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Iterative Approximation Of Best Proximity Pairs Of Asymptotically Relatively Nonexpansive Mappings*

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Abstract

We prove the strong convergence of iterative approximation of best proximity pairs of an asymptotically relatively nonexpansive mappings on a uniformly convex Banach spaces using Noor's iteration schemes under various control conditions on iteration parameters. We also provide an example to support our results.

1 Introduction

Let U be a nonempty subset of a Banach space X and S be a mapping from U into U. Iterative approximation of fixed points on nonexpansive mappings was studied by various authors (see [5], [6], [9], [10], [13], [14], [17]) using Mann iteration schemes ($x_0 \in U, x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sx_n, n \geq 0$, where $\{\alpha_n\} \subseteq [0,1]$) and Ishikawa iteration schemes ($x_0 \in U, x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n, y_n = (1-\beta_n)x_n + \beta_n Sx_n, n \geq 0$, where $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$). Thereafter modification of Mann and Ishikawa [18] iteration schemes was introduced to approximate fixed points of mapping with asymptotic behaviour. Later, Xu and Noor [22] in 2002 introduced three steps (Noor) iterative schemes ($x_0 \in U, x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n; y_n = (1-\beta_n)x_n + \beta_n S^n z_n; z_n = (1-\gamma_n)x_n + \gamma_n S^n x_n, n \geq 0$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[0,1]$) to approximate fixed points of asymptotically nonexpansive mappings.

Let U and V be nonempty subsets of a Banach space X. A mapping $S : U \cup V \to X$ such that $S(U) \subseteq U, S(V) \subseteq V$ is said to be *relatively nonexpansive* [2] if $\|Sx - Sy\| \leq \|x - y\|$ for all $(x,y) \in U \times V$ and *asymptotically relatively nonexpansive* [16] if $\|S^n x - S^n y\| \leq k_n \|x - y\|$ for all $(x,y) \in U \times V$. Under this weaker assumption over S, the existence of the so-called *best proximity pair*, that is, a point $(p,q) \in U \times V$ such that

$$p = Sp, \quad q = Sq \quad \text{and } d(p,q) = \text{dist}(U,V),$$

was studied by various authors (see [1], [3], [4], [7], [15]). Recently, S. Rajesh and P. Veeramani [16] have proved the following theorem which ensures the existence of best proximity pair for asymptotically relatively nonexpansive mappings.

Theorem 1 ([16, Theorem 3.2]) *Let (U,V) be a nonempty bounded closed convex proximal parallel pair in a nearly uniformly convex (NUC) Banach space. Suppose $S : U \cup V \to U \cup V$ is a continuous and asymptotically relatively nonexpansive mappings satisfying $S(U) \subseteq U$ and $S(V) \subseteq V$. Further, assume that (U,V) has the rectangle property and the property UC. Then there exist $u \in U$ and $v \in V$ such that $Su = u$, $Sv = v$ and $\|u - v\| = dist(U,V)$.*

Here we establish the strong convergence of best proximity pairs for Theorem 1 with the help of Noor iteration schemes under variety of control conditions.

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2 Preliminaries

Here we recall some important definitions, notations and results which are necessary for our main results.

Definition 1 ([2]) *Let U and V be nonempty subsets of a Banach space X and $(u,v) \in U \times V$. Then a point u (or v) is said to be a* proximal point *of v (or u) if $\|u-v\| = dist(U,V)$.*

Definition 2 ([2, 8]) *Let U and V be nonempty subsets of a Banach space X. The pair (U,V) is said to be proximal pair if and only if for each (u,v) in $U \times V$, there exists (u_1,v_1) in $U \times V$ such that $\|u-v_1\| = dist(U,V) = \|v-u_1\|$. If (u_1,v_1) in $U \times V$ is unique then the pair (U,V) is called sharp proximal pair.*

Definition 3 ([8]) *The pair (U,V) is said to be a proximal parallel pair if the pair (U,V) is sharp proximal pair and there exists a unique $h \in X$ such that $V = U + h$.*

The proximal pair of (U,V) denoted as (U_0,V_0) which are given by

$$U_0 = \{u \in U : \|u-v'\| = \text{dist}(U,V) \text{ for some } v' \in V\},$$

$$V_0 = \{v \in V : \|u'-v\| = \text{dist}(U,V) \text{ for some } u' \in U\}.$$

Also

$$\mathcal{P}(x) = \begin{cases} y \in U_0 : \|x-y\| = \text{dist}(U,V) & \text{if } x \in V_0, \\ y \in V_0 : \|x-y\| = \text{dist}(U,V) & \text{if } x \in U_0, \end{cases}$$

and $Fix(S) = \{x \in U \cup V / S(x) = x\}$.

