E_0}\ m\big(\|x - y\|\big) \quad \text{for all} \quad x, y \in X. \tag{6}$$

Then the dynamic nonlocal problem (1)–(2) has at least one solution.

Proof. Select $r > 0$ such that for every $x \in X$,

$$r > E_0\big|\Phi(x)\big| + E \int_0^T h_r(s)\Delta s.$$

Then B_r is a closed, convex subset of X and $0 \in B_r$. Here 0 is a null element of X.
We define the mapping $F : B_r \to X$ by

$$F(x)(t) := F_1(x)(t) + F_2(x)(t), \quad t \in [0, T]_{\mathbb{T}}, \tag{7}$$

where $F_i : B_r \to X$ $(i = 1, 2)$ are given by

$$F_1(x)(t) := \int_0^t e_{\ominus p}(t, s) f\big(s, x(s)\big)\Delta s \tag{8}$$

and

$$F_2(x)(t) := e_{\ominus p}(t, 0)\Phi(x). \tag{9}$$

Obviously, the fixed points of F are the solutions of (1)–(2). We verify that the conditions of Theorem 3 are fulfilled.

1) $F(B_r)$ is bounded. Let $x \in B_r$ be arbitrary. Then for each $t \in [0, T]_{\mathbb{T}}$,

$$F(x)(t) = e_{\ominus p}(t, 0)\Phi(x) + \int_0^t e_{\ominus p}(t, s) f\big(s, x(s)\big)\Delta s, \tag{10}$$

and we see that

$$\begin{aligned}
\big|F(x)(t)\big| &\le \sup_{t\in[0,T]_{\mathbb{T}}} \big|F(x)(t)\big| \\
&= \sup_{t\in[0,T]_{\mathbb{T}}} \left|e_{\ominus p}(t, 0)\Phi(x) + \int_0^t e_{\ominus p}(t, s) f\big(s, x(s)\big)\Delta s\right| \\
&\le \sup_{t\in[0,T]_{\mathbb{T}}} \left\{\big|e_{\ominus p}(t, 0)\big|\big|\Phi(x)\big| + \int_0^t \big|e_{\ominus p}(t, s)\big|\big|f\big(s, x(s)\big)\big|\Delta s\right\} \\
&\le E_0\big|\Phi(x)\big| + E \int_0^T h_r(s)\Delta s \\
&< r.
\end{aligned}$$

Thus, $\|F(x)\| < r$ and hence $F(B_r)$ is bounded.

2) $F_1 : B_r \to X$ is continuous and completely continuous. Let (x_n) be a sequence of elements of B_r converges to x in B_r. Then we see that,

$$\begin{aligned}
\|F_1(x_n) - F_1(x)\| &= \sup_{t\in[0,T]_{\mathbb{T}}} \big|F_1(x_n)(t) - F_1(x)(t)\big| \\
&= \sup_{t\in[0,T]_{\mathbb{T}}} \left| \int_0^t e_{\ominus p}(t,s) f\big(s, x_n(s)\big)\Delta s - \int_0^t e_{\ominus p}(t,s) f\big(s, x(s)\big)\Delta s \right| \\
&\le \sup_{t\in[0,T]_{\mathbb{T}}} \left\{ \int_0^t \big|e_{\ominus p}(t,s)\big| \big|f\big(s, x_n(s)\big) - f\big(s, x(s)\big)\big|\Delta s \right\} \\
&\le E \int_0^T \big|f\big(s, x_n(s)\big) - f\big(s, x(s)\big)\big|\Delta s,
\end{aligned}$$

which yields by rd-continuity of f that

$$\|F_1(x_n) - F_1(x)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, F_1 is uniformly continuous on B_r and hence is continuous on B_r.

