Applied Mathematics E-Notes, 21(2021), 307-319 © ISSN 1607-2510
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On The Affine Equivalence And Minimal Rational Bases Of Planar Cubic Differential Systems*

Anis Hezzam†, Abderrahmane Turqui‡

Received 19 May 2020

Abstract

In this paper, we establish necessary and sufficient invariants conditions for the affine equivalence of some classes of planar cubic differential systems with respect to affine group $SL(2,\mathbb{R})$ via invariant theory. Moreover, we deduce the minimal rational base for each one of these classes after having constructed normal forms.

1 Introduction

The algebraic invariant theory of differential equations plays a major role in the qualitative theory of polynomial differential systems. A great contribution in the development of this theory is due to Sibirsky and his school [15, 19, 23]. Besides, many important results have been obtained with the help of invariant theory such as, the number and nature of singular points, normal forms, the geometrical and topological classification of quadratic and cubic differential systems, see for instance [5, 12, 18, 17, 19].

It is worth noting that the invariants computation is rather cumbersome, since it is a combination of complicated polynomials in several indeterminates of higher degrees. In [2, 6], the authors gave an algorithmic methods to describe the algebra of invariants by using the fundamental theorem of classical invariant [8].

The main motivation of this work is the numerous applications of classical invariant theory in the study of differential systems (see for example [3, 10, 15, 16, 20]), more specifically, the affine equivalence of two systems with respect to the group $SL(n,\mathbb{R})$ implies that they have the same topological and geometric properties, and this makes it an important and useful property in the qualitative study of differential systems (see [4, 7]). In [19], Sibirsky showed that for two $n-$dimensional polynomial differential systems to be affine equivalent with respect to the group $SL(n,\mathbb{R})$ (or $O(n,\mathbb{R})$) it is necessary and sufficient that their absolute invariants with respect to this group coincide. In [23], the authors obtained a complete classification to the affine equivalence of planar homogeneous quadratic differential systems, which is characterized by algebraic invariants conditions (see also [1]). In [14], Popa established a necessary and sufficient invariants conditions for the affine equivalence of the class of planer homogeneous quadratic differential systems with linear parts, which were used later by Sibirsky [19] for proving the existence of two foci of nonzero cyclicity for the same class of differential systems in case when the origin is a center.

Inspired by the aforementioned works, the aim of the present paper is to give necessary and sufficient invariants conditions for the affine equivalence of some classes of planar cubic differential systems with respect to the affine group $SL(2,\mathbb{R})$, we deduce the minimal rational base for each one of these classes after having constructed normal forms. These kind of bases is a useful tool for qualitative study of polynomial differential systems[13]. For instance, in [11] the authors showed how to reduce the expression of the conditions for existence of a center (center-focus problem) of bidimensional polynomial systems of differential equations with nonlinearities of fourth degree with the help of minimal rational basis associated with this system.

The article is organized as follows. In section 2, we present some preliminary results and notations. In section 3, we construct canonical forms for the general planar cubic differential systems, then we establish

*Mathematics Subject Classifications: 34C20, 15A72, 13A50.

†Faculty of Mathematics, University of Sciences and Technology Houari Boumediene, PO Box 32 El Alia 16111 Bab Ezzouar Algiers Algeria

‡Department of Mathematics and Computer Science, University Center Aflou, PO Box 306 Aflou Laghouat Algeria

necessary and sufficient invariants conditions for the affine equivalence of some classes with respect to the affine group $SL(2,\mathbb{R})$. In the last section, we deduce the minimal rational base for each one of these classes.

2 Preliminaries and Notations

Using Einstein's notation [19], a polynomial differential system of degree at most k with coefficients in $\mathbb{R}$ is written as:

$$\frac{dx^j}{dt} = a^j + \sum_{r=1}^{k} a^j_{\alpha_1\alpha_2\ldots\alpha_r} x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_r}, \quad j, \alpha_1, \cdots, \alpha_r \in \{1, 2, ..., n\}, \tag{1}$$

where for $j = \overline{1,n}$ and for $r = \overline{0,k}$, $a^j_{\alpha_1\alpha_2\ldots\alpha_r} \in \mathcal{T}^1_r$, here $\mathcal{T}^1_r$ denotes the space of tensors 1 time contravariant and r times covariant symmetric with respect to the lower subscripts. The space $\mathcal{T}^1_r$ corresponds to the homogeneous part of degree r of the polynomials of the right hand side of system (1). We denote by $\mathcal{C}(n,k,\mathbb{R})$ the dimensional coefficient space of system (1) and by a the vector of coefficients $a = \left(a^1, a^2, ..., a^n_{nn\ldots n}\right)$.

The action of the general linear group $GL(n,\mathbb{R})$ on $\mathbb{R}^n : (Q,x) \mapsto Qx$, induces the representation:

$$\begin{array}{llll} \rho: & GL(n,\mathbb{R}) & \longrightarrow & GL(\mathcal{C}(n,k,\mathbb{R})) \\ & Q & \longmapsto & \rho(Q), \end{array}$$

defined by

$$\rho(Q)a^j_{\alpha_1\alpha_2\ldots\alpha_r} \quad = \quad Q^j_i P^{\beta_1}_{\alpha_1} P^{\beta_2}_{\alpha_2} \ldots P^{\beta_r}_{\alpha_r} a^i_{\beta_1\beta_2\ldots\beta r}, \tag{2}$$

where $j, \alpha_1\alpha_2 \ldots \alpha_r = \overline{1,n}$, $r = \overline{0,k}$, and Q is a matrix of $GL(n,\mathbb{R})$ and P its inverse. The formula (2) is called the formula of the centro-affine transformations.

Definition 1 *A polynomial function $C(a,x) : \mathcal{C}\ (n,k,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ is called a covariant with respect to the group $GL(n,\mathbb{R})$ or $GL(n,\mathbb{R})$-covariant of (1) if there exists a character Λ of the linear group $GL(n,\mathbb{R})$ such that*

$$\forall Q \in GL(n,\mathbb{R}), \forall a \in \mathcal{C}(n,k,\mathbb{R}),\ C(\rho(Q)a, \rho(Q)x) = \Lambda(Q)C(a,x).$$

If $\Lambda \equiv 1$, the covariant is said to be absolute, otherwise it is said to be relative. If the $C(a,x)$ is constant with respect to x, then it is said to be a $GL(n,\mathbb{R})$-invariant.

According to [19], the character $\Lambda\ (Q)$ has the form $\Lambda\ (Q) = \det(Q)^{-w}$, where w is an integer, called the weight of covariant $C(a,x)$.

Definition 2 *A $GL(n,\mathbb{R})$-covariant $C(a,x)$ is said to be reducible if it can be expressed as a polynomial function of $GL(n,\mathbb{R})$-covariants of lower degree. If $C(a,x)$ is reducible, we write $C(a,x) \equiv 0$ (modulo $GL(n,\mathbb{R})$).*

Definition 3 *A finite family B of $GL(n,\mathbb{R})$-covariants of (1) is called a system of generators if any $GL(n,\mathbb{R})$-covariant of (1) is reducible to zero modulo B. A system B of generators is said to be minimal if none of them is generated by the others.*

Now, let us consider the tensor product (see [2]):

$$(\mathcal{T}^1_0)^{\otimes d_0} \otimes (\mathcal{T}^1_1)^{\otimes d_1} \otimes \cdots \otimes (\mathcal{T}^1_r)^{\otimes d_r} \otimes \mathbb{R}^{\otimes \delta},\ 1 \leq r \leq k. \tag{3}$$

Then, we have the following result from [8].

