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Integral Inequalities For Exponentially (m,h)-Convex Functions*

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Abstract

In this paper we define exponentially (m,h)-convex functions and make some estimates to the Hadamard's inequality for functions whose absolute values of second derivatives are exponentially (m,h)-convex. Some special cases are also discussed.

1 Introduction

The Hermite-Hadamard inequalities [7, 6] for a convex function $\digamma : \mathcal{W} \to \mathbb{R}$ on an interval $\mathcal{W}$ is defined as:

$$\digamma\left(\frac{l_1+l_2}{2}\right) \le \frac{1}{l_2-l_1}\int_{l_1}^{l_2} \digamma(u)du \le \frac{\digamma(l_1)+\digamma(l_2)}{2}, \tag{1}$$

for all $l_1, l_2 \in \mathcal{W}$ with $l_1 < l_2$. Above inequalities are true in opposite direction if $\digamma$ is concave. We observe that Hadamard's inequality can be viewed as an improvement of the notion of convexity and it ensure simply from Jensens inequality. Hadamard's inequality for convex functions has acquired additional awareness in recent years and an outstanding diversity of improvements and refinements have been obtained. For example see [1, 2, 3, 4, 8, 9, 10, 11, 12, 13] and the references cited therein.

Dragomir et al. [5] proved the following useful result using Hermite-Hadamard inequalities for convex function.

Theorem 1 ([5]) *Consider a twice differentiable function $\digamma : \mathcal{W} \to \mathbb{R}$ on $\mathcal{W}^o$ and let $-\infty < k < \digamma'' < K < \infty$ for all $u \in [l_1, l_2]$. Then we have inequality*

$$k\frac{(l_2-l_1)^2}{12} \le \frac{\digamma(l_1)+\digamma(l_2)}{2} - \frac{1}{l_2-l_1}\int_{l_1}^{l_1} \digamma(u)du \le K\frac{(l_2-l_1)^2}{12}. \tag{2}$$

Definition 1 ([8]) *A function $\digamma : \mathcal{W} \subset [0,\infty) \to [0,\infty)$ is called s-convex in the second sense, if*

$$\digamma(\iota l_1 + (1-\iota)l_2) \le \iota^s \digamma(l_1) + (1-\iota)^s \digamma(l_2), \tag{3}$$

for all $l_1, l_2 \in \mathcal{W}$ and $\iota \in [0,1]$, with $s \in (0,1]$.

Definition 2 ([19]) *Let $h : \mathcal{H} \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. A non-negative function $\digamma : \mathcal{W} \to \mathbb{R}$ on an interval $\mathcal{W} \subset (0,\infty)$ is called h-convex, if*

$$\digamma(\iota l_1 + (1-\iota)l_2) \le h(\iota)\digamma(l_1) + h(1-\iota)\digamma(l_2), \tag{4}$$

holds, for all $l_1, l_2 \in \mathcal{W}$ and $\iota \in [0,1]$. If $-\digamma$ is h-convex then $\digamma$ is called h-concave.

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Definition 3 ([18]) *Let $m \in [0,1]$. A function $F : [0,b] \to \mathbb{R}$ is called m-convex, if we have*

$$F(\iota l_1 + m(1-\iota)l_2) \leq \iota F(l_1) + m(1-\iota)F(l_2), \tag{5}$$

for all $l_1, l_2 \in [0,b]$ and $\iota \in [0,1]$.

Sarikaya et al. [16] and Özdemir et al. [14] proved Hadamard's inequalities for h- and m-convex functions, respectively.

Awan et al. [2], Mehreen and Anwar [12] defined some exponentially convex functions and proved number of Hadamard's type inequalities.

Definition 4 ([2]) *Let $F : \mathcal{W} \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $\alpha \in \mathbb{R}$. Then F is called exponentially convex, if*

$$F(\iota l_1 + (1-\iota)l_2) \leq \iota \frac{F(l_1)}{e^{\alpha l_1}} + (1-\iota)\frac{F(l_2)}{e^{\alpha l_2}}, \tag{6}$$

for all $l_1, l_2 \in \mathcal{W}$, $\iota \in [0,1]$. If the inequality (6) is in opposite direction then F is called exponentially concave.