Next, we show that F_1 is completely continuous. For this, we show that for each bounded subset D of B_r, $F_1(D)$ is in D. We claim that there exists a positive number r_1 such that $F_1(B_{r_1}) \subseteq B_{r_1}$. Suppose not, then we see that for each positive number r there is $x \in B_r$ satisfying $\big|F_1(x)(t)\big| > r$ for all $t \in [0,T]_{\mathbb{T}}$. Now

$$\begin{aligned}
r &< \big|F_1(x)(t)\big| \\
&= \left| \int_0^t e_{\ominus p}(t,s) f\big(s, x(s)\big)\Delta s \right| \\
&\le \int_0^t \big|e_{\ominus p}(t,s)\big| \big|f\big(s, x(s)\big)\big|\Delta s \\
&\le \sup_{t\in[0,T]_{\mathbb{T}}} \left\{ \int_0^t \big|e_{\ominus p}(t,s)\big| \big|f\big(s, x(s)\big)\big|\Delta s \right\} \\
&\le E \int_0^T h_r(s)\Delta s.
\end{aligned}$$

Thus,

$$1 < \frac{E}{r} \int_0^T h_r(s)\Delta s.$$

Taking limit as $r \to \infty$ on both sides, we obtain $1 < E\beta$, which is contradiction. Hence there exists a positive number r_1 such that $F_1(B_{r_1}) \subseteq B_{r_1}$. Let $t_1, t_2 \in [0,T]_{\mathbb{T}}$ with $t_1 < t_2$ and fix $x \in B_r$. Then we compute that

$$\begin{aligned}
&\big|F_1(x)(t_2) - F_1(x)(t_1)\big| \\
= \ &\left| \int_0^{t_2} e_{\ominus p}(t_2,s) f\big(s, x(s)\big)\Delta s - \int_0^{t_1} e_{\ominus p}(t_1,s) f\big(s, x(s)\big)\Delta s \right| \\
= \ &\Big| e_{\ominus p}(t_2,0) \int_0^{t_1} e_p(s,0) f\big(s, x(s)\big)\Delta s + e_{\ominus p}(t_2,0) \int_{t_1}^{t_2} e_p(s,0) f\big(s, x(s)\big)\Delta s \\
&- e_{\ominus p}(t_1,0) \int_0^{t_1} e_p(s,0) f\big(s, x(s)\big)\Delta s \Big| \\
\le \ &\big|e_{\ominus p}(t_2,0) - e_{\ominus p}(t_1,0)\big| \int_0^{t_1} \big|e_p(s,0)\big| \big|f\big(s, x(s)\big)\big|\Delta s \\
&+ \big|e_{\ominus p}(t_2,0)\big| \int_{t_1}^{t_2} \big|e_p(s,0)\big| \big|f\big(s, x(s)\big)\big|\Delta s.
\end{aligned}$$

Since $e_{\ominus p}(t,\cdot)$ is continuous on $[0,T]_{\mathbb{T}}$, the right hand side tends to zero as $t_1 \to t_2$ or $t_2 \to t_1$. Thus, by definition 5 and Theorem 2, we obtain that F_1 is completely continuous. Hence $F_1 : B_r \to X$ is continuous and completely continuous.

3) $F_2 : B_r \to X$ is nonlinear contraction map. For $x, y \in B_r$ and for each $t \in [0,T]_{\mathbb{T}}$, consider

$$F_2(x)(t) - F_2(y)(t) = e_{\ominus p}(t,0)\Phi(x) - e_{\ominus p}(t,0)\Phi(y).$$

Then we see that

$$\begin{aligned}\big\|F_2(x)(t) - F_2(y)(t)\big\| &= \sup_{t\in[0,T]_{\mathbb{T}}} \big|F_2(x)(t) - F_2(y)(t)\big| \\ &= \sup_{t\in[0,T]_{\mathbb{T}}} \big|e_{\ominus p}(t,0)\big|\big|\Phi(x) - \Phi(y)\big| \\ &\le E_0\big|\Phi(x) - \Phi(y)\big| \\ &\le m\big(\|x-y\|\big).\end{aligned}$$

Thus, we have

$$\big|F_2(x)(t) - F_2(y)(t)\big| \le m\big(\|x-y\|\big),$$

where $m : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function satisfying $m(z) < z$ for $z > 0$. Hence, in view of definition 6, F_2 is nonlinear contraction map.