Theorem 1 ([8]) *The expressions obtained with the help of successive alternations and complete contraction over the tensor products (3) form a system of generators of centro-affine covariants of (1).*

Definition 4 *A centro-affine covariant of (1) is said to be of type $(d_0, d_1, \ldots, d_r, \delta)$ if it is homogeneous of degree d_i $(i = 0, 1, ..., r)$ with respect to coordinates of $a^j_{\alpha_1\alpha_2\cdots\alpha_i}$ and of degree δ with respect to coordinates of $x \in \mathbb{R}^n$.*

Remark 1 *An invariant constructed with the help of Theorem 1 is called generator invariant.*

We denote by $\mathcal{A}(n,k,\mathbb{R})$ the $\mathbb{R}-$algebra of the centro-affine covariants of (1), this algebra is multigraded of finite type [9], hence

$$\mathcal{A}(n,k,\mathbb{R}) = \oplus_{d_0,d_1,d_2,\ldots,d_r\in\mathbb{N}}\mathcal{A}_{(d_0,d_1,d_2,\ldots,d_r,\delta)} \tag{4}$$

and $\mathcal{A}_{(d_0,d_1,d_2,\ldots,d_r,\delta)}$ is the finite linear space of the centro-affine covariants of type $(d_0,d_1,d_2,...,d_r,\delta)$.

Example 1 *If $k=n=2$, the polynomials $a^{\beta}a^{\alpha}_{\alpha\beta}$ and $a^{\alpha}_{p}a^{\beta}_{\beta q}a^{\gamma}_{\alpha\gamma}\varepsilon^{pq}$ are generator invariants with respect to the linear group $GL(2,\mathbb{R})$ belonging respectively to $\mathcal{A}_{(1,0,1,0)}$ and $\mathcal{A}_{(0,1,2,0)}$.*

The minimal system of generators for (1) in the case $k=n=2$ was established by Vulpe [21], we shall use the following elements of this basis :

$$\begin{array}{lll}
I_1 = a^{\alpha}_{\alpha}, & I_{17} = a^{\beta}a^{\alpha}_{\alpha\beta}, & I_{22} = a^{\alpha}_{\alpha\beta}a^{\beta}_{\gamma\delta}a^{\gamma}a^{\delta}, \\
I_2 = a^{\alpha}_{\beta}a^{\beta}_{\alpha}, & I_{18} = a^{\alpha}a^{q}a^{p}_{\alpha}\varepsilon_{pq}, & I_{25} = a^{\alpha}_{\alpha p}a^{\beta}_{\gamma q}a^{\gamma}_{\beta\delta}a^{\delta}\varepsilon^{pq}, \\
I_4 = a^{\alpha}_{p}a^{\beta}_{\beta q}a^{\gamma}_{\alpha\gamma}\varepsilon^{pq}, & I_{20} = a^{\alpha}_{\gamma}a^{\beta}_{\alpha\beta}a^{\gamma}, & I_{26} = a^{\alpha}_{\alpha p}a^{\beta}_{\gamma q}a^{\gamma}_{\beta\delta}a^{\delta}\varepsilon^{pq}. \\
I_9 = a^{\alpha}_{pr}a^{\beta}_{\beta q}a^{\gamma}_{\gamma s}a^{\delta}_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs} & I_{21} = a^{p}_{\alpha\beta}a^{\alpha}a^{\beta}a^{q}\varepsilon_{pq}, &
\end{array}$$

Let us consider the family $\{J_1,\ldots,J_{61}\}$ of $GL\,(2,\mathbb{R})-$invariants for general cubic differential system given by

$$\begin{array}{lll}
J_1 = a^{\alpha}_{p}a^{\beta}_{\beta\alpha q}\varepsilon^{pq}, & J_{11} = a^{\alpha}_{\beta\gamma}a^{\beta}_{pr}a^{\gamma}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{21} = a^{\alpha}a^{\beta}_{\delta p}a^{\gamma}_{\gamma\beta r}a^{\delta}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, \\
J_2 = a^{\alpha}_{\alpha pr}a^{\beta}_{\beta qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{12} = a^{\alpha}_{\gamma pr}a^{\beta}_{\alpha qk}a^{\gamma}_{\beta sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, & J_{22} = a^{\alpha}a^{\beta}_{\beta p}a^{\gamma}_{\gamma\delta r}a^{\delta}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, \\
J_3 = a^{\alpha}_{\beta pr}a^{\beta}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{13} = a^{\alpha}_{\gamma\beta p}a^{\beta}_{\alpha rk}a^{\gamma}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, & J_{23} = a^{\alpha}a^{\beta}_{pr}a^{\gamma}_{\gamma\delta q}a^{\delta}_{\alpha\beta s}\varepsilon^{pq}\varepsilon^{rs}, \\
J_4 = a^{\alpha}a^{\beta}a^{\gamma}_{\gamma\alpha\beta}, & J_{14} = a^{\alpha}a^{\beta}a^{\gamma}_{\beta}a^{\delta}_{\delta\alpha\gamma}, & J_{24} = a^{\alpha}_{\delta p}a^{\beta}_{\beta q}a^{\gamma}_{\gamma rk}a^{\delta}_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_5 = a^{\alpha}a^{\beta}_{\gamma p}a^{\gamma}_{\beta\alpha q}\varepsilon^{pq}, & J_{15} = a^{\alpha}a^{\beta}a^{\gamma}_{\alpha\beta p}a^{\delta}_{\delta\gamma q}\varepsilon^{pq}, & J_{25} = a^{\alpha}_{\beta p}a^{\beta}_{\gamma q}a^{\gamma}_{\alpha rk}a^{\delta}_{\delta sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_6 = a^{\alpha}a^{\beta}_{\alpha p}a^{\gamma}_{\beta\gamma q}\varepsilon^{pq}, & J_{16} = a^{\alpha}a^{\beta}_{\delta}a^{\gamma}_{\gamma p}a^{\delta}_{\beta\alpha q}\varepsilon^{pq}, & J_{26} = a^{\alpha}_{\delta p}a^{\beta}_{\beta r}a^{\gamma}_{\gamma qk}a^{\delta}_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_7 = a^{\alpha}a^{\beta}_{\beta p}a^{\gamma}_{\gamma\alpha q}\varepsilon^{pq}, & J_{17} = a^{\alpha}a^{\beta}_{\gamma}a^{\gamma}_{\delta p}a^{\delta}_{\beta\alpha q}\varepsilon^{pq}, & J_{27} = a^{\alpha}_{\gamma p}a^{\beta}_{\beta r}a^{\gamma}_{\delta qk}a^{\delta}_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_8 = a^{\alpha}_{\gamma p}a^{\beta}_{\alpha r}a^{\gamma}_{\beta qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{18} = a^{\alpha}a^{\beta}_{\delta}a^{\gamma}_{\alpha p}a^{\delta}_{\gamma\beta q}\varepsilon^{pq}, & J_{28} = a^{\alpha}_{rk}a^{\beta}_{\delta p}a^{\gamma}_{\gamma sl}a^{\delta}_{\beta\alpha q}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_9 = a^{\alpha}_{\alpha p}a^{\beta}_{\gamma r}a^{\gamma}_{\beta qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{19} = a^{\alpha}a^{\beta}_{\gamma}a^{\gamma}_{\alpha p}a^{\delta}_{\delta\beta q}\varepsilon^{pq}, & J_{29} = a^{\alpha}_{pr}a^{\beta}_{\delta k}a^{\gamma}_{qsl}a^{\delta}_{\gamma\beta\alpha}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{10} = a^{\alpha}_{\beta p}a^{\beta}_{\alpha r}a^{\gamma}_{\gamma qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{20} = a^{\alpha}a^{\beta}_{\alpha}a^{\gamma}_{\delta p}a^{\delta}_{\gamma\beta q}\varepsilon^{pq}, & J_{30} = a^{\alpha}a^{\beta}a^{\gamma}a^{p}a^{q}_{\alpha\beta\gamma}\varepsilon_{pq},
\end{array}$$