Definition 4 ([20]) *The pair (U,V) is said to satisfy the property UC if and only if the following holds: If $\{x_n\}$ and $\{y_n\}$ are sequences in U and $\{z_n\}$ be a sequence in V such that $\lim_{n\to\infty} d(x_n,z_n) = dist(U,V)$ and $\lim_{n\to\infty} d(y_n,z_n) = dist(U,V)$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

Definition 5 ([16]) *Let (U,V) be a nonempty convex parallel pair in a Banach space X. The pair (U,V) is said to have the rectangle property if and only if $\|x+h-y\| = \|y+h-x\|$, for any $x,y \in U$, where $h \in X$ such that $V = U + h$.*

Lemma 2 ([19]) *Let X be a normed linear space. Then for all $x,y \in X$ and $t \in [0,1]$,*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2.$$

Lemma 3 ([23]) *Assume $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ be a strictly increasing map. If a sequence $\{x_n\}$ in $[0,\infty)$ satisfies $\lim_{n\to\infty} f(x_n) = 0$, then $\lim_{n\to\infty} x_n = 0$.*

Lemma 4 ([21]) *A Banach space X is uniformly convex (UC) if and only if for each fixed number $r > 0$, there exists a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(t) = 0 \Leftrightarrow t = 0$, such that*

$$\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\| - 2\lambda(1-\lambda)\varphi(\|x-y\|),$$

for all $\lambda \in [0,1]$ and all $x,y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

Lemma 5 ([12]) *Let U be a nonempty convex subset of a normed linear space X and $S : U \to U$ be a uniformly k-Lipschitzian mapping. For $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subseteq [0,1]$ and $x_1 \in U$, define $x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n$, $y_n = (1-\beta_n)x_n + \beta_n S^n z_n$ and $z_n = (1-\gamma_n)x_n + \gamma_n S^n x_n, n \geq 1$. Then*

$$\| x_n - S(x_n) \| \leq c_n + c_{n-1}k(2 + 2k + 2k^2 + k^3),$$

where $c_n = \| x_n - S^n(x_n) \|$, for all $n \geq 1$.

3 Convergence Results

Let (U,V) be a nonempty closed and convex pair in a strictly convex Banach space X. For $x_1 \in U_0$, put $y_1 := \mathcal{P}(x_1) \in V_0$. Define the sequence pair $\{(x_n, y_n) \in U_0 \times V_0\}$ as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n x_n^{'}, \quad y_{n+1} = (1-\alpha_n)y_n + \alpha_n S^n y_n^{'}, \\ x_n^{'} = (1-\beta_n)x_n + \beta_n S^n x_n^{''}, \quad y_n^{'} = (1-\beta_n)y_n + \beta_n S^n y_n^{''}, \\ x_n^{''} = (1-\gamma_n)x_n + \gamma_n S^n x_n, \quad y_n^{''} = (1-\gamma_n)y_n + \gamma_n S^n y_n, \quad n = 1,2,3..., \end{cases} \tag{1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying one of the following conditions:

(A) $0 < \epsilon \le \alpha_n(1-\alpha_n) \le 1; \beta_n \to 0$ and $0 \le \gamma_n \le 1$ as $n \to \infty$,

(B) $0 < \epsilon \le \alpha_n \le 1$; $0 < \epsilon \le \beta_n(1-\beta_n) \le 1$ and $\gamma_n \to 0$ as $n \to \infty$,

(C) $0 < \epsilon \le \alpha_n, \beta_n \le 1$ and $0 < \epsilon \le \gamma_n(1-\gamma_n) \le 1$ as $n \to \infty$.