Definition 5 ([12]) *Let $\alpha \in \mathbb{R}$. A function $F : \mathcal{W} \subset [0,\infty) \to \mathbb{R}$ is called exponentially s-convex in the second sense on an interval $\mathcal{W}$, if*

$$F(\iota l_1 + (1-\iota)l_2) \leq \iota^s \frac{F(l_1)}{e^{\alpha l_1}} + (1-\iota)^s \frac{F(l_2)}{e^{\alpha l_2}}, \tag{7}$$

for all $l_1, l_2 \in \mathcal{W}$ and $\iota \in [0,1]$ with $s \in (0,1]$. If $-F$ is exponentially s-convex then F is exponentially s-concave.

Qiang et al. [15] defined the following exponentially convex functions.

Definition 6 ([15]) *Consider $F : \mathcal{W} \to \mathbb{R}$ be a function on an interval $\mathcal{W} \subset [0,\infty)$. Then F is called exponentially (s,m)-convex in the second sense, if*

$$F(\iota l_1 + m(1-\iota)l_2) \leq \iota^s \frac{F(l_1)}{e^{\alpha l_1}} + m(1-\iota)^s \frac{F(l_2)}{e^{\alpha l_2}}, \tag{8}$$

for all $l_1, l_2 \in \mathcal{W}$ and $\iota \in [0,1]$ with $s \in (0,1]$.

Alomari et al. [1] and Sarikaya et al. [17] gave following useful results.

Lemma 1 ([1]) *Consider a twice differentiable function $F : \mathcal{W} \to \mathbb{R}$ on $\mathcal{W}^o$. Let $l_1, l_2 \in \mathcal{W}$ with $l_1 < l_2$ and $F'' \in L_1[l_1, l_2]$. Then we have equality:*

$$\frac{F(l_1)+F(l_2)}{2} - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} F(u)du = \frac{(l_2-l_1)^2}{2}\int_0^1 \iota(1-\iota)F''(\iota l_1 + (1-\iota)l_2)d\iota.$$

Lemma 2 ([17]) *Consider a twice differentiable function $F : \mathcal{W} \to \mathbb{R}$ on $\mathcal{W}^o$. Let $l_1, l_2 \in \mathcal{W}$ with $l_1 < l_2$ and $F'' \in L_1[l_1, l_2]$. Then we have equality:*

$$\begin{aligned} &\frac{1}{l_2 - l_1}\int_{l_1}^{l_2} F(u)du - F\left(\frac{l_1+l_2}{2}\right) \\ = &\frac{(l_2-l_1)^2}{2}\int_0^1 n(\iota)(F''(\iota l_1 + (1-\iota)l_2) + F''(\iota l_2 + (1-\iota)l_1))d\iota, \end{aligned}$$

where

$$n(\iota) = \begin{cases} \iota^2, & \iota \in [1, \frac{1}{2}), \\ (1-\iota)^2, & \iota \in [\frac{1}{2}, 1). \end{cases}$$

2 Main Results

First we define exponentially (m, h)-convex functions.

Definition 7 *Let $h : \mathcal{H} \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function and $m \in [0,1]$. A function $F : [0,b] \to \mathbb{R}$ is called exponentially (m,h)-convex, if*

$$F(\iota l_1 + m(1-\iota)l_2) \leq h(\iota)\frac{F(l_1)}{e^{\alpha l_1}} + mh(1-\iota)\frac{F(l_2)}{e^{\alpha l_2}}, \tag{9}$$

for all $l_1, l_2 \in \mathcal{W}$, $\iota \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality (9) is in opposite order then F is called exponentially (m,h)-concave.

Remark 1 *In Definition 9,*
(a) by letting $h(\iota) = \iota^s$, one can get inequality (8) of Definition 6.
(b) by letting $h(\iota) = \iota$ and $m = 1$, one can get inequality (7) of Definition 5.
(c) by letting $h(\iota) = \iota$ and $m = 1$, one can get inequality (6) of Definition 4.
(d) by letting $h(\iota) = \iota$ and $\alpha = 0$, one can get inequality (5) of Definition 3.
(e) by letting $\alpha = 0$ and $m = 1$, one can get inequality (4) of Definition 2.
(f) by letting $h(\iota) = \iota^s$, $\alpha = 0$ and $m = 1$, one can get inequality (3) of Definition 1.
(g) by letting $h(\iota) = \iota$, $\alpha = 0$ and $m = 1$, one can get the definition of convex function.