$$\begin{array}{ll}
J_{31} = a^{\alpha}a^{\beta}a^{\gamma}a^{\delta}_{\delta\mu}a^{\mu}_{\alpha\beta\gamma}, & J_{42} = a^{\alpha}a^{\beta}_{\gamma\delta}a^{\gamma}_{\mu p}a^{\delta}_{\beta r}a^{\mu}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, \\
J_{32} = a^{\alpha}a^{\beta}a^{\gamma}a^{\delta}_{\gamma\beta}a^{\mu}_{\mu\alpha\delta}, & J_{43} = a^{\alpha}a^{\beta}_{rk}a^{\gamma}_{\gamma\delta p}a^{\delta}_{\beta\mu q}a^{\mu}_{\alpha ls}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{33} = a^{\alpha}a^{\beta}a^{\gamma}_{\mu\gamma}a^{\delta}_{\delta p}a^{\mu}_{\beta\alpha q}\varepsilon^{pq}, & J_{44} = a^{\alpha}a^{\beta}_{rk}a^{\gamma}_{\mu\delta p}a^{\delta}_{\beta\alpha q}a^{\mu}_{\gamma ls}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{34} = a^{\alpha}a^{\beta}a^{\gamma}_{\gamma\alpha p}a^{\delta}_{\beta rq}a^{\mu}_{\mu\delta s}\varepsilon^{pq}\varepsilon^{rs}, & J_{45} = a^{\alpha}a^{\beta}_{rk}a^{\gamma}_{\gamma\mu p}a^{\delta}_{\beta\alpha q}a^{\mu}_{\delta ls}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{35} = a^{\alpha}a^{\beta}_{\mu\delta}a^{\gamma}_{\gamma\beta}a^{\delta}_{pr}a^{\mu}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{46} = a^{\alpha}a^{\beta}_{pr}a^{\gamma}_{\mu\delta q}a^{\delta}_{\beta\gamma k}a^{\mu}_{\alpha ls}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{36} = a^{\alpha}a^{\beta}_{\mu\gamma}a^{\gamma}_{\delta\beta}a^{\delta}_{pr}a^{\mu}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{47} = a^{\alpha}_{\mu\delta}a^{\beta}_{\beta\gamma}a^{\gamma}_{\alpha p}a^{\delta}_{rk}a^{\mu}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{37} = a^{\alpha}a^{\beta}_{\beta\mu}a^{\gamma}_{\gamma\delta}a^{\delta}_{pr}a^{\mu}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{48} = a^{\alpha}_{\mu\gamma}a^{\beta}_{\beta\delta}a^{\gamma}_{\alpha p}a^{\delta}_{rk}a^{\mu}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{38} = a^{\alpha}a^{\beta}_{\mu\gamma}a^{\gamma}_{\delta\alpha}a^{\delta}_{pr}a^{\mu}_{\beta qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{49} = a^{\alpha}_{\delta\gamma}a^{\beta}_{\beta\mu}a^{\gamma}_{\alpha p}a^{\delta}_{rk}a^{\mu}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{39} = a^{\alpha}a^{\beta}_{\mu\delta}a^{\gamma}_{\beta\alpha}a^{\delta}_{pr}a^{\mu}_{\gamma qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{50} = a^{\alpha}_{\mu\beta}a^{\beta}_{\gamma\delta}a^{\gamma}_{\alpha p}a^{\delta}_{rk}a^{\mu}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{40} = a^{\alpha}a^{\beta}_{\mu\gamma}a^{\gamma}_{\beta\alpha}a^{\delta}_{pr}a^{\mu}_{\delta qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{51} = a^{\alpha}_{\gamma\beta}a^{\beta}_{\mu\delta}a^{\gamma}_{\alpha p}a^{\delta}_{rk}a^{\mu}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{41} = a^{\alpha}a^{\beta}_{\mu\delta}a^{\gamma}_{\gamma p}a^{\delta}_{\beta r}a^{\mu}_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}, & J_{52} = a^{\alpha}_{\mu\delta}a^{\beta}_{\beta\alpha}a^{\gamma}_{\gamma p}a^{\delta}_{rk}a^{\mu}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{53} = a^{\alpha}_{\mu\gamma}a^{\beta}_{\beta\alpha}a^{\gamma}_{\delta p}a^{\delta}_{rk}a^{\mu}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, & J_{57} = a^{\alpha}a^{\beta}_{\nu\delta}a^{\gamma}_{\mu p}a^{\delta}_{\gamma r}a^{\mu}_{\beta qk}a^{\nu}_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{54} = a^{\alpha}_{\mu\delta}a^{\beta}_{\gamma p}a^{\gamma}_{\beta r}a^{\delta}_{\alpha k}a^{\mu}_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, & J_{58} = a^{\alpha}a^{\beta}_{\nu\gamma}a^{\gamma}_{\mu p}a^{\delta}_{\delta r}a^{\mu}_{\beta qk}a^{\nu}_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{55} = a^{\alpha}a^{\beta}a^{\gamma}a^{\delta}a^{\mu}_{\mu\delta\nu}a^{\nu}_{\alpha\beta\gamma}, & J_{59} = a^{\alpha}a^{\beta}_{\delta\mu}a^{\gamma}_{\nu p}a^{\delta}_{\beta r}a^{\mu}_{\gamma qk}a^{\nu}_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \\
J_{56} = a^{\alpha}a^{\beta}a^{\gamma}a^{\delta}_{\delta\nu}a^{\mu}_{\gamma\mu p}a^{\nu}_{\alpha\beta q}\varepsilon^{pq}, & J_{60} = a^{\alpha}a^{\beta}_{\nu\delta}a^{\gamma}_{\gamma p}a^{\delta}_{\beta r}a^{\mu}_{\mu qk}a^{\nu}_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl},
\end{array}$$

$$J_{61} = a^\alpha a^\beta_{\gamma\mu} a^\gamma_{\nu p} a^\delta_{\beta r} a^\mu_{\delta qk} a^\nu_{\alpha sl} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{kl}.$$

Henceforth, we denote by $\mathcal{B}$ the set

$$\{I_1, I_2, I_4, I_9, I_{17}, I_{18}, I_{20}, I_{21}, I_{22}, I_{25}, I_{26}, J_1, \dots, J_{61}\}.$$

3 Affine Equivalence of Planar Cubic Differential Systems

In this section, by using the constructed invariants in the preceding section, we give necessary and sufficient invariants conditions for the affine equivalence of the general planar cubic differential systems with respect to affine group $SL(2,\mathbb{R})$ in the case $J_4 \neq 0$, $J_7 \neq 0$ and $I_{17} \neq 0$. First, consider the general planar cubic differential systems

$$\frac{dx^j}{dt} = a^j + a^i_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta + a^j_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma,\ j, \alpha, \beta, \gamma = 1, 2, \tag{5}$$

which ia a particular case of systems (1) where $n = 2$ and $k = 3$.

Definition 5 *We say that two families $S\left(a^{(1)}\right)$ and $S\left(a^{(2)}\right)$ of systems (1) (with the same degree k and dimension n) are affine equivalent with respect to affine group $SL(n,\mathbb{R})$ (or $SL(n,\mathbb{R})$-equivalent) if there exists a transformation $Q \in SL(n,\mathbb{R})$ such that $\rho(Q)a^{(1)} = a^{(2)}$.*

Lemma 1 *The elements of the set $\mathcal{B}$ are centro-affine invariants of the cubic differential system 5 which are polynomially independent.*

Proof. We shall prove that it is not possible to write any element of this family polynomially by means of others, to this end, we use an algorithmic method which is based on reducing the polynomial decomposition to a linear one(see [2, 6, 22]), for instance I_2, I_1 are polynomially independent according to their linear decompositions (see Table 3), where $\mathcal{A}_{(0,2,0,0)} = \{I_1^2, I_2\}$, obviously I_2 cannot be generated from I_1 since $2a^1_2 a^2_1$ is a term of I_2 but not of I_1^2. In this case, it suffices to find a product with nonzero coefficients, which do not come into any other invariants of $\mathcal{B}$ or their products, this proof is similar to the one in [19, p. 41-42].

Indeed, if one invariant of type $T = (d_0, d_1, d_2, d_3)$ can be generated by the others, it can be expressed as a linear combination of the generating family $\mathcal{A}_{(d_0,d_1,d_2,d_3)}$, for example $K = a^p_r a^q_s \varepsilon_{pq} \varepsilon^{ps}$ of type $T = (0, 2, 0)$ can be expressed linearly by $\mathcal{A}_{(0,2,0)}$, hence $K = 2det(a^i_j)_{j,i=1,2} = I_1^2 - I_2$.