Lemma 6 *Let (U,V) be a nonempty bounded closed convex proximal parallel pair in a uniformly convex Banach space X also assume that (U,V) have rectangle and UC property. Let $S : U \cup V \to U \cup V$ is a continuous and noncyclic asymptotically relatively nonexpansive mapping with $\sum_{n\ge1}(k_n - 1) < \infty$. Assume $\{x_n\}$ and $\{y_n\}$ are real sequences as defined in (1). Then*

(i) $\lim_{n\to\infty} \|x_n - q\|$ *exists for all* $q \in \mathrm{Fix}(S) \cap V_0$.

(ii) $\lim_{n\to\infty} \|y_n - p\|$ *exists for all* $p \in \mathrm{Fix}(S) \cap U_0$.

Proof. For any $q \in \mathrm{Fix}(S) \cap V_0$, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)x_n + \alpha_n S^n x_n^{'} - q\| \\ &\le (1-\alpha_n)\|x_n - q\| + k_n\alpha_n\|x_n^{'} - q\| \\ &\le (1-\alpha_n)\|x_n - q\| + k_n\alpha_n \left\|(1-\beta_n)x_n + \beta_n S^n x_n^{''} - q\right\| \\ &\le (1-\alpha_n)\|x_n - q\| + k_n\alpha_n(1-\beta_n)\|x_n - q\| + k_n^2\alpha_n\beta_n\|x_n^{''} - q\| \\ &\le (1-\alpha_n)\|x_n - q\| + k_n\alpha_n(1-\beta_n)\|x_n - q \\ &\quad + k_n^2\alpha_n\beta_n\|(1-\gamma_n)x_n + \gamma_n S^n(x_n) - q\| \\ &\le \left\{1 + \alpha_n(k_n - 1)(\beta_n\gamma_n {k_n}^2 + \beta_n k_n + 1)\right\}\|x_n - q\|, \end{aligned}$$

$$\|x_{n+1} - q\| \le (1+\mu_n)\|x_n - q\| \le e^{\sum_{i=1}^{\infty}\mu_i}\|x_1 - q\|,$$

$$\|x_{n+1} - q\| \le e^{\sum_{i=1}^{\infty}{k_i}^3 - 1}\|x_1 - q\|.$$

where $\mu_n = {k_n}^3 - 1$. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ (which implies that $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$), $\{\|x_n - q\|\}$ is a bounded sequence and hence $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in \mathrm{Fix}(S) \cap V_0$. Similarly, we can show that $\lim_{n\to\infty}\|y_n - p\|$ exists for all $p \in \mathrm{Fix}(S) \cap U_0$. ■

Lemma 7 *Let (U,V) be a nonempty bounded closed convex proximal parallel pair in a uniformly convex Banach space X, also assume that (U,V) have rectangle and UC property. Let $S : U \cup V \to U \cup V$ be a continuous uniformly k-Lipschitzian and noncyclic asymptotically relatively nonexpansive mapping with $\sum_{n\ge1}(k_n - 1) < \infty$. For $x_1 \in U_0$, let $y_1 \in V_0$ be a unique proximal point of x_1. Assume $\{x_n\}$ and $\{y_n\}$ are real sequences as defined in (1) and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfy either (A) or (B) or (C), then*

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0 \quad \textit{and} \quad \lim_{n\to\infty}\|y_n - Sy_n\| = 0.$$

Proof. Let $q \in F(T) \cap V_0$, then

$$\begin{aligned}\left\|x_n^{''}-q\right\| &= \|(1-\gamma_n)x_n+\gamma_n S^n x_n - q\| \\ &= \|(1-\gamma_n)(x_n-q)+\gamma_n(S^n x_n - q)\| \\ &\leq (1-\gamma_n)\|x_n-q\|+\gamma_n\|S^n x_n-q\|-2\gamma_n(1-\gamma_n)g(\|S^n x_n-x_n\|) \\ &\leq (1-\gamma_n)\|x_n-q\|+k_n\gamma_n\|x_n-q\|-2\gamma_n(1-\gamma_n)g(\|S^n x_n-x_n\|),\end{aligned}$$

$$\left\|x_n^{''}-q\right\| \leq (1-\gamma_n+k_n\gamma_n)\|x_n-q\|-2\gamma_n(1-\gamma_n)g(\|S^n x_n-x_n\|), \tag{2}$$