We define an interval $\mathcal{W} \subset [0,\infty) = \mathbb{R}_0$ with interior $\mathcal{W}^\circ$ and a positive function $h : \mathcal{H} \subseteq \mathbb{R} \to \mathbb{R}$.

Theorem 2 *Let $m \in (0,1]$ and $\alpha \in \mathbb{R}$. Let $F : \mathcal{W} \to \mathbb{R}$ be such that F'' exits on $\mathcal{W}^o$ and $F'' \in L_1[l_1, l_2]$, here $l_1, l_2 \in \mathcal{W}$ and $l_1 < l_2$. If $|F''|^q$ is exponentially (m,h)-convex function, $q \geq 1$, then we have inequality:*

$$\left|\frac{F(l_1)+F(l_2)}{2} - \frac{1}{l_2-l_1}\int_{l_1}^{l_2} F(u)du\right|$$
$$\leq \frac{(l_2-l_1)^2}{2}\left(\frac{1}{6}\right)^{\frac{q-1}{q}}\left(A_1(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + mA_2(\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|^q\right)^{\frac{1}{q}},$$

where

$$A_1(\iota) = \int_0^1 \iota(1-\iota)h(\iota)d\iota \quad \text{and} \quad A_2(\iota) = \int_0^1 \iota(1-\iota)h(1-\iota)d\iota.$$

Proof. First consider the case $q = 1$. From Lemma 1, we have

$$\left|\frac{F(l_1)+F(l_2)}{2} - \frac{1}{l_2-l_1}\int_{l_1}^{l_2} F(u)du\right| = \frac{(l_2-l_1)^2}{2}\int_0^1 \iota(1-\iota)|F''(\iota l_1 + (1-\iota)l_2)|d\iota.$$

Since $|F''|$ is exponentially (m,h)-convex function, we have

$$|F''(\iota l_1 + (1-\iota)l_2)| \leq h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right| + mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|. \tag{10}$$

Thus

$$\left|\frac{F(p_1)+F(l_2)}{2} - \frac{1}{l_2-l_1}\int_{l_1}^{l_2} F(u)du\right|$$
$$\leq \frac{(l_2-l_1)^2}{2}\int_0^1 \iota(1-\iota)\left[h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right| + mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|\right]$$
$$= \frac{(l_2-l_1)^2}{2}\left(A_1(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right| + mA_2(\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|\right).$$

Thus we get reguired inequality for the case $q = 1$.

Now consider $q > 1$. Then using Lemma 1 and power mean inequality, we obtain

$$\begin{aligned}
&\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right| \\
\leq\ & \frac{(l_2-l_1)^2}{2}\left[\int_0^1 \iota(1-\iota)d\iota\right]^{\frac{q-1}{q}}\left[\int_0^1 \iota(1-\iota)|F''(\iota l_1+(1-\iota)l_2)|^q d\iota\right]^{\frac{1}{q}}.
\end{aligned}$$

Then by using exponentially (m,h)-convexity of $|F''|^q$, we find

$$\begin{aligned}
&\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right| \\
\leq\ & \frac{(l_2-l_1)^2}{2}\left[\int_0^1 \iota(1-\iota)d\iota\right]^{\frac{q-1}{q}}\left[\int_0^1 \iota(1-\iota)\left(h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|^q\right)\right]^{\frac{1}{q}} \\
=\ & \frac{(l_2-l_1)^2}{2}\left(\frac{1}{6}\right)^{\frac{q-1}{q}}\left(A_1(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+mA_2(\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence the proof is completed. ■

Corollary 1 *Consider the similar assumptions of Theorem 2.*
(a) If $h(\iota) = \iota$, then we have

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right| \leq \frac{(l_2-l_1)^2}{12}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+m\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|^q}{2}\right)^{\frac{1}{q}}.$$