Product	I_1^2	I_2	K
$(a^1_1)^2$	1	1	0
$a^1_1 a^2_2$	2	0	2
$a^1_2 a^2_1$	0	2	-2
$(a^2_2)^2$	1	1	0

Table 3:

The same conclusion for $\{J_{12}, J_{13}\} \in \mathcal{A}_{(0,0,0,3)}$ and $\{J_{24}, J_{25}, J_{26}\} \in \mathcal{A}_{(0,0,2,2)}$, which is clear from table 4 that are polynomially independent. It follows easily that $\alpha_1 J_{12} + \alpha_2 J_{13} \equiv 0$ and $\beta_1 J_{24} + \beta_2 J_{25} + \beta_3 J_{26} \equiv 0$ only when $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = \beta_3 = 0$.

Product	J_{13}	J_{12}
$a^1_{111} a^1_{112} a^2_{222}$	1	0
$a^1_{111} a^1_{122} a^2_{122}$	-2	-3
$a^1_{111} a^1_{222} a^2_{112}$	-1	3

Product	J_{26}	J_{25}	J_{24}
$(a^1_{11})^2 a^1_{112} a^2_{222}$	-1	0	0
$a^1_{11} a^1_{22} a^1_{111} a^2_{122}$	-1	1	-1
$a^1_{11} a^2_{12} a^1_{111} a^1_{222}$	-1	-1	-1

Table 4:

For the remaining elements of $\mathcal{B}$, the proof process is the same by refereeing to tables 6 and 7 in the annexe. ■

Lemma 2 *If $I_{17} \neq 0$, then the system (5) can be reduced by means of centro-affine transformations to the form:*

$$\frac{dy^j}{dt} = \sum_{l+h=0}^{3} \frac{1}{I_{17}^{h+j-1}} K^j_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}} \left(y^1\right)^h \left(y^2\right)^l,\ j = 1, 2. \tag{6}$$

where

$$K^1 = 0, \quad K^2_2 = I_{20}, \quad K^1_1 = I_1 I_{17} - I_{20}, \quad K^1_{22} = I_{21},$$

$$K^2 = I_{17}, \quad K^1_{11} = I_{26}, \quad K^1_{12} = I^2_{17} - I_{22}, \quad K^2_{11} = I_9,$$

$$K^2_1 = I_4, \quad K^1_{222} = -J_{30}, \quad K^1_{122} = I_{17}J_4 - J_{31}, \quad K^2_{12} = -I_{26},$$

$$K^1_2 = I_{18}, \quad K^2_{222} = J_{31}, \quad K^1_{112} = J_{33} - I_{17}J_7, \quad K^2_{22} = I_{22},$$

$$K^1_{111} = J_{35} - J_{37} - J_{38} - J_{39} + J_{40} - J_{41} + I_{17}(J_{11} - 2J_8 + 2J_9 + J_{10}),$$

$$K^2_{111} = 2J_{47} - 2J_{48} - J_{49} - J_{50} + 2J_{51} - 3J_{52} + 3J_{53} - J_{54},$$

$$K^2_{122} = -J_{33},$$

$$K^2_{112} = J_{37} - J_{35} + J_{38} + J_{39} - J_{40} + J_{41} - I_{17}J_{11}.$$

Proof. Since $I_{17} \neq 0$, we can consider the following matrix

$$Q_{17}(a) = \begin{pmatrix} a^2 & -a^1 \\ \frac{1}{I_{17}} a^\alpha_{\alpha 1} & \frac{1}{I_{17}} a^\alpha_{\alpha 2} \end{pmatrix}. \tag{7}$$

One can easily check that $\det(Q_{17}(a)) = 1$, therefore the system (5) can be transformed by the centro-affine transformation $y^j = Q_{17}(a)^j_i x^i$, $i, j = 1, 2$, into a new system, where the new coefficients are given by [20]

$$\begin{aligned} \rho(Q_{17}) a^1_{\underbrace{11..1}_{h\ \text{times}}\ \underbrace{22..2}_{l\ \text{times}}} &= \frac{1}{I^h_{17}} a^q a^p_{\beta_1\ldots\beta_h\delta_1\delta_2\ldots\delta_l} \prod_{i=1}^{h} a^{\alpha_i}_{\alpha_i k_i} \varepsilon^{\beta_i k_i} \times \prod_{j=1}^{l} a^{\delta_j} \varepsilon_{pq}, \\ \rho(Q_{17}) a^2_{\underbrace{11..1}_{h\ \text{times}}\ \underbrace{22..2}_{l\ \text{times}}} &= \frac{1}{I^{h+1}_{17}} a^\alpha a^\beta_{\beta\alpha i} a^i_{\beta_1\ldots\beta_h\delta_1\delta_2\ldots\delta_l} \prod_{i=1}^{h} a^{\alpha_i}_{\alpha_i k_i} \varepsilon^{\beta_i k_i} \times \prod_{j=1}^{l} a^{\delta_j}, \end{aligned} \tag{8}$$

where $l + h = 0, 1, 2, 3$ and $h, l = 0, 1, 2, 3$.

The proof is completed by expressing the elements $I^{h+j-1}_{17} \rho(Q_{I_{17}}) a^j_{\underbrace{11..1}_{h\text{times}}\underbrace{22..2}_{l\text{times}}}$ as a polynomial function of the family $\mathcal{B}$. For instance, choosing the element $I^2_{17}\rho(Q_{17})a^1_{112} = a^q a^\alpha a^\beta_{\beta r} a^\gamma_{\gamma k} a^p_{\alpha s l} \varepsilon_{pq} \varepsilon^{rs} \varepsilon^{kl}$ and $I^2_{17}\rho(Q_{17})a^2_{122} = a^\alpha a^\beta a^\gamma_{\gamma\mu} a^\delta_{\delta q} a^\mu_{\beta\alpha q} \varepsilon^{pq}$, which are obtained from (8) at $h = 2, l = 1$ and $h = 1, l = 2$ Respectively and

$$I^2_{17}\rho(Q_{17})a^2_{122}, I^2_{17}\rho(Q_{17})a^1_{112} \in \mathcal{A}_{(2,0,2,1)} = \{I_{17}J_7, I_{17}J_6, I_{17}J_5, J_{33}\}.$$

Now by using the method developed in[6], we can easily express these elements as a linear combination of $\mathcal{A}_{(2,0,2,1)}$ (see table 5)

Product	$I_{17}J_7$	J_{33}	$I_{17}^2\rho(Q_{17})a_{112}^1$	$I_{17}^2\rho(Q_{17})a_{122}^2$
$(a^1)^2(a_{11}^1)^2a_{112}^1$	1	1	0	-1
$(a^1)^2(a_{11}^1)^2a_{122}^2$	1	0	-1	0
$(a^1)^2a_{11}^1a_{12}^1a_{111}^1$	-1	-1	0	1
$(a^1)^2a_{11}^1a_{12}^1a_{112}^2$	-1	1	2	-1
$(a^1)^2a_{11}^1a_{12}^2a_{112}^1$	2	2	0	-2

Table 5:

It is easy to check that: $I_{17}^2\rho(Q_{17})a_{112}^1 = J_{33} - I_{17}J_7 = K_{112}^1$ and $I_{17}^2\rho(Q_{17})a_{122}^2 = -J_{33} = K_{122}^2$. Similarly, we obtain other elements expressions, for $l+h = 0,1,2,3,\ \ h,l = 0,1,2,3$ we find

$$I_{17}^{h+j-1}\rho(Q_{17})a^j_{\underbrace{11..1}_{h \text{ times}}\underbrace{22..2}_{l \text{ times}}} = K^j_{\underbrace{11..1}_{h \text{ times}}\underbrace{22..2}_{l \text{ times}}},\ j = 1,2$$

which concludes the proof. ■

Lemma 3 *If $J_4 \neq 0$, then the system (5) can be reduced by means of centro-affine transformations to the form:*

$$\frac{dy^j}{dt} = \sum_{l+h=0}^{3} \frac{1}{J_4^{h+j-1}} H^j_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}} \left(y^1\right)^h \left(y^2\right)^l,\ j = 1,2. \tag{9}$$

where

$$H^1 = 0,\quad H_{22}^2 = J_{32},\quad H_1^1 = I_1J_4 - J_{14},\quad H_{122}^2 = J_4J_{15} - \tfrac{1}{2}J_2J_{30},$$

$$H^2 = J_4,\quad H_{22}^1 = I_{21},\quad H_{12}^1 = I_{17}J_4 - J_{32},\quad H_{12}^2 = J_4J_6 + \tfrac{1}{2}I_{21}J_2,$$

$$H_2^2 = J_{14},\quad H_{222}^1 = -J_{30},\quad H_{122}^1 = J_4^2 - J_{55},\quad H_{112}^1 = \tfrac{1}{2}J_2J_{30} - J_4J_{15},$$

$$H_2^1 = I_{18},\quad H_{222}^2 = J_{55},\quad H_1^2 = J_1J_4 + \tfrac{1}{2}I_{18}J_2,\quad H_{111}^1 = J_4J_{34} + \tfrac{1}{2}J_2J_{55},$$

$$H_{112}^2 = \frac{1}{2}J_2(J_4^2 - J_{55}) - J_4J_{34}),$$

$$H_{11}^1 = -J_4J_6 + J_4J_7 - \frac{1}{2}I_{21}J_2,$$

$$H_{111}^2 = \frac{1}{4}J_2^2J_{30} - J_4(J_2J_{15} + J_4(J_{12} + J_{13})),$$

$$H_{11}^2 = J_4(J_{23} - J_{21} + J_{22}) + \frac{1}{2}J_2(I_{17}J_4 - J_{32}).$$

Proof. The proof of this lemma is almost the same as that of Lemma 2. Since $J_4 \neq 0$, it suffices to consider the matrix of centro-affine transformations:

$$Q_4(a) = \begin{pmatrix} a^2 & -a^1 \\ \frac{1}{J_4}a^\alpha a^\beta_{\beta\alpha 1} & \frac{1}{J_4}a^\alpha a^\beta_{\beta\alpha 2} \end{pmatrix},$$

clearly $\det\left(Q_4(a)\right) = 1$. In the same manner, by applying the formula of centro-affine transformations, for $l+h = 0,1,2,3$ and $h,l = 0,1,2,3$ we obtain

$$\rho(Q_4)a^1_{\underbrace{11..1}_{h \text{ times}}\underbrace{22..2}_{l \text{ times}}} = \frac{1}{J_4^h}a^q a^p_{\beta_1\ldots\beta_h\delta_1\delta_2\ldots\delta_l}\prod_{i=1}^{h}a^{\alpha_i}a^{\lambda_i}_{\lambda_i\alpha_i k_i}\varepsilon^{\beta_i k_i} \times \prod_{j=1}^{l}a^{\delta_j}\varepsilon_{pq},$$

$$\rho(Q_4)a^2_{\underbrace{11..1}_{h \text{ times}}\underbrace{22..2}_{l \text{ times}}} = \frac{1}{J_4^{h+1}}a^\alpha a^\beta_{\beta\alpha i}a^i_{\beta_1\ldots\beta_h\delta_1\delta_2\ldots\delta_l}\prod_{i=1}^{h}a^{\alpha_i}a^{\lambda_i}_{\lambda_i\alpha_i k_i}\varepsilon^{\beta_i k_i} \times \prod_{j=1}^{l}a^{\delta_j}.$$

The proof is completed by expressing these elements as polynomial function of the family $\mathcal{B}$, which lead to the systems 9

$$J_4^{h+j-1}\rho(Q_4)a^j_{\underbrace{11..1}_{h\ \text{times}}\underbrace{22..2}_{l\ \text{times}}} = H^j_{\underbrace{11..1}_{h\ \text{times}}\underbrace{22..2}_{l\ \text{times}}}.$$

for $l+h=0,1,2,3$ and $h,l=0,1,2,3$ and $j=1,2$. ■

Lemma 4 *If $J_7 \neq 0$, then the system (5) can be reduced by means of centro-affine transformations to the form:*

$$\frac{dy^j}{dt} = \sum_{l+h=0}^{3} \frac{1}{J_7^{h+j-1}} F^j_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}} \left(y^1\right)^h \left(y^2\right)^l,\ j=1,2. \tag{10}$$

where

$$F^2_2 = J_{16} - J_{17} - J_{18} + J_{19} + J_{20} - I_{17}J_1, \quad F^1 = I_{17},$$

$$F^1_1 = I_{17}J_1 + I_1J_7 - J_{16} + J_{17} + J_{18} - J_{19} - J_{20}, \quad F^2 = J_4,$$

$$F^1_{11} = J_4(J_{10} + J_{11} - J_8) - \tfrac{1}{2}I_{22}J_2, \quad F^1_2 = -I_4,$$

$$F^2_{12} = J_4(2J_9 - J_8 - J_{11}) + \tfrac{1}{2}J_2(I_{22} - I_{17}^2), \quad F^1_{22} = I_9,$$

$$F^2_{111} = \tfrac{1}{4}J_2^2J_{30} - J_4^2(J_{12} + J_{13}) - J_2J_4J_{15}, \quad F^2_1 = J_1J_4 + \tfrac{1}{2}I_{18}J_2,$$

$$F^1_{12} = J_{35} - J_{36} - J_{37} + J_{38} + J_{39} - J_{40} + J_{42} - I_{17}J_{11},$$

$$F^2_{22} = -J_{35} + J_{36} + J_{37} - J_{38} - J_{39} + J_{40} - J_{42} + I_{17}J_{11},$$

$$F^2_{11} = J_4(J_{22} + J_{23} - J_{21}) + \frac{1}{2}J_2(I_{17}J_4 - J_{32}),$$

$$F^1_{111} = J_4(\frac{1}{2}(J_{43} + 3J_{44} - J_{45} + J_2J_7) - J_{46} - \frac{3}{4}J_3(J_5 - J_7) - I_{17}(J_{13} + \frac{1}{3}J_{12})) + \frac{1}{2}J_2J_{56},$$

$$F^1_{112} = J_4(\frac{1}{2}(J_{24} - J_{25}) - J_{26} - J_{27} - J_{28} + J_{29}) - \frac{1}{2}J_2J_{33},$$

$$F^1_{122} = 2(J_{58} - J_{57} + J_{59} - J_{61}) + I_{17}(J_{26} + J_{27} + J_{28} - J_{29} - \frac{3}{2}J_{24} - \frac{1}{2}J_{25}) - J_{60} + \frac{1}{2}I_{25}J_2,$$

$$F^1_{222} = J_{49} + J_{50} - 2J_{47} + 2J_{48} - 2J_{51} + 3J_{52} - 3J_{53} + J_{54},$$

$$F^2_{112} = J_4(J_{46} + \frac{1}{2}(J_{45} - J_{43} - 3J_{44}) + \frac{3}{4}J_3(J_5 - J_7) + I_{17}(J_{13} + \frac{1}{3}J_{12})) - \frac{1}{2}J_2J_{56},$$

$$F^2_{122} = J_4(J_{26} + J_{27} + J_{28} - J_{29} + \frac{1}{2}(J_{25} - J_{24})) + \frac{1}{2}J_2(J_{33} - I_{17}J_7),$$

$$\begin{aligned} F^2_{222} &= J_7(2J_9 - 2J_8 + J_{10}) + 2(J_{57} - J_{58} - J_{59} + J_{61}) \\ &\quad + J_{60} + I_{17}(J_{29} - J_{26} - J_{27} - J_{28} + \frac{1}{2}(J_{25} + 3J_{24})) - \frac{1}{2}I_{25}J_2. \end{aligned}$$