$$\begin{aligned}\left\|x_n^{'}-q\right\| &= \|(1-\beta_n)x_n+\beta_n S^n x_n^{''}-q\| \\ &= \|(1-\beta_n)(x_n-q)+\beta_n(S^n x_n^{''}-q)\| \\ &\leq (1-\beta_n)\|x_n-q\|+\beta_n\|S^n x_n^{''}-q\|-2\beta_n(1-\beta_n)g(\|S^n x_n^{''}-x_n\|),\end{aligned}$$

$$\left\|x_n^{'}-q\right\| \leq (1-\beta_n)\|x_n-q\|+k_n\beta_n\|x_n^{''}-q\|-2\beta_n(1-\beta_n)g(\|S^n x_n^{''}-x_n\|), \tag{3}$$

$$\begin{aligned}\|x_{n+1}-q\| &= \|(1-\alpha_n)x_n+\alpha_n S^n x_n^{'}-q\| \\ &= \|(1-\alpha_n)(x_n-q)+\alpha_n(S^n x_n^{'}-q)\| \\ &\leq (1-\alpha_n)\|x_n-q\|+\alpha_n\|S^n x_n^{'}-q\|-2\alpha_n(1-\alpha_n)g(\|S^n x_n^{'}-x_n\|),\end{aligned}$$

$$\|x_{n+1}-q\| \leq (1-\alpha_n)\|x_n-q\|+k_n\alpha_n\|x_n^{'}-q\|-2\alpha_n(1-\alpha_n)g(\|S^n x_n^{'}-x_n\|). \tag{4}$$

Applying equations (2) and (3) in (4), we get

$$\begin{aligned}\|x_{n+1}-p\| \leq\ & \{1+\alpha_n(k_n-1)(\beta_n\gamma_n {k_n}^2+\beta_n k_n+1)\}\|x_n-q\| \\ & -2\alpha_n(1-\alpha_n)g(\|S^n x_n^{'}-x_n\|)-2k_n\alpha_n\beta_n(1-\beta_n)g(\|S^n x_n^{''}-x_n\|) \\ & -2k_n^2\alpha_n\beta_n\gamma_n(1-\gamma_n)g(\|S^n x_n-x_n\|).\end{aligned}$$

This can be transformed into the following three equations:

$$\begin{aligned}2\alpha_n(1-\alpha_n)g(\|S^n x_n^{'}-x_n\|) &\leq \|x_n-q\|-\|x_{n+1}-q\| \\ &\quad +\alpha_n(\beta_n\gamma_n k_n^2+\beta_n k_n+1)(k_n-1)\,\|x_n-q\| \\ &\leq \|x_n-q\|-\|x_{n+1}-q\|+M(k_n-1),\end{aligned} \tag{5}$$

$$2k_n\alpha_n\beta_n(1-\beta_n)g(\|S^n x_n^{''}-x_n\|) \leq \|x_n-q\|-\|x_{n+1}-q\|+M(k_n-1), \tag{6}$$

$$2k_n^2\alpha_n\beta_n\gamma_n(1-\gamma_n)g(\|S^n x_n-x_n\|) \leq \|x_n-q\|-\|x_{n+1}-q\|+M(k_n-1). \tag{7}$$

Now we have to show that for the given conditions, $\lim_{n\to\infty}\|S^n x_n-x_n\| \to 0$.

Case (i). Suppose the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy condition (A).
Then summing up the first m terms of equation (5), we get

$$\sum_{n=1}^{m} 2\alpha_n(1-\alpha_n)g(\|S^n x_n^{'}-x_n\|) \leq \|x_1-q\|-\|x_{m+1}-q\|+M\sum_{n=1}^{m}(k_n^3-1) < \infty,$$

for all $m \geq 1$. Therefore, $\sum_{n=1}^{\infty} 2\alpha_n(1-\alpha_n)g(\|S^n x_n^{'} - x_n\|) < \infty$. Since $\alpha_n(1-\alpha_n) \geq \epsilon$, $g(\|S^n x_n^{'} - x_n\|) \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \|S^n x_n^{'} - x_n\| \to 0$ from Lemma 3. Also