(b) If $h(\iota) = \iota$ and $m = 1$, then we have

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right| \leq \frac{(l_2-l_1)^2}{12}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|^q}{2}\right)^{\frac{1}{q}}.$$

(c) If $h(\iota) = \iota^s$, then we have

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right| \leq \frac{(l_2-l_1)^2}{2}\left(\frac{1}{6}\right)^{\frac{q-1}{q}}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+m\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|^q}{(s+2)(s+3)}\right)^{\frac{1}{q}}.$$

(d) If $h(\iota) = \iota^s$ and $m = 1$, then we have

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right| \leq \frac{(l_2-l_1)^2}{2}\left(\frac{1}{6}\right)^{\frac{q-1}{q}}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|^q}{(s+2)(s+3)}\right)^{\frac{1}{q}}.$$

Remark 2 *In Corollar 1(a),*
(a) by letting $\alpha = 0$, one can get inequality of Theorem 2 in [14].
(b) by letting $\alpha = 0$, $m = 1$ and if $|F''(u)| \leq K$ on $[l_1, l_2]$, one can get the right hand inequality of (2).

Note that, from Corollary 1((b)–(d)), by letting $|F''(u)| \leq K$ on $[l_1, l_2]$, we can have more useful inequalities.

Theorem 3 *Let $m \in (0,1]$ and $\alpha \in \mathbb{R}$. Consider a function $F : \mathcal{W} \to \mathbb{R}$ such that F'' exists on $\mathcal{W}^o$ and $F'' \in L_1[l_1, l_2]$, here $l_1, l_2 \in \mathcal{W}$ and $l_1 < l_2$. If $|F''|^q$ is exponentially (m,h)-convex function, $q, t > 1$ with $1/q + 1/t = 1$. Then we have inequality:*

$$\left|\frac{F(l_1)+F(l_2)}{2} - \frac{1}{l_2-l_1}\int_{l_1}^{l_2} F(u)du\right| \leq \frac{(l_2-l_1)^2}{8}\left(\frac{\Gamma(1+t)}{\Gamma(\frac{3}{2}+t)}\right)^{\frac{1}{t}}\left(h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha \frac{l_2}{m}}}\right|^q\right)^{\frac{1}{q}}.$$

Proof. From Lemma 1 and using Hölder's inequality, we find

$$\left|\frac{F(l_1)+F(l_2)}{2} - \frac{1}{l_2-l_1}\int_{l_1}^{l_2} F(u)du\right| \leq \frac{(l_2-l_1)^2}{2}\left[\int_0^1 (\iota-\iota^2)^t d\iota\right]^{\frac{1}{t}}\left[\int_0^1 |F''(\iota l_1 + (1-\iota)l_2)|^q d\iota\right]^{\frac{1}{q}}.$$

Since we know,

$$\beta(u,w) = \int_0^1 l^{u-1}(1-l)^{w-1}dl,\ u,w>0, \qquad \Gamma(u) = \int_0^\infty e^{-w}w^{u-1}dw,\ u>0,$$

also, we have

$$\int_0^1 (\iota-\iota^2)^t d\iota = \int_0^1 \iota^t(1-\iota)^t d\iota = \beta(t+1,t+1).$$

Furthermore,

$$\beta(w,w) = 2^{1-2w}\beta(\frac{1}{2},w), \qquad \beta(u,w) = \frac{\Gamma(u)\Gamma(w)}{\Gamma(u+w)}.$$

Therefore, we have

$$\beta(t+1,t+1) = 2^{1-2(t+1)}\beta(\frac{1}{2},t+1) = 2^{1-2(t+1)}\frac{\Gamma(\frac{1}{2})\Gamma(t+1)}{\Gamma(\frac{3}{2}+t)}.$$