Proof. Since $J_7 \neq 0$, it suffices to consider the matrix of centro-affine transformations

$$Q_7(a) = \begin{pmatrix} a^\alpha_{\alpha 1} & a^\alpha_{\alpha 2} \\ \frac{1}{J_7}a^\alpha a^\beta_{\alpha\beta 1} & \frac{1}{J_7}a^\alpha a^\beta_{\alpha\beta 2} \end{pmatrix},$$

we see immediately that $\det(Q_7(a)) = 1$, by the centro-affine transformations formula we obtain:

$$\rho(Q_7)a^1_{\underbrace{11..1}_{h\text{ times}}\underbrace{22..2}_{l\text{ times}}} = \frac{1}{J_7^h} a^\alpha_{\alpha i} a^i_{\beta_1\ldots\beta_h\delta_1\delta_2\ldots\delta_l} \prod_{i=1}^{h} a^{\alpha_i} a^{\lambda_i}_{\alpha_i\lambda_i k_i} \varepsilon^{\beta_i k_i} \times \prod_{j=1}^{l} a^{\nu_j}_{\nu_j\omega_j} \varepsilon^{\omega_j\delta_j},$$

$$\rho(Q_7)a^2_{\underbrace{11..1}_{h\text{ times}}\underbrace{22..2}_{l\text{ times}}} = \frac{1}{J_7^{h+1}} a^\alpha a^\beta_{\alpha\beta i} a^i_{\beta_1\ldots\beta_h\delta_1\delta_2\ldots\delta_l} \prod_{i=1}^{h} a^{\alpha_i} a^{\lambda_i}_{\alpha_i\lambda_i k_i} \varepsilon^{\beta_i k_i} \times \prod_{j=1}^{l} a^{\nu_j}_{\nu_j\omega_j} \varepsilon^{\omega_j\delta_j},$$

for $l + h = 0, 1, 2, 3$ and $h, l = 0, 1, 2, 3$. By expressing these elements as polynomial function of $\mathcal{B}$ yields

$$J_7^{h+j-1} \rho(Q_7)a^j_{\underbrace{11..1}_{h\text{ times}}\underbrace{22..2}_{l\text{ times}}} = F^j_{\underbrace{11..1}_{h\text{ times}}\underbrace{22..2}_{l\text{ times}}},$$

for $l + h = 0, 1, 2, 3$, $h, l = 0, 1, 2, 3$ and $j = 1, 2$, which leads to the system (10). ∎

Theorem 2 *The following statements hold:*

1) *Two planar cubic differential systems $S(a^{(1)})$ and $S(a^{(2)})$ such that $I_{17}^{(1)}.I_{17}^{(2)} \neq 0$ are $SL(2,\mathbb{R})-$ equivalent if, and only if,*

$$\left(I_{17}^{(2)}\right)^{h+j-1} K^{j\,(1)}_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}} = \left(I_{17}^{(1)}\right)^{h+j-1} K^{j\,(2)}_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}}.$$

2) *Two planar cubic differential systems $S(a^{(1)})$ and $S(a^{(2)})$ such that $J_4^{(1)}.J_4^{(2)} \neq 0$ are $SL(2,\mathbb{R})$-equivalent if, and only if,*

$$\left(J_4^{(2)}\right)^{h+j-1} H^{j\,(1)}_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}} = \left(J_4^{(1)}\right)^{h+j-1} H^{j\,(2)}_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}}.$$

3) *Two planar cubic differential systems $S(a^{(1)})$ and $S(a^{(2)})$ such that $J_7^{(1)}.J_7^{(2)} \neq 0$ are $SL(2,R)-$equivalent if, and only if,*

$$\left(J_7^{(2)}\right)^{h+j-1} F^{j\,(1)}_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}} = \left(J_7^{(1)}\right)^{h+j-1} F^{j\,(2)}_{\underbrace{11..1}_{h\ times}\underbrace{22..2}_{l\ times}},$$

for all $l + h = 1, 2, 3$, $h, l = 0, 1, 2, 3$.

Proof. Let $S(a^{(1)})$ and $S(a^{(2)})$ be two planar cubic differential systems such that $I_{17}^{(1)}.I_{17}^{(2)} \neq 0$. We assume $S(a^{(1)})$ and $S(a^{(2)})$ are $SL(2,\mathbb{R})-$equivalent (i.e. $\rho(Q)a^{(1)} = a^{(2)}$ for some Q in $SL(2,\mathbb{R})$), because the expressions

$$(I_{17})^{h+j-1} K^j_{\underbrace{11..1}_{h\text{ times}}\underbrace{22..2}_{l\text{ times}}},$$

are centro-affines invariants for $l + h = 1, 2, 3; h, l = 0, 1, 2, 3$, then, it is clear that

$$\left(I_{17}^{(2)}\right)^{h+j-1} K^{j\,(1)}_{\underbrace{11..1}_{h\text{ times}}\underbrace{22..2}_{l\text{ times}}} = \left(I_{17}^{(1)}\right)^{h+j-1} K^{j\,(2)}_{\underbrace{11..1}_{h\text{ times}}\underbrace{22..2}_{l\text{ times}}}. \tag{11}$$

In the same way, we prove the sufficient condition in the cases (2) and (3). Conversely, if we have two planar cubic differential systems $S(a^{(1)})$ and $S(a^{(2)})$ such that $I_{17}^{(1)}.I_{17}^{(2)} \neq 0$ and satisfy the equalities (11), then by denoting $Q_{17}^{(i)}$ $(i=1,2)$ the matrix $Q_{17}\left(a^{(i)}\right)$ associated with the system $S(a^{(1)})$, the equalities (11) mean that

$$\rho(Q_{17}^{(1)})a^{(1)} = \rho(Q_{17}^{(2)})a^{(2)}$$

therefore,

$$\begin{aligned}
\rho\left(\left(Q_{17}^{(2)}\right)^{-1} Q_{17}^{(1)})\right) a^{(1)} &= \rho\left(\left(Q_{17}^{(2)}\right)^{-1}\right) \rho\left(Q_{17}^{(1)})\right) a^{(1)} \\
&= \rho\left(\left(Q_{17}^{(2)}\right)^{-1}\right) \rho\left(Q_{17}^{(2)})\right) a^{(2)} \\
&= \rho\left(I_2\right) a^{(2)}, \qquad (I_2 \text{ denotes the identity matrix of order } 2) \\
&= a^{(2)}.
\end{aligned}$$

Hence, the systems $S(a^{(1)})$ and $S(a^{(2)})$ are $SL(2,\mathbb{R})-$equivalent by the transformation $\left(Q_{17}^{(2)}\right)^{-1} Q_{17}^{(1)}$. Similar proof for the case (2) (Resp. (3)) shows that the systems $S(a^{(1)})$ and $S(a^{(2)})$ are $SL(2,\mathbb{R})-$equivalent by the transformation $\left(Q_{4}^{(2)}\right)^{-1} Q_{4}^{(1)}$ (Resp. $\left(Q_{7}^{(2)}\right)^{-1} Q_{7}^{(1)}$). ■

4 Minimal Rational Basis of Cubic Polynomial Differential Systems

In this section, we shall deduce the minimal rational bases associated with the normal forms which have been constructed in the previous section.