$$\begin{aligned}
\|S^n x_n - \mathcal{P}x_n\| &\leq \|S^n x_n - \mathcal{P}S^n x_n^{'}\| + \|\mathcal{P}S^n x_n^{'} - \mathcal{P}x_n\| \\
&= \|S^n x_n - S^n(\mathcal{P}x_n^{'})\| + \|S^n x_n^{'} - x_n\|, \\
&\leq k_n\|x_n - \mathcal{P}x_n^{'}\| + \|S^n x_n^{'} - x_n\| \\
&\leq k_n\|x_n - \mathcal{P}\{(1-\beta_n)x_n + \beta_n S^n x_n^{''}\}\| + \|S^n x_n^{'} - x_n\| \\
&= k_n\|x_n - \mathcal{P}x_n + \beta_n(\mathcal{P}x_n - \mathcal{P}S^n x_n^{''})\| + \|S^n x_n^{'} - x_n\| \\
&\leq k_n\|x_n - \mathcal{P}x_n\| + k_n\beta_n\|\mathcal{P}x_n - \mathcal{P}S^n x_n^{''}\| + \|S^n x_n^{'} - x_n\|,
\end{aligned}$$

$$\|S^n x_n - \mathcal{P}x_n\| \leq k_n\|x_n - \mathcal{P}x_n\| + k_n\beta_n\|x_n - S^n x_n^{''}\| + \|S^n x_n^{'} - x_n\|.$$

That is, $\lim_{n\to\infty} \|S^n x_n - \mathcal{P}x_n\| = \text{dist}(A, B)$ which implies that $\lim_{n\to\infty} \|S^n x_n - x_n\| \to 0$.

Case (ii). If the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy condition (B).
Then adding the first m terms of equation (6), we get

$$\sum_{n=1}^{m} k_n\alpha_n\beta_n(1-\beta_n)g(\|S^n x_n^{''} - x_n\|) \leq \|x_1 - q\| - \|x_{m+1} - q\| + M\sum_{n=1}^{m}(k_n^3 - 1) < \infty,$$

for all $m \geq 1$. As $m \to \infty$, we get $\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)g(\|S^n x_n^{''} - x_n\|) < \infty$. Then by Lemma 3, we have that $\lim_{n\to\infty} \|S^n x_n^{''} - x_n\| \to 0$. Now

$$\begin{aligned}
\|S^n x_n - \mathcal{P}x_n\| &\leq \|S^n x_n - \mathcal{P}S^n x_n^{''}\| + \|\mathcal{P}S^n x_n^{''} - \mathcal{P}x_n\| \\
&\leq k_n\|x_n - \mathcal{P}x_n^{''}\| + \|S^n x_n^{''} - x_n\| \\
&\leq k_n\|x_n - \mathcal{P}x_n + \gamma_n(\mathcal{P}x_n - \mathcal{P}S^n x_n)\| + \|S^n x_n^{''} - x_n\|,
\end{aligned}$$

$$\|S^n x_n - \mathcal{P}x_n\| \leq k_n\|x_n - \mathcal{P}x_n\| + k_n\gamma_n\|x_n - S^n x_n\| + \|S^n x_n^{''} - x_n\|.$$

We obtain that $\lim_{n\to\infty} \|S^n x_n - \mathcal{P}x_n\| = \text{dist}(A, B)$ and hence $\lim_{n\to\infty} \|S^n x_n - x_n\| = 0$.

Case (iii). Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy condition (C).
Then taking summation on the first m terms of the equation (7), we get

$$\sum_{n=1}^{m} k_n^2\alpha_n\beta_n\gamma_n(1-\gamma_n)g(\|S^n x_n - x_n\|) \leq \|x_1 - q\| - \|x_{m+1} - q\| + M\sum_{n=1}^{m}(k_n^3 - 1) < \infty,$$

for all $m \geq 1$. As $m \to \infty$, $\sum_{n=1}^{m} k_n^2\alpha_n\beta_n\gamma_n(1-\gamma_n)g(\|S^n x_n - x_n\|) \to 0$ which gives

$$k_n^2\alpha_n\beta_n\gamma_n(1-\gamma_n)g(\|S^n x_n - x_n\|) \to 0 \text{ as } n \to \infty.$$