Then by using exponentially (m,h)-convexity of $|F''|^q$ along with above calculations and the fact $\Gamma(\frac{1}{2}) = \sqrt{\pi} < 2$, we get

$$\begin{aligned}
&\left|\frac{F(l_1)+F(l_2)}{2} - \frac{1}{l_2-l_1}\int_{l_1}^{l_2} F(u)dudu\right| \\
\leq\ & \frac{(l_2-l_1)^2}{2}\left[\int_0^1 (\iota-\iota^2)^t d\iota\right]^{\frac{1}{t}}\left(h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha \frac{l_2}{m}}}\right|^q\right)^{\frac{1}{q}} \\
=\ & \frac{(l_2-l_1)^2}{2}\left[2^{1-2(t+1)}\frac{\Gamma(\frac{1}{2})\Gamma(t+1)}{\Gamma(\frac{3}{2}+t)}\right]^{\frac{1}{t}}\left(h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha \frac{l_2}{m}}}\right|^q\right)^{\frac{1}{q}} \\
\leq\ & \frac{(l_2-l_1)^2}{8}\left(\frac{\Gamma(1+t)}{\Gamma(\frac{3}{2}+t)}\right)^{\frac{1}{t}}\left(h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha \frac{l_2}{m}}}\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

■

Corollary 2 *Consider the similar assumptions of Theorem 3.*
(a) If $h(\iota) = \iota$, then we have

$$\left|\frac{F(l_1)+F(l_2)}{2} - \frac{1}{l_2-l_1}\int_{l_1}^{l_2} F(u)du\right| \leq \frac{(l_2-l_1)^2}{8}\left(\frac{\Gamma(1+t)}{\Gamma(\frac{3}{2}+t)}\right)^{\frac{1}{t}}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + m\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha \frac{l_2}{m}}}\right|^q}{2}\right)^{\frac{1}{q}}.$$

(*b*) *If* $h(\iota)=\iota$ *and* $m=1$*, then we have*

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right|\leq\frac{(l_2-l_1)^2}{8}\left(\frac{\Gamma(1+t)}{\Gamma(\frac{3}{2}+t)}\right)^{\frac{1}{t}}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|^q}{2}\right)^{\frac{1}{q}}.$$

(*c*) *If* $h(\iota)=\iota^s$*, then we have*

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right|\leq\frac{(l_2-l_1)^2}{8}\left(\frac{\Gamma(1+t)}{\Gamma(\frac{3}{2}+t)}\right)^{\frac{1}{t}}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+m\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|^q}{(s+2)(s+3)}\right)^{\frac{1}{q}}.$$

(*d*) *If* $h(\iota)=\iota^s$ *and* $m=1$*, then we have*

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right|\leq\frac{(l_2-l_1)^2}{8}\left(\frac{\Gamma(1+t)}{\Gamma(\frac{3}{2}+t)}\right)^{\frac{1}{t}}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|^q}{(s+2)(s+3)}\right)^{\frac{1}{q}}.$$

Remark 3 *In Corollar 2(a),*
(*a*) *by letting* $\alpha=0$*, one can get inequality of Theorem 3 in [14].*
(*b*) *by letting* $\alpha=0$ *and* $|F''(u)|\leq K$ *on* $[l_1,l_2]$*, one can get the inequality of Corollary 1 in [14].*

Remark 4 *From Corollar 1(a) and Corollar 2(a), we can have the inequality of Corollary 2 in [14].*

Note that, from Corollary 2((b)–(d)), by letting $|F''(u)|\leq K$ on $[l_1,l_2]$, we can have more useful inequalities.

Theorem 4 *Let* $m\in(0,1]$ *and* $\alpha\in\mathbb{R}$*. Consider a function* $F:\mathcal{W}\to\mathbb{R}$ *such that* F'' *exists on* $\mathcal{W}^o$ *and* $F''\in L_1[l_1,l_2]$*, here* $l_1,l_2\in\mathcal{W}$ *and* $l_1<l_2$*. If* $|F''|^q$ *is exponentially* (m,h)*-convex function,* $q,t>1$ *with* $1/q+1/t=1$*. Then we have inequality:*

$$\begin{aligned}&\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right|\\ \leq\;&\frac{(l_2-l_1)^2}{2}\left(B_1(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+mB_2(\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|^q\right)^{\frac{1}{q}},\end{aligned}\tag{11}$$

where

$$B_1(\iota)=\int_0^1(1-\iota)^qh(\iota)d\iota\quad\text{and}\quad B_2(\iota)=\int_0^1(1-\iota)^qh(1-\iota)d\iota.$$

Proof. From Lemma 1 and using Hölder's inequality, we find

$$\begin{aligned}&\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right|\\ \leq\;&\frac{(l_2-l_1)^2}{2}\left[\int_0^1\iota^td\iota\right]^{\frac{1}{t}}\left[\int_0^1(1-\iota)^q|F''(\iota l_1+(1-\iota)l_2)|^qd\iota\right]^{\frac{1}{q}}\\ =\;&\frac{(l_2-l_1)^2}{2}\left[\int_0^1\frac{1}{1+t}d\iota\right]^{\frac{1}{t}}\left[\int_0^1(1-\iota)^q|F''(\iota l_1+(1-\iota)l_2)|^qd\iota\right]^{\frac{1}{q}}.\end{aligned}$$

Then by using exponentially (m,h)-convexity of $|F''|^q$ and the fact that $(\frac{1}{1+t})^{\frac{1}{t}}<1$, we get the required inequality (11). ■

Corollary 3 *Consider the similar assumptions of Theorem 4.*
(a) If $h(\iota)=\iota$, then we have

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right|\le\frac{(l_2-l_1)^2}{2}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+m(q+1)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}. \tag{12}$$

(b) If $h(\iota)=\iota$ and $m=1$, then we have

$$\left|\frac{F(l_1)+F(l_2)}{2}-\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du\right|\le\frac{(l_2-l_1)^2}{2}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q+\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}.$$

Remark 5 *By letting $\alpha=0$ in (12), one can get inequality of Theorem 4 in [14].*

Theorem 5 *Let $m\in(0,1]$ and $\alpha\in\mathbb{R}$. Consider a function $F:\mathcal{W}\to\mathbb{R}$ such that F'' exists on $\mathcal{W}^o$ and $F''\in L_1[l_1,l_2]$, here $l_1,l_2\in\mathcal{W}$ and $l_1<l_2$. If $|F''|$ is exponentially (m,h)-convex function, then we have inequality:*

$$\left|\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du-F\left(\frac{l_1+l_2}{2}\right)\right|\le\frac{(l_2-l_1)^2}{2}C(\iota)\left(\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|+\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|\right), \tag{13}$$

where

$$C(\iota)=\int_0^1 n(\iota)(h(\iota)+mh(1-\iota))d\iota,$$

with $n(\iota)$ is as given in Lemma 2.

Proof. Using Lemma 2 and exponentially (m,h)-convexity of $|F''|$, we have

$$\begin{aligned}&\left|\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du-F\left(\frac{l_1+l_2}{2}\right)\right|\\ \le\ &\frac{(l_2-l_1)^2}{2}\left\{\int_0^1|n(\iota)||F''(\iota l_1+(1-\iota)l_2)|d\iota+\int_0^1|n(\iota)||F''(\iota l_2+(1-\iota)l_1)|d\iota\right\}\\ \le\ &\frac{(l_2-l_1)^2}{2}\Bigg\{\int_0^1|n(\iota)|\left[h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|+mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|\right]d\iota\\ &+\int_0^1|n(\iota)|\left[h(\iota)\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|+mh(1-\iota)\left|\frac{F''(\frac{l_1}{m})}{e^{\alpha\frac{l_1}{m}}}\right|\right]d\iota\Bigg\}\\ =\ &\frac{(l_2-l_1)^2}{2}C(\iota)\left(\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|+\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha\frac{l_2}{m}}}\right|\right).\end{aligned}$$

■

Corollary 4 *Consider the similar assumptions of Theorem 5. If $h(\iota)=\iota$ and $m=1$, then we have*

$$\left|\frac{1}{l_2-l_1}\int_{l_1}^{l_2}F(u)du-F\left(\frac{l_1+l_2}{2}\right)\right|\le\frac{(l_2-l_1)^2}{2}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|+\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|}{24}\right). \tag{14}$$

From inequality (13) one can get several other refinements for some exponentially convex functions.

Remark 6 *By letting $\alpha = 0$ in (14), one can get inequality (7) of Theorem 5 in [17].*

Theorem 6 *Let $m \in (0,1]$ and $\alpha \in \mathbb{R}$. Consider a function $F : \mathcal{W} \to \mathbb{R}$ such that F'' exists on $\mathcal{W}^o$ and $F'' \in L_1[l_1, l_2]$, here $l_1, l_2 \in \mathcal{W}$ and $l_1 < l_2$. If $|F''|^q$ is exponentially (m,h)-convex function, $q, t > 1$ with $1/q + 1/t = 1$. Then we have inequality:*

$$\begin{aligned} &\left|\frac{1}{l_2 - l_1}\int_{l_1}^{l_2} F(u)du - F\left(\frac{l_1 + l_2}{2}\right)\right| \\ \leq\ & \frac{(l_2 - l_1)^2}{8(2t+1)^{\frac{1}{t}}}\left\{\left(\int_0^1 \left[h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha \frac{l_2}{m}}}\right|^q\right] d\iota\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\int_0^1 \left[h(\iota)\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|^q + mh(1-\iota)\left|\frac{F''(\frac{l_1}{m})}{e^{\alpha \frac{l_1}{m}}}\right|^q\right] d\iota\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{15}$$

Proof. Using Lemma 2 and Hölder's inequality, we have

$$\begin{aligned} &\left|\frac{1}{l_2 - l_1}\int_{l_1}^{l_2} F(u)du - F\left(\frac{l_1 + l_2}{2}\right)\right| \\ \leq\ & \frac{(l_2 - l_1)^2}{2}\left(\int_0^1 |n(\iota)|^t d\iota\right)^{\frac{1}{t}}\left\{\left(\int_0^1 |F''(\iota l_1 + (1-\iota)l_2)|^q d\iota\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\int_0^1 |F''(\iota l_2 + (1-\iota)l_1)|^q d\iota\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Since $|F''|^q$ is exponentially (m,h)-convex, we get

$$\begin{aligned} &\left|\frac{1}{l_2 - l_1}\int_{l_1}^{l_2} F(u)du - F\left(\frac{l_1 + l_2}{2}\right)\right| \\ \leq\ & \frac{(l_2 - l_1)^2}{2}\left(\frac{1}{4^t(2t+1)}\right)^{\frac{1}{t}}\left\{\left(\int_0^1 \left[h(\iota)\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + mh(1-\iota)\left|\frac{F''(\frac{l_2}{m})}{e^{\alpha \frac{l_2}{m}}}\right|^q\right] d\iota\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\int_0^1 \left[h(\iota)\left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|^q + mh(1-\iota)\left|\frac{F''(\frac{l_1}{m})}{e^{\alpha \frac{l_1}{m}}}\right|^q\right] d\iota\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Since $\int_0^1 |n(\iota)|^t d\iota = \frac{1}{4^t(2t+1)}$. This completes the proof. ■

Corollary 5 *Consider the similar assumptions of Theorem 6. If $h(\iota) = \iota$ and $m = 1$, then we have*

$$\left|\frac{1}{l_2 - l_1}\int_{l_1}^{l_2} F(u)du - F\left(\frac{l_1 + l_2}{2}\right)\right| \leq \frac{(l_2 - l_1)^2}{8(2t+1)^{\frac{1}{t}}}\left(\frac{\left|\frac{F''(l_1)}{e^{\alpha l_1}}\right|^q + \left|\frac{F''(l_2)}{e^{\alpha l_2}}\right|^q}{2}\right)^{\frac{1}{q}}. \tag{16}$$

From inequality (15) one can get several other refinements for some exponentially convex functions.

Remark 7 *By letting $\alpha = 0$ in (16), one can get inequality (11) of Theorem 6 in [17].*

Conclusion

In this article, we find new Hadamard's inequalities for exponentially (m, h)-convex functions defined in a new way. With the help of these inequalities we find new Hadamard's inequalities for exponentially h-convex, exponentially (m, s)-convex, exponentially m-convex and exponentially convex functions. Our work may open new doors to more useful results for above parameters. For instance, we may find new inequalities for exponentially (m, h)-convex functions via fractional integrals (including Riemann-Liouville, Hadamard, Katugampola, conformable and new fractional conformable integrals etc.).

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