Definition 6 *([3]) A set S of $GL(n,\mathbb{R})-$invariants is called a rational on $M \subset \mathcal{C}(n,k,\mathbb{R})$ basis of invariants for system (1) with respect to the group $GL(n,\mathbb{R})$ if any invariants of system (1) with respect to the group $GL(n,\mathbb{R})$ can be expressed as a rational function of elements of the set S. And a rational basis on $M \subset A$ of invariants for system (1) with respect to a group $GL(n,\mathbb{R})$ is called minimal if by the removal from it of any comitant it ceases to be a rational basis.*

Theorem 3 *The following statements hold:*

1) *The set of $GL(2,\mathbb{R})$-invariants*

$$\mathcal{B}_1 = \left\{ \begin{array}{c} I_1, I_4, I_{17}, I_{20}, I_{21}, I_{26}, I_9, I_{22}, J_8, J_9, J_{10}, \\ J_{11}, J_{30}, J_{31}, J_{33}, J_4, J_7, J_{35}, J_{37}, J_{38}, J_{39}, \\ J_{40}, J_{41}, J_{47}, J_{48}, J_{49}, J_{50}, J_{51}, J_{52}, J_{53}, J_{54} \end{array} \right\}$$

is a minimal rational basis of the $GL(2,\mathbb{R})$-invariants of system (5) on

$$\mathcal{M} = \left\{a \in \mathcal{C}(2,3,\mathbb{R}); I_{17} \neq 0\right\}.$$

2) *The set of $GL(2,\mathbb{R})$-invariants*

$$\mathcal{B}_2 = \left\{ \begin{array}{c} J_4, J_6, I_{21}, J_2, J_{34}, J_{55}, J_{30}, J_{15}, I_1, J_{14}, J_1, J_{12}, \\ J_{13}, J_2, I_{18}, J_7, I_{17}, J_{32}, J_{21}, J_{22}, J_{23}, J_{32}, J_{55} \end{array} \right\}$$

is a minimal rational basis of the $GL(2,\mathbb{R})$-invariants of system (5) on

$$\mathcal{M} = \left\{a \in \mathcal{C}(2,3,\mathbb{R}); J_4 \neq 0\right\}.$$

3) The set of $GL(2,\mathbb{R})$-invariants

$$\mathcal{B}_3 = \left\{ \begin{array}{c} I_1, I_4, I_9, I_{17}, I_{18}, I_{22}, I_{25}, J_1, J_2, J_4, J_5, J_7, J_{11}, J_8, J_9, J_{10}, J_{11}, \\ J_{12}, J_{13}, J_{15}, J_{16}, J_{17}, J_{18}, J_{19}, J_{20}, J_{21}, J_{22}, J_{23}, J_{24}, J_{25}, J_{26}, J_{27}, \\ J_{28}, J_{29}, J_{30}, J_{32}, J_{33}, J_{35}, J_{36}, J_{37}, J_{38}, J_{39}, J_{40}, J_{42}, J_{43}, J_{44}, J_{45}, \\ J_{46}, J_{47}, J_{48}, J_{49}, J_{50}, J_{51}, J_{52}, J_{53}, J_{54}, J_{56}, J_{57}, J_{58}, J_{59}, J_{60}, J_{61} \end{array} \right\}$$

is a minimal rational basis of the $GL(2,\mathbb{R})$-invariants of system (5) on

$$\mathcal{M} = \{a \in \mathcal{C}(2,3,\mathbb{R}); J_7 \neq 0\}.$$

Proof. Let us consider a planar cubic differential systems $S(a)$ such that $I_{17} \neq 0$ and Q_{17} the matrix defined by (7). From Lemma 2 the linear transformation $y = Q_{17}x$ can brought system (1) to the normal form (6). Now if $F(a)$ is a $GL(2,\mathbb{R})-$invariant of system (1) then by using the fact that $\det(Q_{17}) = 1$ we obtain

$$F(\rho(Q_{17})a) = F\left(\left\{\frac{1}{I_{17}^{h+j-1}} K^j \underbrace{11..1}_{h \text{ times}} \underbrace{22..2}_{l \text{ times}} (a)\right\}\right) = F(a).$$

Which means that any invariant can be expressed as a rational function of elements of the set $\mathcal{B}_1$. By lemma 1, the elements of $\mathcal{B}_1$ are polynomially independents, thus the set $\mathcal{B}_1$ is a minimal rational basis of the $GL(2,\mathbb{R})$-invariants of system (5) on

$$\mathcal{M} = \{a \in \mathcal{C}(2,3,\mathbb{R}); I_{17} \neq 0\}.$$

In the same way, we can prove the case (2) and (3). ∎

Acknowledgment. The authors would like to thank the handling editor and anonymous referee for their valuable remarks and suggestions to improve this work.

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5 Annexe

Table 6: A minimal polynomial basis of center-affine invariants of cubic differential systems