But (as $n \to \infty$) $k_n \to 1$, $\alpha_n, \beta_n \geq \epsilon > 0$ and $\gamma_n(1-\gamma_n) \geq \epsilon > 0$,

$$g(\|S^n x_n - x_n\|) \to 0 \quad \text{as} \quad n \to \infty.$$

Then by Lemma 3, we get that $\lim_{n\to\infty} \|S^n x_n - x_n\| \to 0$. From all the above cases, $\|S^n x_n - x_n\| \to 0$ as $n \to \infty$. Further, since the mapping S is k-Lipschitzian, we have the following from Lemma 5,

$$\| Sx_n - x_n \| \leq c_n + c_{n-1}k(2 + 2k + 2k^2 + k^3),$$

where $c_n = \| x_n - S^n(x_n) \|$, for all $n \geq 1$. Hence

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Similarly we can show that, $\lim_{n\to\infty} \|Sy_n - y_n\| = 0$. ■

Theorem 8 *Let (U, V) be a nonempty bounded closed convex proximal parallel pair in a uniformly convex Banach space X, also assume that (U, V) have rectangle and UC property. Let $S : U \cup V \to U \cup V$ be a continuous uniformly k-Lipschitzian and noncyclic asymptotically relatively nonexpansive mapping with $S(U)$ contained in a compact subset and $\sum_{n\geq 1}(k_n - 1) < \infty$. For $x_1 \in U_0$, let $y_1 \in V_0$ be a unique proximal point of x_1. Assume $\{x_n\}$ and $\{y_n\}$ are real sequences as defined in (1), and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfy either (A) or (B) or (C). Then the sequence pair $\{(x_n, y_n)\}$ converges to (p, q), where $p = Sp \in U$ and $q = Sq \in V$.*

Proof. Let $S(U)$ lie in a compact subset. Then there exists a subsequence $\{Sx_{n_k}\}$ of $\{Sx_n\}$ which converges to some point $u \in U_0$. Then by Lemma 7, we have $\lim_{n\to\infty} \|Sx_{n_k} - x_{n_k}\| = 0$ which implies that $x_{n_k} \to u$. Thus, $Su = u$ and therefore $S(\mathcal{P}u) = \mathcal{P}u$. Then by Lemma 6, $\lim_n \|x_n - \mathcal{P}u\|$ exists and

$$\lim_n \|x_n - \mathcal{P}u\| = \lim_k \|x_{n_k} - \mathcal{P}u\| = \|u - \mathcal{P}u\| = \text{dist}(U, V),$$

which gives that $x_n \to u \in U$. Similarly we can show that, as $n \to \infty$, $\|Sy_n - y_n\| = 0$ and $y_n \to v \in V$. For a given $x_1 \in U$, there exists an element $y_1 = \mathcal{P}(x_1) \in V$ such that $\|x_1 - y_1\| = dist(U, V)$. Here

$$\begin{aligned}
\|x_2 - y_2\| &= \|(1-\alpha_1)x_1 + \alpha_1 Sx_1' - ((1-\alpha_1)y_1 + \alpha_1 Sy_1')\| \\
&\leq (1-\alpha_1)\|x_1 - y_1\| + \alpha_1\|Sx_1' - Sy_1'\| \\
&\leq (1-\alpha_1)\|x_1 - y_1\| + k_1\alpha_1\|x_1' - y_1'\| \\
&= (1-\alpha_1)\|x_1 - y_1\| + k_1\alpha_1\|(1-\beta_1)x_1 + \beta_1 Sx_1'' - ((1-\beta_1)y_1 + \beta_1 Sy_1'')\| \\
&\vdots \\
&\leq \left\{1 + \alpha_1(k_1 - 1)(\beta_1\gamma_1 k_1^2 + \beta_1 k_1 + 1)\right\}\|x_1 - y_1\|,
\end{aligned}$$

$$\|x_2 - y_2\| \leq k_1^3\|x_1 - y_1\|.$$

In general,

$$\|x_n - y_n\| \leq k_n^3\|x_1 - y_1\|.$$

As $n \to \infty$,

$$\|x_n - y_n\| \to dist(U, V).$$

Finally,

$$\|u - v\| = \lim_{n\to\infty} \|x_n - y_n\| = \text{dist}(U, V),$$

which deduces that $(u, v) \in \text{Prox}_{U\times V}(S)$. This completes the proof. ■

If we choose $\gamma_n = 0$ in Theorem 8, then Noor's type (three steps) iteration schemes reduces to Ishikawa's type iteration schemes. In this case conditions (A) and (B) are still valid, but (C) is not.

Corollary 9 *Let (U, V) be a nonempty bounded closed convex proximal parallel pair in a uniformly convex Banach space X, also assume that (U, V) have rectangle and UC property. Let $S : U \cup V \to U \cup V$ be a continuous uniformly k-Lipschitzian and noncyclic asymptotically relatively nonexpansive mapping with*

$S(U)$ contained in a compact subset and $\sum_{n\geq 1}(k_n-1)<\infty$. For $x_1\in U_0$, let $y_1\in V_0$ be a unique proximal point of x_1, define the sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$\begin{cases} x_{n+1}=(1-\alpha_n)x_n+\alpha_n=S^n x_n^{'}, \quad y_{n+1}=(1-\alpha_n)y_n+\alpha_n S^n y_n^{'} \\ x_n^{'}=(1-\beta_n)x_n+\beta_n S^n x_n, \quad y_n^{'}=(1-\beta_n)y_n+\beta_n S^n y_n \quad (n=1,2,3...), \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ satisfying one of the following conditions:

(A_1) $0<\epsilon\leq\alpha_n(1-\alpha_n)\leq 1$ and $\beta_n\to 0$ as $n\to\infty$,

(B_1) $0<\epsilon\leq\alpha_n\leq 1$; $0<\epsilon\leq\beta_n(1-\beta_n)\leq 1$ as $n\to\infty$.

Then the sequence pair $\{(x_n,y_n)\}$ converges to (p,q), where $p=Sp\in U$ and $q=Sq\in V$.

Putting $\beta_n=\gamma_n=0$ in Theorem 8, then Noor's type (three steps) iteration schemes reduces to Mann's type iteration schemes. In such cases, condition (A) alone valid.

Corollary 10 *Let (U,V) be a nonempty, bounded closed convex proximal parallel pair in a uniformly convex Banach space X also assume that (U,V) have rectangle and UC property. Let $S:U\cup V\to U\cup V$ is a continuous and noncyclic asymptotically relatively nonexpansive mapping with $S(U)$ contained in a compact subset and $\sum_{n\geq 1}(k_n-1)<\infty$. For $x_1\in U_0$, let $y_1\in V_0$ be a unique proximal point of x_1, define the sequences $\{x_n\}$ and $\{y_n\}$ as follows:*

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n S^n x_n,\ y_{n+1}=(1-\alpha_n)y_n+\alpha_n S^n y_n \quad (n=1,2,3...),$$

where $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the condition $0<\epsilon\leq\alpha_n(1-\alpha_n)\leq 1$. Then the sequence pair $\{(x_n,y_n)\}$ converges to (p,q), where $p=Sp\in U$ and $q=Sq\in V$.

Example 1 *Let (U,V) be a nonempty pair of subsets of the Hilbert space l^2 such that*

$$U=\{(0,x_1,x_2,x_3,...)/\sum_{i=1}^{\infty}\mid x_i\mid\leq 1\}$$

and

$$V=\{(1,y_1,y_2,y_3,...)/\sum_{i=1}^{\infty}\mid y_i\mid\leq 1\}.$$

It is evident that U and V are closed, convex and compact subsets of l^2. Define a mapping $S:U\cup V\to U\cup V$ by

$$Sx=\begin{cases} (0,0,{x_1}^2,A_2x_2,A_3x_3,...) & \text{if } x\in U, \\ (1,0,{x_1}^2,A_2x_2,A_3x_3,...) & \text{if } x\in V, \end{cases}$$

where $\{A_i\}=\left\{\frac{1}{2^{1/2^{i-1}}}\right\}$. It is easy to verify that S is an asymptotically relatively nonexpansive mapping but not relatively nonexpansive(refer [11]). Clearly $dist(U,V)=1$. Let us consider the point $(u,v)\in U\times V$. Then $u=(0,x_1,x_2,x_3,...)\in U$ and $v=(1,y_1,y_2,y_3,...)\in V$. If we choose $u^{'}=(0,y_1,y_2,y_3,...)\in U$ and $v^{'}=(1,x_1,x_2,x_3,...)\in V$, then $\|u-v^{'}\|=\|u^{'}-v\|=dist(U,V)$. Since the point is arbitrary, the pair (U,V) is a proximal pair of a Banach space l^2 and also $U=U_0$ and $V=V_0$. Since $X=l^2$ is strictly convex Banach space, the pair (U,V) is a proximinal parallel pair (refer Lemma 3.1 in [16]) and so the pair (U,V) posses the rectangle property (refer Example 2.1 in [16]). Obviously U is a convex set and therefore the pair (U,V) satisfies the property UC (refer Proposition 3 in [20]). Hence by Theorem 8, the sequence pair $\{(x_n,y_n)\}$ under the mapping S, converges to the best proximity pair, say (p,q) of (U,V), where $(p,q)=((0,0,0,...),(1,0,0,...))$.

Take $\{\alpha_n\} = \{\frac{2n}{3n+1}\}, \{\beta_n\} = \{\frac{1}{n^2+1}\}$ *and* $\{\gamma_n\} = \{\frac{n}{n+1}\}$*. Then clearly the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfy condition (A). Assume the initial guess as* $x_1 = (0, 0.001, 0.019, 0.080, 0, 0, ...) \in U_0$*. Putting this initial value in equation (1), we get the following sequence of iterations in 3 decimal places:*

Iterations S. No	*Corresponding Iteration values*
I	$x_1^{''} = (0, 0, 0.001, 0.046, 0.033, 0, 0, 0, 0, 0, ...)$ $x_1^{'} = (0, 0, 0.001, 0.04, 0.002, 0.015, 0, 0, 0, 0, ...)$ $x_2 = (0, 0, 0.001, 0.04, 0.016, 0.009, 0.007, 0, ...)$
II	$x_2^{''} = (0, 0, 0, 0.001, 0.005, 0.023, 0.011, 0.005, 0.005, 0....)$ $x_2^{'} = (0, 0, 0.008, 0.032, 0.013, 0.007, 0.006, 0.004, 0.002, 0.001, 0.001, 0, ...)$ $x_3 = (0, 0, 0, 0.017, 0.007, 0.017, 0.009, 0.004, 0.003, 0.002, 0.001, 0, ...)$
III	$x_3^{''} = (0, 0, 0, 0.004, 0.002, 0.004, 0.011, 0.005, 0.012, 0.006, 0.003, 0.002, 0.001, 0, ...)$ $x_3^{'} = (0, 0, 0, 0.015, 0.006, 0.015, 0.008, 0.003, 0.003, 0.002, 0.002, 0, ...)$ $x_4 = (0, 0, 0, 0.007, 0.003, 0.007, 0.001, 0.004, 0.009, 0.001, 0.002, 0.002, 0.001, 0.001, 0, ...)$
IV	$x_4^{''} = (0, 0, 0, 0.001, 0.0008, 0.005, 0.001, 0.005, 0, 0.003, 0.007, 0, 0.001, 0.001, 0, ...)$ $x_4^{'} = (0, 0, 0, 0.001, 0.001, 0.0007, 0.005, 0.001, 0.005, 0.002, 0.001, 0.001, 0, ...)$ $x_5 = (0, 0, 0, 0.002, 0.001, 0.002, 0, 0.001, 0.003, 0.003, 0, 0.002, 0.001, 0.003, 0, ...)$
V	$x_5^{''} = (0, 0, 0, 0, 0, 0, 0, 0, 0.001, 0.001, 0, 0.0008, 0.002, 0.002, 0, 0.0016, 0.0008, 0.002, 0, ...)$ $x_5^{'} = (0, 0, 0, 0.002, 0.0009, 0.002, 0, 0.0009, 0.003, 0.003, 0.002, 0.001, 0.003, 0, ...)$ $x_6 = (0, 0, 0, 0, 0, 0, 0, 0.001, 0.002, 0, 0.001, 0, 0.001, 0.001, 0.001, 0.001, 0, 0.001, 0, ...)$
VI	$x_6^{''} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.0008, 0.0017, 0, 0.0008, 0.001, 0, 0.001, 0.001, 0.001, 0.001, 0, 0.001, 0, ...)$ $x_6^{'} = (0, 0, 0, 0, 0, 0, 0, 0.001, 0.002, 0.001, 0, 0.001, 0.001, 0.001, 0.001, 0, 0.001, 0....)$ $x_7 = (0, 0, 0, 0, 0, ...)$

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