4) There is no $u \in \partial B_r$ such that $u = \lambda F(u)$ for $\lambda \in (0,1)$.
Suppose there is such a $u \in \partial B_r$. Then for each $t \in [0,T]_{\mathbb{T}}$,

$$u(t) = \lambda F_1(u)(t) + \lambda F_2(u)(t).$$

We compute that,

$$\begin{aligned}\|u\| &= \sup_{t\in[0,T]_{\mathbb{T}}} \big|u(t)\big| \\ &\le \sup_{t\in[0,T]_{\mathbb{T}}} \left|e_{\ominus p}(t,0)\Phi(u) + \int_0^t e_{\ominus p}(t,s) f\big(s,u(s)\big)\Delta s\right| \\ &\le \sup_{t\in[0,T]_{\mathbb{T}}} \left\{\big|e_{\ominus p}(t,0)\big|\big|\Phi(u)\big| + \int_0^t \big|e_{\ominus p}(t,s)\big|\big|f\big(s,u(s)\big)\big|\Delta s\right\} \\ &\le E_0\big|\Phi(u)\big| + E\int_0^T h_r(s)\Delta s.\end{aligned}$$

Since $u \in \partial B_r$, we obtain

$$r \le E_0\big|\Phi(u)\big| + E\int_0^T h_r(s)\Delta s,$$

which contradicts to the choice of $r > 0$. Hence there is no $u \in \partial B_r$ for which $u = \lambda F(u)$ for $\lambda \in (0,1)$. Consequently, we can deduce from Theorem 3 that F has a fixed point in B_r. Hence there exists at least one solution of (1)–(2). This completes the proof. ■

Now, we shall turn our attention to the existence of solutions to the dynamic nonlocal problem (3)–(2). Notice that the simple useful formula transform (3) into (1). Hence, we will use Theorem 4 to formulate the existence result for dynamic nonlocal problem (3)–(2).

Theorem 5 *Consider the dynamic nonlocal problem (3)–(2). Let q be rd-continuous, regressive function such that $1-\mu(t)q(t) > 0$ for all $t \in [0,T]_{\mathbb{T}}$. Assume that $g : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous function such that for all $t \in [0,T]_{\mathbb{T}}$ and for each $x \in B_r$,*

$$\big|g(t,x)\big| \le h_r(t)\big(1-\mu q(t)\big), \tag{11}$$

where the functions h_r and Φ are defined as in Theorem 4. Then the dynamic nonlocal problem (3)–(2) has at least one solution.

Proof. Replacing x in (3) by the simple useful formula $x^\sigma := x + \mu x^\Delta$, we obtain

$$x^\Delta(t) + \left(\frac{q(t)}{1-\mu q(t)}\right)x^\sigma(t) = \frac{g\big(t, x(t)\big)}{1-\mu q(t)}.$$

This equation is essentially in the form (1) with $p(t) := \dfrac{q(t)}{1-\mu q(t)}$ and $f(t,x) := \dfrac{g(t,x)}{1-\mu q(t)}$. Also, we notice that p is regressive and rd-continuous and f is rd-continuous. Now, it is not difficult to show that (11) reduces to $|f(t,x)| \le h_r(t)$ for all $t \in [0,T]_{\mathbb{T}}$ and for each $x \in B_r$. Hence the result follows from Theorem 4. ■

Next, again Theorem 4 will be used to guarantee the existence of at least one solution to the nonlocal dynamic Cauchy problem of the type

$$x^\Delta(t) = F(t,x), \quad t \in [0,T]_{\mathbb{T}} \tag{12}$$

subject to (2), where $F : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous and possibly nonlinear function.

Theorem 6 *Consider the dynamic nonlocal Cauchy problem (12)–(2). Let q be rd-continuous, regressive function such that $1-\mu(t)q(t) > 0$ for all $t \in [0,T]_{\mathbb{T}}$. Assume that $F : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous function such that for all $t \in [0,T]_{\mathbb{T}}$ and for each $x \in B_r$,*

$$\big|F(t,x) + q(t)x\big| \le h_r(t)\big(1 - q(t)x\big),$$

where the functions h_r and Φ are defined as in Theorem 4. Then the dynamic nonlocal Cauchy problem (12)–(2) has at least one solution.

Proof. We write (12) as

$$x^\Delta(t) = -q(t)x^\sigma(t) + q(t)x^\sigma(t) + F\big(t, x(t)\big),$$

which by using so-called simple useful formula yields that

$$x^\Delta(t) = \frac{-q(t)}{1-\mu(t)q(t)}x^\sigma(t) + \frac{q(t)x(t) + F\big(t,x(t)\big)}{1-\mu(t)q(t)}.$$

Let $p : [0,T]_{\mathbb{T}} \to \mathbb{R}$ be defined by

$$p(t) := \frac{q(t)}{1-\mu(t)q(t)}.$$

Then clearly, p is rd-continuous and regressive. Again, define $f : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,x) := \frac{q(t)x + F(t,x)}{1-\mu(t)q(t)}$$

which is rd-continuous. With these notations, (12) may be written as

$$x^\Delta(t) + p(t)x^\sigma(t) = f\big(t, x(t)\big) \tag{13}$$

which is essentially (1). Now for all $t \in [0,T]_{\mathbb{T}}$ and for each $x \in B_r$,

$$\big|f(t,x)\big| = \frac{\big|q(t)x + F\big(t,x\big)\big|}{1-\mu(t)q(t)} \le \frac{h_r(t)\big(1-\mu(t)q(t)\big)}{1-\mu(t)q(t)}.$$

Hence $\big|f(t,x)\big| \le h_r(t)$. Thus, we are able to apply Theorem 4 to the dynamic nonlocal problem (13)–(2) and obtain that the dynamic nonlocal Cauchy problem (12)–(2) has at least one solution. This completes the proof. ■

Remark 1 *We remark that Theorems 4, 5, and 6 also hold and can be proved by using the same analysis, even if f is Δ-Carathéodory function. In this case, we get Δ-Carathéodory type solutions. For such solutions, the dynamic nonlocal problem (1)–(2) is equivalent to the integral equation (4) Δ- a.e. $t \in [0,T]_{\mathbb{T}}$. For more details of Δ-Carathéodory function and solutions, the readers are referred to [11, 20].*

The dependency of solutions to (1)–(2) with respect to the initial conditions can be obtained as follows.

Theorem 7 *Suppose $f : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ and $\Phi_1, \Phi_2 : X \to \mathbb{R}$ satisfy the hypotheses of Theorem 4. Also, suppose for all $t \in [0,T]_{\mathbb{T}}$ and for all $x, y \in X$,*

$$\big|f(t,x) - f(t,y)\big| \leq \frac{1}{E}\ \|x-y\|. \tag{14}$$

Then for the corresponding solutions x_1, x_2 of the dynamic equation (1) subject to the initial conditions

$$x(0) = \Phi_i(x) \qquad (i = 1,2), \tag{15}$$

the inequality

$$\|x_1 - x_2\| \leq E_0\ \big|\Phi_1(x_1) - \Phi_2(x_2)\big| \sup_{t\in[0,T]_{\mathbb{T}}} \big|e_1(t,0)\big|.$$

holds. Additionally, if $\big|\Phi_1(x_1) - \Phi_2(x_2)\big| \leq \delta$ for some $\delta > 0$, then we have

$$\|x_1 - x_2\| \leq E_0\ \delta \sup_{t\in[0,T]_{\mathbb{T}}} \big|e_1(t,0)\big|. \tag{16}$$

Proof. Let x_i $(i = 1,2)$ be the corresponding solutions to the dynamic problem (1)-(15). Then from Lemma 1, these solutions are given by

$$x_i(t) = e_{\ominus p}(t,0)\Phi_i(x_i) + \int_0^t e_{\ominus p}(t,s) f\big(s, x_i(s)\big)\Delta s \qquad (i = 1,2). \tag{17}$$

Therefore, for each $t \in [0,T]_{\mathbb{T}}$, we obtain

$$\begin{aligned}
|x_1(t) - x_2(t)| &= \bigg| e_{\ominus p}(t,0)\Phi_1(x_1) + \int_0^t e_{\ominus p}(t,s) f\big(s, x_1(s)\big)\Delta s \\
&\qquad - e_{\ominus p}(t,0)\Phi_2(x_2) + \int_0^t e_{\ominus p}(t,s) f\big(s, x_2(s)\big)\Delta s \bigg| \\
&\leq \big|e_{\ominus p}(t,0)\big|\big|\Phi_1(x_1) - \Phi_2(x_2)\big| + \int_0^t \big|e_{\ominus p}(t,s)\big|\big|f\big(s,x_1(s)\big) - f\big(s,x_2(s)\big)\big|\Delta s \\
&\leq E_0\big|\Phi_1(x_1) - \Phi_2(x_2)\big| + E\int_0^t \big|f\big(s,x_1(s)\big) - f\big(s,x_2(s)\big)\big|\Delta s.
\end{aligned}$$

By inequality (14), we can write

$$|x_1(t) - x_2(t)| \leq E_0\big|\Phi_1(x_1) - \Phi_2(x_2)\big| + \int_0^t \big|x_1(s) - x_2(s)\big|\Delta s. \tag{18}$$

Now for $t \in [0,T]_{\mathbb{T}}$, let

$$F(t) = \int_0^t \big|x_1(s) - x_2(s)\big|\Delta s.$$

Then, we see from (18) that

$$F^{\Delta}(t) - F(t) \leq E_0\big|\Phi_1(x_1) - \Phi_2(x_2)\big|.$$

Dividing both sides by $e_1(\sigma(t),0)$ and integrating from 0 to t, $t\in[0,T]_{\mathbb{T}}$, we obtain

$$e_1(0,t)F(t)\le E_0\big|\Phi_1(x_1)-\Phi_2(x_2)\big|\big[e_1(0,0)-e_1(0,t)\big],$$

which yields that

$$F(t)\le E_0\big|\Phi_1(x_1)-\Phi_2(x_2)\big|\big[e_1(t,0)-1\big].$$

Now (18) becomes

$$\begin{aligned}|x_1(t)-x_2(t)|&\le E_0\big|\Phi_1(x_1)-\Phi_2(x_2)\big|+E_0\big|\Phi_1(x_1)-\Phi_2(x_2)\big|\big[e_1(t,0)-1\big]\\&=E_0\big|\Phi_1(x_1)-\Phi_2(x_2)\big|e_1(t,0).\end{aligned}$$

Thus,

$$\|x_1-x_2\|\le E_0\big|\Phi_1(x_1)-\Phi_2(x_2)\big|\sup_{t\in[0,T]_{\mathbb{T}}}\big|e_1(t,0)\big|.$$

Finally, inequality (16) is consequence of the above inequality. This completes the proof. ■

4 Examples

In this section, we present two examples to complement and expound our main result. Throughout this section, we take $\mathbb{T}=[0,1]\cup[2,3]$.
Consider the dynamic problem (1)–(2) on $[0,3]_{\mathbb{T}}$.
(a) In (1)–(2), we take $p\equiv 1$, $f(t,x)=\dfrac{x^3}{x^2+3}+t$, and $\Phi(x)=\sqrt{1+x}$.
Then for $x\in B_r$, we see that $\big|f(t,x)\big|\le r+t$. Also, we get $\beta=1$.
Now, for $x,y\in B_r$,

$$\big|\Phi(x)-\Phi(y)\big|\le 1+\sqrt{\|x-y\|}.$$

We take $m(z)=1+\sqrt{z}$. Then m is decreasing function such that

$$\big|\Phi(x)-\Phi(y)\big|\le m\big(\|x-y\|\big).$$

It is easy to see that $E=1/e$ and $E\beta<1$. From Theorem 4, we conclude that the dynamic problem (1)–(2) has a solution on $[0,3]_{\mathbb{T}}$.
(b) We can take $p\equiv 1$, $f(t,x)=\dfrac{x^2}{x^2+3}+t$, and $\Phi(x)=x^{2/3}$. Then we see that $\beta=0$ and the conditions of Theorem 4 are satisfied. Thus the dynamic problem (1)–(2) has a solution on $[0,3]_{\mathbb{T}}$.

5 Conclusions

Throughout this paper, we extended the concept of nonlocal initial value problems to time scales. We obtained certain results concerning the existence of solutions; and discussed the continuous dependence of solutions to the initial conditions. Moreover, we complemented and expounded our result through examples. We believe that, in the context of general time scales, the results presented here are promising in simultaneous modelling of both continuous and discrete processes. Further, it is possible to extend the results of this paper easily to systems of equations and Banach spaces. Also, as a continuation of this work, the other aspects like the approximation of solutions, boundedness of solutions, upper and lower solutions can be studied in the near future.

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