Invariant Expression	Type (d_0, d_1, d_2, d_3)	Product	Cofficient
$J_1 = a^\alpha_p a^\beta_{\beta\alpha q}\varepsilon^{pq}$	$(0,1,0,1)$	$a^1_1 a^1_{112}$	1
$J_2 = a^\alpha_{\alpha pr} a^\beta_{\beta qs}\varepsilon^{pq}\varepsilon^{rs}$	$(0,0,0,2)$	$a^1_{111} a^2_{222}$	2
$J_3 = a^\alpha_{\beta pr} a^\beta_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}$	$(0,0,0,2)$	$a^1_{222} a^2_{111}$	2
$J_4 = a^\alpha a^\beta a^\gamma_{\gamma\alpha\beta}$	$(2,0,0,1)$	$(a^1)^2 a^1_{111}$	1
$J_5 = a^\alpha a^\beta_{\gamma p} a^\gamma_{\beta\alpha q}\varepsilon^{pq}$	$(1,0,1,1)$	$a^1 a^1_{22} a^2_{111}$	-1
$J_6 = a^\alpha a^\beta_{\alpha p} a^\gamma_{\beta\gamma q}\varepsilon^{pq}$	$(1,0,1,1)$	$a^1 a^2_{11} a^2_{222}$	1
$J_7 = a^\alpha a^\beta_{\beta p} a^\gamma_{\gamma\alpha q}\varepsilon^{pq}$	$(1,0,1,1)$	$a^1 a^2_{22} a^1_{111}$	-1
$J_8 = a^\alpha_{\gamma p} a^\beta_{\alpha r} a^\gamma_{\beta qs}\varepsilon^{pq}\varepsilon^{rs}$	$(0,0,2,1)$	$a^1_{11} a^2_{12} a^1_{122}$	-1
$J_9 = a^\alpha_{\alpha p} a^\beta_{\gamma r} a^\gamma_{\beta qs}\varepsilon^{pq}\varepsilon^{rs}$	$(0,0,2,1)$	$a^1_{11} a^2_{22} a^1_{112}$	-1
$J_{10} = a^\alpha_{\beta p} a^\beta_{\alpha r} a^\gamma_{\gamma qs}\varepsilon^{pq}\varepsilon^{rs}$	$(0,0,2,1)$	$(a^1_{11})^2 a^2_{222}$	1
$J_{11} = a^\alpha_{\beta\gamma} a^\beta_{pr} a^\gamma_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}$	$(0,0,2,1)$	$a^1_{11} a^1_{22} a^1_{111}$	1
$J_{13} = a^\alpha_{\gamma\beta p} a^\beta_{\alpha rk} a^\gamma_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(0,0,0,3)$	$(a^1_{111})^2 a^1_{222}$	1
$J_{14} = a^\alpha a^\beta a^\gamma_\beta a^\delta_{\delta\alpha\gamma}$	$(2,1,0,1)$	$(a^1)^2 a^2_1 a^2_{122}$	1
$J_{15} = a^\alpha a^\beta a^\gamma_{\alpha\beta p} a^\delta_{\delta\gamma q}\varepsilon^{pq}$	$(2,0,0,2)$	$(a^1)^2 a^1_{111} a^2_{122}$	1
$J_{16} = a^\alpha a^\beta_\delta a^\gamma_{\gamma p} a^\delta_{\beta\alpha q}\varepsilon^{pq}$	$(1,1,1,1)$	$a^1 a^2_1 a^2_{22} a^1_{112}$	-1
$J_{20} = a^\alpha a^\beta_\alpha a^\gamma_{\delta p} a^\delta_{\gamma\beta q}\varepsilon^{pq}$	$(1,1,1,1)$	$a^1 a^2_1 a^2_{22} a^2_{122}$	-1
$J_{21} = a^\alpha a^\beta_{\delta p} a^\gamma_{\gamma\beta r} a^\delta_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}$	$(1,0,1,2)$	$a^1 a^2_{11} a^1_{112} a^2_{222}$	-1
$J_{22} = a^\alpha a^\beta_{\beta p} a^\gamma_{\gamma\delta r} a^\delta_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}$	$(1,0,1,2)$	$a^1 a^2_{22} a^1_{111} a^2_{122}$	1
$J_{23} = a^\alpha a^\beta_{pr} a^\gamma_{\gamma\delta q} a^\delta_{\alpha\beta s}\varepsilon^{pq}\varepsilon^{rs}$	$(1,0,1,2)$	$a^1 a^1_{12} a^1_{111} a^1_{112}$	-2
$J_{27} = a^\alpha_{\gamma p} a^\beta_{\beta r} a^\gamma_{\delta qk} a^\delta_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(0,0,2,2)$	$(a^1_{11})^2 a^1_{222} a^2_{112}$	-1
$J_{28} = a^\alpha_{rk} a^\beta_{\delta p} a^\gamma_{\gamma sl} a^\delta_{\beta\alpha q}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(0,0,2,2)$	$(a^1_{11})^2 a^1_{112} a^1_{122}$	1
$J_{29} = a^\alpha_{pr} a^\beta_{\delta k} a^\gamma_{qsl} a^\delta_{\gamma\beta\alpha}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(0,0,2,2)$	$(a^1_{11})^2 a^1_{111} a^1_{222}$	1
$J_{30} = a^\alpha a^\beta a^\gamma a^p a^q_{\alpha\beta\gamma}\varepsilon_{pq}$	$(4,0,0,1)$	$(a^1)^4 a^2_{111}$	1
$J_{31} = a^\alpha a^\beta a^\gamma a^\delta_{\delta\mu} a^\mu_{\alpha\beta\gamma}$	$(3,0,1,1)$	$(a^1)^3 a^2_{22} a^2_{111}$	1
$J_{32} = a^\alpha a^\beta a^\gamma a^\delta_{\gamma\beta} a^\mu_{\mu\alpha\delta}$	$(3,0,1,1)$	$(a^1)^3 a^2_{11} a^2_{122}$	1
$J_{33} = a^\alpha a^\beta a^\gamma_{\mu\gamma} a^\delta_{\delta p} a^\mu_{\beta\alpha q}\varepsilon^{pq}$	$(2,0,2,1)$	$(a^1)^2 (a^1_{12})^2 a^2_{111}$	-1
$J_{34} = a^\alpha a^\beta a^\gamma_{\gamma\alpha p} a^\delta_{\beta rq} a^\mu_{\mu\delta s}\varepsilon^{pq}\varepsilon^{rs}$	$(2,0,0,3)$	$(a^1)^2 a^1_{112} a^2_{111} a^2_{222}$	-1
$J_{41} = a^\alpha a^\beta_{\mu\delta} a^\gamma_{\gamma p} a^\delta_{\beta r} a^\mu_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}$	$(1,0,3,1)$	$a^2 (a^2_{12})^3 a^1_{222}$	1
$J_{42} = a^\alpha a^\beta_{\gamma\delta} a^\gamma_{\mu p} a^\delta_{\beta r} a^\mu_{\alpha qs}\varepsilon^{pq}\varepsilon^{rs}$	$(1,0,3,1)$	$a^1 a^2_{11} (a^2_{22})^2 a^1_{112}$	-1
$J_{43} = a^\alpha a^\beta_{rk} a^\gamma_{\gamma\delta p} a^\delta_{\beta\mu q} a^\mu_{\alpha ls}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(1,0,1,3)$	$a^1 a^1_{12} a^1_{112} a^2_{111} a^2_{222}$	2
$J_{46} = a^\alpha a^\beta_{pr} a^\gamma_{\mu\delta q} a^\delta_{\beta\gamma k} a^\mu_{\alpha ls}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(1,0,1,3)$	$a^2 a^1_{11} a^1_{111} a^1_{112} a^1_{222}$	1
$J_{49} = a^\alpha_{\delta\gamma} a^\beta_{\beta\mu} a^\gamma_{\alpha p} a^\delta_{rk} a^\mu_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(0,0,4,1)$	$(a^1_{11})^3 a^2_{22} a^2_{222}$	1
$J_{52} = a^\alpha_{\mu\delta} a^\beta_{\beta\alpha} a^\gamma_{\gamma p} a^\delta_{rk} a^\mu_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(0,0,4,1)$	$(a^1_{11})^3 a^2_{22} a^1_{122}$	-1
$J_{54} = a^\alpha_{\mu\delta} a^\beta_{\gamma p} a^\gamma_{\beta r} a^\delta_{\alpha k} a^\mu_{qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(0,0,4,1)$	$(a^1_{12})^4 a^2_{111}$	-1
$J_{55} = a^\alpha a^\beta a^\gamma a^\delta a^\mu_{\mu\delta\nu} a^\nu_{\alpha\beta\gamma}$	$(4,0,0,2)$	$(a^1)^4 a^2_{111} a^2_{122}$	1
$J_{56} = a^\alpha a^\beta a^\gamma a^\delta_{\delta\nu} a^\mu_{\gamma\mu p} a^\nu_{\alpha\beta q}\varepsilon^{pq}$	$(3,0,1,2)$	$(a^1)^3 a^1_{12} a^1_{112} a^2_{111}$	-1
$J_{60} = a^\alpha a^\beta_{\nu\delta} a^\gamma_{\gamma p} a^\delta_{\beta r} a^\mu_{\mu qk} a^\nu_{\alpha sl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}$	$(1,0,3,2)$	$a^2 (a^1_{11})^3 a^1_{222} a^2_{122}$	1

Product	J_{36}	J_{35}	J_{37}	J_{39}	J_{40}	J_{38}
$a^1a^1_{22}(a^2_{11})^2a^1_{122}$	1	0	0	0	0	0
$a^2a^1_{11}a^1_{22}a^2_{11}a^2_{222}$	1	1	0	0	0	0
$a^1a^2_{11}(a^2_{22})^2a^2_{122}$	1	1	1	0	0	0
$a^1a^1_{22}(a^2_{11})^2a^2_{222}$	0	0	0	1	1	0
$a^1a^1_{22}a^2_{11}a^2_{22}a^2_{112}$	0	0	0	1	1	1
$a^2(a^1_{22})^2a^2_{12}a^2_{111}$	0	0	0	1	2	1

Product	J_{18}	J_{19}	J_{17}
$a^2a^1_2a^2_{12}a^2_{122}$	1	1	1
$a^2a^2_1a^1_{11}a^1_{222}$	0	0	1
$a^2a^2_1a^1_{22}a^2_{122}$	0	-1	-1

Product	J_{45}	J_{44}
$a^1a^1_{12}a^1_{112}a^1_{122}a^2_{111}$	2	0
$a^2a^1_{22}a^1_{112}a^1_{122}a^2_{111}$	0	1
$a^2a^2_{11}a^1_{112}(a^1_{122})^2$	-1	-1

Product	J_{53}	J_{51}	J_{50}	J_{47}	J_{48}
$(a^1_{11})^4a^1_{222}$	1	1	1	1	1
$(a^1_{11})^2a^1_{22}a^2_{22}a^1_{111}$	-1	0	0	0	0
$(a^1_{22})^2(a^2_{11})^2a^1_{112}$	0	1	0	0	0
$(a^1_{12})^3a^2_{22}a^2_{111}$	0	0	-1	0	-1
$(a^1_{11})^3a^2_{12}a^1_{222}$	1	0	0	1	1

Product	J_{61}	J_{58}	J_{57}	J_{59}	J_{60}
$a^1(a^1_{12})^3a^1_{112}a^2_{111}$	1	-1	-1	1	-1
$a^1(a^1_{22})^2a^2_{11}a^1_{111}a^2_{112}$	1	0	0	1	0
$a^1(a^1_{22})^2a^2_{11}(a^2_{112})^2$	0	0	1	1	0
$a^1(a^1_{12})^2a^2_{22}a^1_{112}a^2_{111}$	-1	0	0	0	-1
$a^1(a^1_{12})^3a^2_{111}a^2_{122}$	1	1	1	1	-1

Table 7: