Applied Mathematics E-Notes, 21(2021), 291-306 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/
ISSN 1607-2510

Boundary Value Problem For A Coupled System Of Nonlinear Fractional Differential Equations Involving Erdélyi-Kober Derivative*

Yacine Arioua†, Maria Titraoui‡

Received 1 May 2020

Abstract

In this paper, we focus on the boundary value problem (BVP) for a coupled system of nonlinear fractional differential equations (SFDEs) involving the Erdélyi–Kober derivatives on an infinite interval. First, we define the integral solution of the BVP for Erdélyi-Kober (SFDEs). Then, by using the Banach contraction principle and the Leray-Schauder nonlinear alternative fixed point theorem in a special Banach space, existence and uniqueness theorems of the given problem are demonstrated, respectively. Finally, several indispensable examples are presented to illustrate the usefulness of our main results.

1 Introduction

Fractional-order differential equations have been used in the study of models of many phenomena in various fields of science and engineering, such as viscoelasticity, fluid mechanics, electrochemistry, control, porous media, mathematical biology, and electromagnetic bioengineering. More details are available, for instance, in the books by Samko et al. 1993 [27], Podlubny 1999 [25], Kilbas et al. 2006 [12], Sabatier et al. 2007 [26], Das 2008 [9], Diethelm 2010 [10], and Mathai and Haubold 2018 [20].

The classical fractional calculus is based on several definitions for the operators of integration and differentiation of arbitrary order [13]. Among the various definitions of fractional differentiation, the Riemann–Liouville and Caputo fractional derivatives are widely used in the literature. The most useful classical fractional integrals, however, seem to be the Erdélyi–Kober operators. These were introduced by Sneddon (see, for example, [29, 30, 31]), who studied their basic properties and emphasized their useful applications to generalized axially symmetric potential theory and other physical problems, such as in electrostatics and elasticity.

The theory of boundary value problems on infinite intervals arises quite naturally and has many applications [4]; it is important and several authors have done much work on this topic [2, 3, 15, 16, 17, 23, 28, 32, 34, 35, 37, 39]. For instance, in [22], Ahmad and Ntouyas, studied a three point boundary value problem for a coupled system of nonlinear fractional differential equations given by

$$\begin{cases} \mathcal{D}^{\alpha}_{0^+}u(t) = f(t, v(t), \mathcal{D}^{p}_{0^+}v(t)),\ t \in (0,1), \\ \mathcal{D}^{\beta}_{0^+}v(t) = g\left(t, u(t), \mathcal{D}^{q}_{0^+}u(t)\right),\ t \in (0,1), \\ u(0) = 0,\ u(1) = \gamma u(\eta),\ v(0) = 0,\ v(1) = \gamma v(\eta), \end{cases}$$

where $1 < \alpha, \beta < 2$, $p, q, \gamma > 0$, $0 < \eta < 1$, $\mathcal{D}$ is the standard Riemann–Liouville fractional derivative and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

*Mathematics Subject Classifications: 34A08, 34A37, 47H10.

†Department of Mathematics, Laboratory for Pure and Applied Mathematics, University of M'sila, Bp 166 M'sila, Algeria
‡Laboratory for Pure and Applied Mathematics, University of M'sila, Bp 166 M'sila, Algeria

In [36], Zhai and Jiang considered a new coupled system of fractional differential equations with integral boundary conditions

$$\begin{cases} \mathcal{D}^{\alpha}u(t) + f(t, v(t)) = a, \ 0 < t < 1, \\ \mathcal{D}^{\beta}v(t) + g(t, u(t)) = b, \ 0 < t < 1, \\ u(0) = 0, \ u(1) = \int_0^1 \phi(t)u(t)dt, \\ v(0) = 0, \ v(1) = \int_0^1 \psi(t)v(t)dt, \end{cases}$$

where $1 < \alpha, \beta \leq 2$, a, b are constants. $\mathcal{D}$ denotes the usual Riemann–Liouville fractional derivative. $f, g \in C([0,1] \times \mathbb{R} \times \mathbb{R})$, $\phi, \psi \in L^1[0,1]$.

The aim of this study is to investigate the existence and uniqueness of a positive solution to boundary value problem for a coupled system of nonlinear fractional differential equation involving Erdélyi–Kober differential operators on an infinite interval:

$$\begin{cases} \mathcal{D}_{\beta}^{\gamma,\delta_1} u\left(t\right) + f_1\left(t, u\left(t\right), v\left(t\right)\right) = 0, \ t > 0, \\ \mathcal{D}_{\beta}^{\gamma,\delta_2} v\left(t\right) + f_2\left(t, u\left(t\right), v\left(t\right)\right) = 0, \ t > 0, \end{cases} \tag{1}$$

with the boundary conditions

$$\lim_{t \to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta_1+\gamma, 2-\delta_1} u(t) = 0, \ \lim_{t \to \infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta_1+\gamma, 2-\delta_1} u(t) = 0, \tag{2}$$

$$\lim_{t \to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta_2+\gamma, 2-\delta_2} v(t) = 0, \ \lim_{t \to \infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta_2+\gamma, 2-\delta_2} v(t) = 0, \tag{3}$$

where $\mathcal{D}_{\beta}^{\gamma,\delta_k}$, $(k = 1, 2)$, denotes the Erdélyi–Kober fractional derivative operator of order δ_k and $\mathcal{I}^{\delta_k+\gamma, 2-\delta_k}$ is the Erdélyi–Kober fractional integral of order $2 - \delta_k$, with $1 < \delta_k \leq 2$, $-2 < \gamma < -1$, $\beta > 0$, and f_k are given continuous functions.

We obtain several existence and uniqueness results for a coupled system of the nonlinear fractional boundary value problem (1)–(3). The methods used in this work are the Leray–Schauder nonlinear alternative fixed point theorem and Banach contraction principle in a special Banach space.

Throughout this paper, we will refer to the following hypotheses:

(H1) $f_k : (0, \infty) \times \mathbb{R}^2 \longrightarrow (0, \infty)$ are continuous functions.

(H2) For all $(t, u, v) \in (0, \infty) \times \mathbb{R}^2$,

$$F_1(t, u, v) = t^{\beta(1+\gamma)-1} f_1\left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u, \left(1 + t^{-\beta(1+\gamma)}\right) v\right),$$

$$F_2(t, u, v) = t^{\beta(1+\gamma)-1} f_2\left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u, \left(1 + t^{-\beta(1+\gamma)}\right) v\right),$$

such that

$$F_1(t, u, v) \leq \varphi_1(t)\omega_1(|u|) + \psi_1(t)\omega_2(|v|),$$
$$F_2(t, u, v) \leq \varphi_2(t)\tilde{\omega}_1(|u|) + \psi_2(t)\tilde{\omega}_2(|v|),$$

with $\omega_k, \tilde{\omega}_k \in C\left((0, \infty), (0, \infty)\right)$ nondecreasing and $\varphi_k, \psi_k \in L^1\left(0, \infty\right), k = 1, 2$.

(H3) There exist a positive functions $q_k, \tilde{q}_k$ with

$$q_k^* = \int_0^{+\infty} \left(1 + t^{-\beta(1+\gamma)}\right) q_k\left(t\right) dt < \infty,$$

$$\tilde{q}_k^* = \int_0^{+\infty} \left(1 + t^{-\beta(1+\gamma)}\right) \tilde{q}_k\left(t\right) dt < \infty,$$

such that

$$t^{\beta(\gamma+1)-1} \left|f_k\left(t, u, v\right) - f_k\left(t, \tilde{u}, \tilde{v}\right)\right| \leq q_k\left(t\right) \left|u - \tilde{u}\right| + \tilde{q}_k\left(t\right) \left|v - \tilde{v}\right|,$$

for any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $t \in (0, \infty)$.

The remainder of this paper is organized as follows. In Section 2, we recall some necessary preliminary facts. In Section 3, we prove our main results, after we have established sufficient conditions for the existence and uniqueness results for the solution to the problem (1)–(3). In Section 4, two examples are presented to explain the application of our main results.

2 Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory that will be used to derive our main results.

Definition 1 ([18]) *The space of functions C_{α}^{n}, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, consists of all functions $f(t)$, $t > 0$, that can be represented in the form $f(t) = t^{p} f_{1}(t)$ with $p > \alpha$ and $f_{1} \in C^{n}([0,\infty))$.*

Definition 2 (*Erdélyi–Kober* fractional integrals [18]) *The right-hand Erdélyi-Kober fractional integrals of the orders δ of the function $u \in C_{\alpha}$ are defined by*

$$\left(\mathcal{I}_{\beta}^{\gamma,\delta} u\right)(t) = \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+\delta)} \int_{0}^{t} \left(t^{\beta} - s^{\beta}\right)^{\delta-1} s^{\beta(\gamma+1)-1} u(s)\, ds,\ \ \delta, \beta > 0,\ \ \gamma \in \mathbb{R}, \tag{4}$$

where Γ is the Euler gamma function.

Definition 3 (*Erdélyi–Kober* fractional derivatives [18]) *Let $n-1 < \delta \leq n$, $n \in \mathbb{N}$. The right-hand Erdélyi-Kober fractional derivative of the order δ of the function $u \in C_{\alpha}^{n}$ is defined by*

$$\left(\mathcal{D}_{\beta}^{\gamma,\delta} u\right)(t) = \prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_{\beta}^{\gamma+\delta, n-\delta} u\right)(t), \tag{5}$$

where

$$\prod_{j=1}^{n} \left(\gamma + j + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_{\beta}^{\gamma+\delta, n-\delta} u\right) = \left(\gamma + 1 + \frac{1}{\beta} t \frac{d}{dt}\right) \ldots \left(\gamma + n + \frac{1}{\beta} t \frac{d}{dt}\right) \left(\mathcal{I}_{\beta}^{\gamma+\delta, n-\delta} u\right).$$

Remark 1 *Let $\delta, \beta > 0$ and $\gamma \in \mathbb{R}$. Then we have*

$$\mathcal{D}_{\beta}^{\gamma,\delta} t^{p} = \frac{\Gamma\left(\gamma + \delta + \frac{p}{\beta} + 1\right)}{\Gamma\left(\gamma + \frac{p}{\beta} + 1\right)} t^{p},\ \ p + \beta(\gamma+1) > 0.$$

In particular,

$$\mathcal{D}_{\beta}^{\gamma,\delta} t^{-\beta(\gamma-i)} = 0,\ \ \text{for each } i = \{1, 2, \ldots, n\}.$$

Definition 4 *Let E be a Banach space; a subset P in $C(E)$ is called equicontinuous if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall u, v \in E,\ \forall \mathcal{A} \in P,\ \ \|u - v\| < \delta \Rightarrow |\mathcal{A}(u) - \mathcal{A}(v)| < \varepsilon.$$

Theorem 1 (Ascoli–Arzela [7]) *Let E be a compact metric space. If $\mathcal{P}$ is an equicontinuous, bounded subset of $C(E)$, then $\mathcal{P}$ is relatively compact.*

Definition 5 ([4]) *Let E be a Banach space. A map $\mathcal{A} : X \subseteq E \to E$ is said to be completely continuous if $\mathcal{A}(Y)$ is relatively compact for all bounded sets $Y \subseteq X$.*

The following fixed-point theorems are fundamental in the proofs of our main results.

Theorem 2 (Leray–Schauder Nonlinear Alternative theorem [14]) *Let E be a Banach space, and Ω a bounded open subset of E with $0 \in \Omega$. Then, every completely continuous map $\mathcal{A} : \overline{\Omega} \to E$ has at least one of the following two properties:*

(i) $\mathcal{A}$ has a fixed point in $\overline{\Omega}$.

(ii) There is an $x \in \partial\Omega$ and $\lambda \in (0,1)$ with $x = \lambda \mathcal{A} x$.

Theorem 3 (Banach's fixed point theorem [1]) *Let E be a Banach space, D be closed subset of E, and $\mathcal{A} : D \to D$ be a strict contraction, i.e.,*

$$\|\mathcal{A}u - \mathcal{A}v\| \leq k \|u - v\| \text{ for some } k \in (0,1) \text{ and all } u, v \in D.$$

Then $\mathcal{A}$ has a unique fixed point.

3 Main Results

In this section, we prove a preparatory lemma for the boundary value problem of nonlinear fractional differential equations with an Erdélyi-Kober derivative.

Lemma 1 ([5]) *Let $1 < \delta \leq 2$, $-2 < \gamma < -1$, $\beta > 0$ and $y \in C_\alpha^2$, with $\int_0^\infty s^{\beta(\gamma+m)-1} y(s) ds < \infty$, $m = \{1, 2\}$. Then, the fractional differential equation*

$$\mathcal{D}_\beta^{\gamma,\delta} u(t) + y(t) = 0, \; t > 0, \tag{6}$$

with the conditions

$$\lim_{t \to 0} t^{\beta(2+\gamma)} \mathcal{I}^{\delta+\gamma, 2-\delta} u(t) = 0, \tag{7}$$

$$\lim_{t \to \infty} t^{\beta(1+\gamma)} \mathcal{I}^{\delta+\gamma, 2-\delta} u(t) = 0, \tag{8}$$

has a unique solution given by

$$u(t) = \int_0^\infty G_\delta(t,s) \, s^{\beta(\gamma+1)-1} y(s) \, ds, \tag{9}$$

where

$$G_\delta(t,s) = \begin{cases} \frac{\beta}{\Gamma(\delta)} \left[t^{-\beta(\gamma+1)} - t^{-\beta(\delta+\gamma)} \left(t^\beta - s^\beta\right)^{\delta-1} \right], & 0 < s \leq t < \infty, \\ \frac{\beta}{\Gamma(\delta)} t^{-\beta(\gamma+1)}, & 0 < t \leq s < \infty, \end{cases} \tag{10}$$

is called the Green function of the boundary value problem (6)–(8).

Now, we present some properties of Green's function that form the basis of our main work.

Lemma 2 ([5]) *For $1 < \delta \leq 2$, $-2 < \gamma < -1$, and $\beta > 0$, the function $G_\delta(t,s)$ in Lemma 1 satisfies the following conditions:*

1. $\frac{G_\delta(t,s)}{1+t^{-\beta(1+\gamma)}} > 0, \; \forall t, s \in (0, \infty)$.
2. $\frac{G_\delta(t,s)}{1+t^{-\beta(1+\gamma)}} \leq \frac{\beta}{\Gamma(\delta)}, \; \forall t, s \in (0, \infty)$.
3. *For all $0 < \frac{\tau}{\lambda} \leq t \leq \tau$ and $\forall s > \frac{\tau}{\lambda^2}$, where $\lambda > 1$, $\tau > 0$, we have*

$$\frac{G_\delta(t,s)}{1+t^{-\beta(1+\gamma)}} \geq \frac{\beta(\delta-1)\tau^{-\beta(1+\gamma)}}{\Gamma(\delta)\lambda^{\beta(1-\gamma)}\left(1+\tau^{-\beta(1+\gamma)}\right)} = \frac{\beta}{\Gamma(\delta)} p(\tau).$$

Based on the previous Lemma 1, we will define the integral solution of the problem (1)–(3).

Lemma 3 *Let $0 < \delta_1, \delta_2 \leq 1$. We give $f_k \in C^2_\alpha, (k = 1, 2)$ with*

$$\int_0^\infty s^{\beta(\gamma+m)-1} f_k(s, u(s), v(s))ds < \infty, \quad m = \{1, 2\}.$$

Then the problem (1)–(3) has a unique solution given by

$$u(t) = \int_0^\infty G_{\delta_1}(t, s)\, s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s))ds, \tag{11}$$

$$v(t) = \int_0^\infty G_{\delta_2}(t, s)\, s^{\beta(\gamma+1)-1} f_2(s, u(s), v(s))ds, \tag{12}$$

where

$$G_{\delta_1}(t, s) = \begin{cases} \frac{\beta}{\Gamma(\delta_1)} \left[t^{-\beta(\gamma+1)} - t^{-\beta(\delta_1+\gamma)} \left(t^\beta - s^\beta\right)^{\delta_1 - 1} \right], & 0 < s \leq t < \infty, \\ \frac{\beta}{\Gamma(\delta_1)} t^{-\beta(\gamma+1)}, & 0 < t \leq s < \infty, \end{cases} \tag{13}$$

$$G_{\delta_2}(t, s) = \begin{cases} \frac{\beta}{\Gamma(\delta_2)} \left[t^{-\beta(\gamma+1)} - t^{-\beta(\delta_2+\gamma)} \left(t^\beta - s^\beta\right)^{\delta_2 - 1} \right], & 0 < s \leq t < \infty, \\ \frac{\beta}{\Gamma(\delta_2)} t^{-\beta(\gamma+1)}, & 0 < t \leq s < \infty, \end{cases} \tag{14}$$

Proof. By Lemma 1, for $\delta = \delta_1$, and $\delta = \delta_2$, respectively. ■

We now turn to the question of existence for the boundary value problem (1)–(3).

In this work, we use the space $E = X \times Y$, with the norm $\|(u, v)\|_E = \|u\|_X + \|v\|_Y$, Obviously $(E, \|(u, v)\|_E)$, is a Banach space, where

$$X = \left\{ u \in C((0, \infty), \mathbb{R}) \;\middle|\; \begin{array}{l} \lim_{t\to 0} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \text{ and} \\ \lim_{t\to+\infty} \frac{u(t)}{1+t^{-\beta(1+\gamma)}} \text{ exist} \end{array} \right\},$$

with the norm

$$\|u\|_X = \sup_{t>0} \left| \frac{u(t)}{1 + t^{-\beta(1+\gamma)}} \right|.$$

and

$$Y = \left\{ v \in C((0, \infty), \mathbb{R}) \;\middle|\; \begin{array}{l} \lim_{t\to 0} \frac{v(t)}{1+t^{-\beta(1+\gamma)}} \text{ and} \\ \lim_{t\to+\infty} \frac{v(t)}{1+t^{-\beta(1+\gamma)}} \text{ exist} \end{array} \right\},$$

with the norm

$$\|v\|_Y = \sup_{t>0} \left| \frac{v(t)}{1 + t^{-\beta(1+\gamma)}} \right|.$$

Define an integral operator $\mathcal{A} : E \to E$ by

$$\mathcal{A}(u, v)(t) = \begin{pmatrix} \mathcal{A}_1(u, v)(t) \\ \mathcal{A}_2(u, v)(t) \end{pmatrix}, \tag{15}$$

where

$$\mathcal{A}_1(u, v)(t) = \int_0^\infty G_{\delta_1}(t, s)\, s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s))ds,$$

$$\mathcal{A}_2(u, v)(t) = \int_0^\infty G_{\delta_2}(t, s)\, s^{\beta(\gamma+1)-1} f_2(s, u(s), v(s))ds,$$

with $G_{\delta_k}(t, s), (k = 1.2)$ is defined by (13) and (14).

Clearly, from Lemma 3, the fixed points of the operator $\mathcal{A}$ coincide with the solutions of the problem (1)–(3).

Lemma 4 *If (H1)–(H2) hold, then $\mathcal{A}: E \to E$ is completely continuous.*

Proof. First, for all $(u,v) \in E$, we have

$$\begin{aligned}
\left\| \mathcal{A}_1(u,v)(t) \right\|_X &= \sup_{t>0} \frac{|\mathcal{A}_1(u,v)(t)|}{1+t^{-\beta(1+\gamma)}} \\
&= \sup_{t>0} \left| \int_0^\infty \frac{G_{\delta_1}(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))ds \right| \\
&\leq \frac{\beta}{\Gamma(\delta)} \int_0^\infty \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds;
\end{aligned}$$

together with conditions (H1) and (H2), it then follows that

$$\begin{aligned}
& \int_0^\infty \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds \\
=& \int_0^\infty \left| s^{\beta(\gamma+1)-1} f_1\left(s, \frac{\left(1+s^{-\beta(1+\gamma)}\right)u(s)}{1+s^{-\beta(1+\gamma)}}, \frac{\left(1+s^{-\beta(1+\gamma)}\right)v(s)}{1+s^{-\beta(1+\gamma)}}\right) \right| ds \\
=& \int_0^\infty \left| F_1\left(s, \frac{u(s)}{1+s^{-\beta(1+\gamma)}}, \frac{v(s)}{1+s^{-\beta(1+\gamma)}}\right) \right| ds \\
\leq& \int_0^\infty \left(\varphi_1(s)\omega_1\left(\frac{|u(s)|}{1+s^{-\beta(1+\gamma)}}\right) + \psi_1(s)\omega_2\left(\frac{|v(s)|}{1+s^{-\beta(1+\gamma)}}\right)\right) ds \\
\leq& \omega_1\left(\|u\|_X\right) \int_0^\infty \varphi_1(s)ds + \omega_2\left(\|v\|_Y\right) \int_0^\infty \psi_1(s)ds < \infty.
\end{aligned}$$

Similarly, one can find that

$$\left\| \mathcal{A}_2(u,v)(t) \right\|_Y \leq \tilde{\omega}_1\left(\|u\|_X\right) \int_0^\infty \varphi_2(s)ds + \tilde{\omega}_2\left(\|v\|_Y\right) \int_0^\infty \psi_2(s)ds < \infty.$$

Hence, $\mathcal{A}: E \to E$ is well-defined. Let $\Omega = \{(u,v) \in E, \|(u,v)\|_E \leq r,\ r > 0\}$ be a bounded subset of E. In the following, we divide the proof into several steps.

Step 1: $\mathcal{A}$ is continuous.
Let $(u_n,v_n)_{n\in\mathbb{N}} \in E$ be a convergent sequence to (u,v) in E such that $\|(u,v)\|_E \leq r$, from Lemma 2, we obtain that

$$\begin{aligned}
\left\| \mathcal{A}_1(u_n,v_n) - \mathcal{A}_1(u,v) \right\|_X &= \sup_{t\in(0,\infty)} \left| \frac{\mathcal{A}_1(u_n,v_n)(t) - \mathcal{A}_1(u,v)(t)}{1+t^{-\beta(1+\gamma)}} \right| \\
&\leq \frac{\beta}{\Gamma(\delta_1)} \sup_{t\in(0,\infty)} \left| \int_0^\infty s^{\beta(\gamma+1)-1} f_1(s,u_n(s),v_n(s))ds \right. \\
&\quad \left. - \int_0^\infty s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))ds \right|.
\end{aligned}$$

By the condition (H2), we obtain

$$\left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| \leq \omega_1(r)\varphi_1(s) + \omega_2(r)\psi_1(s) \in L^1(0,\infty).$$

Together with the continuity of the function $s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))$, the Lebesgue dominated convergence theorem (Theorem 12.12, page 199 in [6]) yields $(u,v) \to \int_0^\infty s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))ds$, is continuous and it follows that

$$\int_0^\infty s^{\beta(\gamma+1)-1} f_1(s,u_n(s),v_n(s))ds \to \int_0^\infty s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))ds \text{ as } n \to \infty.$$

Therefore,

$$\|\mathcal{A}_1(u_n,v_n)-\mathcal{A}_1(u,v)\|_X \to 0, \text{ as } n\to\infty.$$

Similarly, one can find that

$$\|\mathcal{A}_2(u_n,v_n)-\mathcal{A}_2(u,v)\|_Y \to 0, \text{ as } n\to\infty.$$

Thus

$$\|\mathcal{A}(u_n,v_n)-\mathcal{A}(u,v)\|_E \to 0, \text{ as } n\to\infty.$$

Step2: $\mathcal{A}(\Omega)$ is relatively compact.
First, we show that $\mathcal{A}(\Omega)$ is uniformly bounded. Let $(u,v)\in\Omega$, by the condition (H2), we obtain

$$\begin{aligned}
\frac{|\mathcal{A}_1(u,v)(t)|}{1+t^{-\beta(1+\gamma)}} &= \left|\int_0^\infty \frac{G_{\delta_1}(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))ds\right| \\
&\le \frac{\beta}{\Gamma(\delta_1)}\int_0^\infty \left|s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))\right| ds \\
&\le \frac{\beta}{\Gamma(\delta_1)}\omega_1(r)\int_0^\infty \varphi_1(s)ds + \frac{\beta}{\Gamma(\delta_1)}\omega_2(r)\int_0^\infty \psi_1(s)ds<\infty,
\end{aligned}$$

consequently,

$$\|\mathcal{A}_1(u,v)\|_X \le \frac{\beta}{\Gamma(\delta_1)}\omega_1(r)\int_0^\infty \varphi_1(s)ds + \frac{\beta}{\Gamma(\delta_1)}\omega_2(r)\int_0^\infty \psi_1(s)ds<\infty, \tag{16}$$

Similarly, one can find that

$$\|\mathcal{A}_2(u,v)\|_Y \le \frac{\beta}{\Gamma(\delta_2)}\tilde{\omega}_1(r)\int_0^\infty \varphi_2(s)ds + \frac{\beta}{\Gamma(\delta_2)}\tilde{\omega}_2(r)\int_0^\infty \psi_2(s)ds<\infty, \tag{17}$$

for all $(u,v)\in\Omega$. Hence, $\mathcal{A}(\Omega)$ is uniformly bounded.

Next, letting $V=\left\{\frac{\mathcal{A}(u,v)}{1+t^{-\beta(1+\gamma)}}\,|(u,v)\in\Omega\right\}$, we show that V is equicontinuous on any compact interval of $\mathbb{R}_+$. For all $(u,v)\in\Omega$, $t_1,t_2\in[a,b]$, $0<a<b<\infty$, and $t_1\le t_2$, we can find

$$\begin{aligned}
&\left|\frac{\mathcal{A}_1(u,v)(t_2)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}_1(u,v)(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\le& \int_0^\infty \left|\frac{G_{\delta_1}(t_2,s)}{1+t_2^{-\beta(1+\gamma)}}-\frac{G_{\delta_1}(t_1,s)}{1+t_1^{-\beta(1+\gamma)}}\right|\left|s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))\right| ds \\
\le& \int_0^\infty \left|\frac{G_{\delta_1}(t_2,s)}{1+t_2^{-\beta(1+\gamma)}}-\frac{G_{\delta_1}(t_1,s)}{1+t_2^{-\beta(1+\gamma)}}+\frac{G_{\delta_1}(t_1,s)}{1+t_2^{-\beta(1+\gamma)}}\right. \\
&\left.-\frac{G_{\delta_1}(t_1,s)}{1+t_1^{-\beta(1+\gamma)}}\right|\left|s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))\right| ds \\
\le& \int_0^\infty \left|\frac{G_{\delta_1}(t_2,s)-G_{\delta_1}(t_1,s)}{1+t_2^{-\beta(1+\gamma)}}-\frac{G_{\delta_1}(t_1,s)\left(t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}\right)}{\left(1+t_2^{-\beta(1+\gamma)}\right)\left(1+t_1^{-\beta(1+\gamma)}\right)}\right| \\
&\times\left|s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))\right| ds \\
\le& \int_0^\infty \frac{|G_{\delta_1}(t_2,s)-G_{\delta_1}(t_1,s)|}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))\right| ds \\
&+\int_0^\infty \frac{G_{\delta_1}(t_1,s)\left(t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}\right)}{\left(1+t_2^{-\beta(1+\gamma)}\right)\left(1+t_1^{-\beta(1+\gamma)}\right)}\left|s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s))\right| ds. \\
\to& \ 0, \text{ uniformly as } t_1\to t_2 \text{ for all } (u,v)\in\Omega.
\end{aligned}$$

and

$$
\begin{aligned}
& \left|\frac{\mathcal{A}_2(u,v)\,(t_2)}{1+t_2^{-\beta(1+\gamma)}}-\frac{\mathcal{A}_2(u,v)\,(t_1)}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\leq\ & \int_0^{\infty}\frac{\left|G_{\delta_2}\,(t_2,s)-G_{\delta_2}\,(t_1,s)\right|}{1+t_2^{-\beta(1+\gamma)}}\left|s^{\beta(\gamma+1)-1}f_2(s,u(s),v(s))\right|ds \\
& +\int_0^{\infty}\frac{G_{\delta_2}\,(t_1,s)\left(t_2^{-\beta(1+\gamma)}-t_1^{-\beta(1+\gamma)}\right)}{\left(1+t_2^{-\beta(1+\gamma)}\right)\left(1+t_1^{-\beta(1+\gamma)}\right)}\left|s^{\beta(\gamma+1)-1}f_2(s,u(s),v(s))\right|ds. \\
\to\ & 0,\ \text{uniformly as } t_1\to t_2 \text{ for all } (u,v)\in\Omega.
\end{aligned}
$$

Hence, V is locally equicontinuous on $(0,\infty)$.

Finally, we show that V is equiconvergent at ∞. We know that

$$
\begin{aligned}
\mathcal{A}_1(u,v)(t) \;=\;& \frac{\beta}{\Gamma(\delta_1)}t^{-\beta(1+\gamma)}\int_0^{\infty}s^{\beta(\gamma+1)-1}f_1(s,u(s),v(s))ds \\
& \frac{-\beta}{\Gamma(\delta_1)}t^{-\beta(\gamma+\delta_1)}\int_0^{t}\left(t^{\beta}-s^{\beta}\right)^{\delta_1-1}s^{\beta(\gamma+1)-1}f_1(s,u(s),v(s))ds,
\end{aligned}
\tag{18}
$$

observing that for any $(u,v)\in\Omega$, the condition (H2) gives

$$
\int_0^{\infty}\left|s^{\beta(\gamma+1)-1}f_1(s,u(s),v(s))\right|ds<\infty, \tag{19}
$$

for a given $\varepsilon>0$, there exists a constant $L>0$, such that

$$
\int_L^{\infty}\left|s^{\beta(\gamma+1)-1}f_1(s,u(s),v(s))\right|ds<\varepsilon. \tag{20}
$$

However, because $\lim\limits_{t\to+\infty}\frac{t^{-\beta(1+\gamma)}}{1+t^{-\beta(1+\gamma)}}=1$, there exists a constant $T_1>0$, such that for any $t_1,t_2\geq T_1$, and we obtain

$$
\left|\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|\leq\left|1-\frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}}\right|<\varepsilon. \tag{21}
$$

Similarly, $\lim\limits_{t\to+\infty}\frac{t^{-\beta(\delta_1+\gamma)}\left(t^{\beta}-s^{\beta}\right)^{\delta_1-1}}{1+t^{-\beta(1+\gamma)}}=1$, and thus there exists a constant $T_2>L>0$, such that for any $t_1,t_2\geq T_2$ and $0<s\leq L$, it holds that

$$
\begin{aligned}
& \left|\frac{t_2^{-\beta(\delta_1+\gamma)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta_1-1}}{1+t_2^{-\beta(1+\gamma)}}-\frac{t_1^{-\beta(\delta_1+\gamma)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta_1-1}}{1+t_1^{-\beta(1+\gamma)}}\right| \\
\leq\ & \left|1-\frac{t_1^{-\beta(\delta_1+\gamma)}\left(t_1^{\beta}-s^{\beta}\right)^{\delta_1-1}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(\delta_1+\gamma)}\left(t_2^{\beta}-s^{\beta}\right)^{\delta_1-1}}{1+t_2^{-\beta(1+\gamma)}}\right| \\
\leq\ & \left|1-\frac{t_1^{-\beta(\delta_1+\gamma)}\left(t_1^{\beta}-L^{\beta}\right)^{\delta_1-1}}{1+t_1^{-\beta(1+\gamma)}}\right|+\left|1-\frac{t_2^{-\beta(\delta_1+\gamma)}\left(t_2^{\beta}-L^{\beta}\right)^{\delta_1-1}}{1+t_2^{-\beta(1+\gamma)}}\right| \\
<\ & \varepsilon.
\end{aligned}
\tag{22}
$$

Now, we choose $T > \max\{T_1, T_2\}$, for all $t_1, t_2 \geq T$, by (18), we can obtain

$$
\begin{aligned}
&\left| \frac{\mathcal{A}_1(u,v)(t_2)}{1+t_2^{-\beta(1+\gamma)}} - \frac{\mathcal{A}_1(u,v)(t_1)}{1+t_1^{-\beta(1+\gamma)}} \right| \\
\leq\ & \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds \\
&+ \frac{\beta}{\Gamma(\delta_1)} \left| \int_0^{t_2} \frac{t_2^{-\beta(\gamma+\delta_1)} \left(t_2^\beta - s^\beta\right)^{\delta_1-1}}{1+t_2^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds \right. \\
&\left. - \int_0^{t_1} \frac{t_1^{-\beta(\gamma+\delta_1)} \left(t_1^\beta - s^\beta\right)^{\delta_1-1}}{1+t_1^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds \right| \\
\leq\ & \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds \\
&+ \frac{\beta}{\Gamma(\delta_1)} \left| \int_0^{L} \frac{t_2^{-\beta(\gamma+\delta_1)} \left(t_2^\beta - s^\beta\right)^{\delta_1-1}}{1+t_2^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds \right. \\
&\left. - \int_0^{L} \frac{t_1^{-\beta(\gamma+\delta_1)} \left(t_1^\beta - s^\beta\right)^{\delta_1-1}}{1+t_1^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds \right| \\
&+ \frac{\beta}{\Gamma(\delta_1)} \left| \int_L^{t_2} \frac{t_2^{-\beta(\gamma+\delta_1)} \left(t_2^\beta - s^\beta\right)^{\delta_1-1}}{1+t_2^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds \right. \\
&\left. - \int_L^{t_1} \frac{t_1^{-\beta(\gamma+\delta_1)} \left(t_1^\beta - s^\beta\right)^{\delta_1-1}}{1+t_1^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) ds \right|,
\end{aligned}
$$

a direct calculation yields

$$
\begin{aligned}
&\left| \frac{\mathcal{A}_1(u,v)(t_2)}{1+t_2^{-\beta(1+\gamma)}} - \frac{\mathcal{A}_1(u,v)(t_1)}{1+t_1^{-\beta(1+\gamma)}} \right| \\
\leq\ & \frac{\beta}{\Gamma(\delta_1)} \left| \frac{t_2^{-\beta(1+\gamma)}}{1+t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(1+\gamma)}}{1+t_1^{-\beta(1+\gamma)}} \right| \int_0^\infty \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds \\
&+ \frac{\beta}{\Gamma(\delta_1)} \int_0^{L} \left| \frac{t_2^{-\beta(\gamma+\delta_1)} \left(t_2^\beta - s^\beta\right)^{\delta_1-1}}{1+t_2^{-\beta(1+\gamma)}} - \frac{t_1^{-\beta(\gamma+\delta_1)} \left(t_1^\beta - s^\beta\right)^{\delta_1-1}}{1+t_1^{-\beta(1+\gamma)}} \right| \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds \\
&+ \frac{\beta}{\Gamma(\delta_1)} \int_L^{t_2} \frac{t_2^{-\beta(\gamma+\delta_1)} \left(t_2^\beta - s^\beta\right)^{\delta_1-1}}{1+t_2^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds \\
&+ \frac{\beta}{\Gamma(\delta_1)} \int_L^{t_1} \frac{t_1^{-\beta(\gamma+\delta_1)} \left(t_1^\beta - s^\beta\right)^{\delta_1-1}}{1+t_1^{-\beta(1+\gamma)}} \left| s^{\beta(\gamma+1)-1} f_1(s,u(s),v(s)) \right| ds.
\end{aligned}
$$

From (19), (20), (21), (22) and for $t_1, t_2 \to \infty$, we obtain

$$\begin{aligned} & \left| \frac{\mathcal{A}_1(u,v)(t_2)}{1+t_2^{-\beta(1+\gamma)}} - \frac{\mathcal{A}_1(u,v)(t_1)}{1+t_1^{-\beta(1+\gamma)}} \right| \\ < & \frac{\beta}{\Gamma(\delta_1)} \varepsilon \int_0^\infty \left| s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s)) \right| ds \\ & + \frac{\beta}{\Gamma(\delta_1)} \varepsilon \int_0^L \left| s^{\beta(\gamma+1)-1} f_1(s, u(s), v(s)) \right| ds + \frac{2\beta}{\Gamma(\delta_1)} \varepsilon. \end{aligned}$$

Analogously, we can obtain

$$\begin{aligned} & \left| \frac{\mathcal{A}_2(u,v)(t_2)}{1+t_2^{-\beta(1+\gamma)}} - \frac{\mathcal{A}_2(u,v)(t_1)}{1+t_1^{-\beta(1+\gamma)}} \right| \\ < & \frac{\beta}{\Gamma(\delta_2)} \varepsilon \int_0^\infty \left| s^{\beta(\gamma+1)-1} f_2(s, u(s), v(s)) \right| ds \\ & + \frac{\beta}{\Gamma(\delta_2)} \varepsilon \int_0^L \left| s^{\beta(\gamma+1)-1} f_2(s, u(s), v(s)) \right| ds + \frac{2\beta}{\Gamma(\delta_2)} \varepsilon. \end{aligned}$$

Hence, V is equiconvergent at ∞. Consequently, V is relatively compact.
Therefore, $A : E \to E$ is completely continuous. ■

3.1 Existence of At Least One Solution

Now, to prove the first existence result, we use the Leray-Schauder nonlinear alternative fixed point theorem.

Theorem 4 *Assume that hypotheses (H1)–(H2) hold, and*

$$\begin{aligned} & \frac{\beta}{\Gamma(\delta_1)} \omega_1(r) \int_0^\infty \varphi_1(s) ds + \frac{\beta}{\Gamma(\delta_1)} \omega_2(r) \int_0^\infty \psi_1(s) ds \\ & + \frac{\beta}{\Gamma(\delta_2)} \tilde{\omega}_1(r) \int_0^\infty \varphi_2(s) ds + \frac{\beta}{\Gamma(\delta_2)} \tilde{\omega}_2(r) \int_0^\infty \psi_2(s) ds < r. \end{aligned} \tag{23}$$

Then, the fractional boundary value problem (1)–(3) has at least one solution $(u, v) \in \Omega$.

Proof. From Lemma 4, we know that $\mathcal{A}$ is a completely continuous operator. We apply the nonlinear alternative of Leray-Schauder to prove that $\mathcal{A}$ has at least one nontrivial solution in Ω. Let $(u, v) \in \partial\Omega$, such that $(u, v) = \lambda \mathcal{A}(u, v)$, $\lambda \in (0, 1)$, from (16) and (17), we obtain

$$\begin{aligned} & \|(u,v)\|_E = \|u\|_X + \|v\|_Y \\ = & \lambda \|\mathcal{A}_1(u,v)\|_X + \lambda \|\mathcal{A}_2(u,v)\|_Y \\ \le & \frac{\beta}{\Gamma(\delta_1)} \omega_1(r) \int_0^\infty \varphi_1(s) ds + \frac{\beta}{\Gamma(\delta_1)} \omega_2(r) \int_0^\infty \psi_1(s) ds \\ & + \frac{\beta}{\Gamma(\delta_2)} \tilde{\omega}_1(r) \int_0^\infty \varphi_2(s) ds + \frac{\beta}{\Gamma(\delta_2)} \tilde{\omega}_2(r) \int_0^\infty \psi_2(s) ds, \end{aligned}$$

and thus

$$\begin{aligned} r \le & \frac{\beta}{\Gamma(\delta_1)} \omega_1(r) \int_0^\infty \varphi_1(s) ds + \frac{\beta}{\Gamma(\delta_1)} \omega_2(r) \int_0^\infty \psi_1(s) ds \\ & + \frac{\beta}{\Gamma(\delta_2)} \tilde{\omega}_1(r) \int_0^\infty \varphi_2(s) ds + \frac{\beta}{\Gamma(\delta_2)} \tilde{\omega}_2(r) \int_0^\infty \psi_2(s) ds, \end{aligned}$$

which contradicts (23). By Theorem 2 and Lemma 4, the boundary value problem (1)–(3) has at least one solution $(u, v) \in \Omega$. ■

3.2 Uniqueness of Solution

The last result of the existence is based on the Banach contraction principle theorem.

Theorem 5 *Assume that hypotheses (H1), (H2) and (H3) hold. If*

$$\frac{\beta(q_1^* + \tilde{q}_1^*)}{\Gamma(\delta_1)} + \frac{\beta(q_2^* + \tilde{q}_2^*)}{\Gamma(\delta_2)} < 1, \tag{24}$$

then the boundary value problem (1)–(3) has a unique solution.

Proof. We shall show that the operator $\mathcal{A}$ defined by (15) is a contraction mapping. Let $(u,v), (\tilde{u},\tilde{v}) \in E$; from Lemma 2 and by the condition (H3), we can obtain that

$$\begin{aligned}
&\left|\frac{\mathcal{A}_1(u,v)(t) - \mathcal{A}_1(\tilde{u},\tilde{v})(t)}{1+t^{-\beta(1+\gamma)}}\right| \\
= &\int_0^\infty \frac{G_{\delta_1}(t,s)}{1+t^{-\beta(1+\gamma)}} s^{\beta(\gamma+1)-1} \left|f_1(s,u(s),v(s)) - f_1(s,\tilde{u}(s),\tilde{v}(s))\right| ds \\
\leq &\frac{\beta}{\Gamma(\delta_1)} \int_0^\infty s^{\beta(\gamma+1)-1} \left|f_1(s,u(s),v(s)) - f_1(s,\tilde{u}(s),\tilde{v}(s))\right| ds \\
\leq &\frac{\beta}{\Gamma(\delta_1)} \int_0^\infty q_1(s)\left|u-\tilde{u}\right| + \tilde{q}_1(s)\left|v-\tilde{v}\right| ds \\
\leq &\frac{\beta}{\Gamma(\delta_1)} \int_0^\infty q_1(s)\left(1+s^{-\beta(1+\gamma)}\right)\left|\frac{u-\tilde{u}}{1+s^{-\beta(1+\gamma)}}\right| ds \\
&+\frac{\beta}{\Gamma(\delta_1)} \int_0^\infty \tilde{q}_1(s)\left(1+s^{-\beta(1+\gamma)}\right)\left|\frac{v-\tilde{v}}{1+s^{-\beta(1+\gamma)}}\right| ds,
\end{aligned}$$

this implies that

$$\begin{aligned}
\left\|\mathcal{A}_1(u,v) - \mathcal{A}_1(\tilde{u},\tilde{v})\right\|_X &\leq \frac{\beta}{\Gamma(\delta_1)} \left\|u-\tilde{u}\right\|_X \int_0^\infty q_1(s)\left(1+s^{-\beta(1+\gamma)}\right) ds \\
&+ \frac{\beta}{\Gamma(\delta_1)} \left\|v-\tilde{v}\right\|_Y \int_0^\infty \tilde{q}_1(s)\left(1+s^{-\beta(1+\gamma)}\right) ds,
\end{aligned}$$

then

$$\left\|\mathcal{A}_1(u,v) - \mathcal{A}_1(\tilde{u},\tilde{v})\right\|_X \leq \frac{\beta(q_1^* + \tilde{q}_1^*)}{\Gamma(\delta_1)}(\left\|u-\tilde{u}\right\|_X + \left\|v-\tilde{v}\right\|_Y). \tag{25}$$

Similarly, one can find that

$$\left\|\mathcal{A}_2(u,v) - \mathcal{A}_2(\tilde{u},\tilde{v})\right\|_Y \leq \frac{\beta(q_2^* + \tilde{q}_2^*)}{\Gamma(\delta_2)}(\left\|u-\tilde{u}\right\|_X + \left\|v-\tilde{v}\right\|_Y). \tag{26}$$

Thus it follows from (25) and (26), that

$$\left\|\mathcal{A}(u,v) - \mathcal{A}(\tilde{u},\tilde{v})\right\|_E \leq \left[\frac{\beta(q_1^* + \tilde{q}_1^*)}{\Gamma(\delta_1)} + \frac{\beta(q_2^* + \tilde{q}_2^*)}{\Gamma(\delta_2)}\right] \left\|(u,v) - (\tilde{u},\tilde{v})\right\|_E .$$

It follows from the assumption (24) and the preceding estimate that $\mathcal{A}$ is a contraction mapping. Applying Banach's fixed point Theorem 3, the operator $\mathcal{A}$ has a fixed point that corresponds to the unique solution of problem (1)–(3). ■

4 Examples

In this section, we present some examples to illustrate the usefulness of our main results.

Example 1 *Consider the following boundary value problem for a coupled System:*

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}}u(t)+t^{\frac{3}{2}}e^{-t}\sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|}+e^{-t}\ln\left(\arctan\left(\left|\frac{v}{1+t^{\frac{1}{2}}}\right|\right)\right)=0,\ t>0,\\ \mathcal{D}_1^{-\frac{3}{2},\frac{7}{6}}v(t)+t^{\frac{5}{2}}\ln\left(\left(\frac{v}{1+t^{\frac{1}{2}}}\right)^2+1\right)e^{-2t^2+1}+t^{\frac{3}{2}}\ln\left(\frac{u}{1+t^{\frac{1}{2}}}\right)^2e^{-t}=0, t>0,\\ \lim_{t\to 0}t_1^{\frac{1}{2}}\mathcal{I}_1^{0,\frac{1}{2}}u(t)=0,\ \lim_{t\to\infty}t^{-\frac{1}{2}}\mathcal{I}_1^{0,\frac{1}{2}}u(t)=0,\\ \lim_{t\to 0}t^{\frac{1}{2}}\mathcal{I}_1^{\frac{-1}{3},\frac{5}{6}}v(t)=0,\ \lim_{t\to\infty}t^{-\frac{1}{2}}\mathcal{I}_1^{\frac{-1}{3},\frac{5}{6}}v(t)=0. \end{cases} \tag{27}$$

Here,

$$f_1(t,u,v)=t^{\frac{3}{2}}\left(\sqrt{\left|\frac{u}{1+t^{\frac{1}{2}}}\right|}+\ln\left(\arctan\left(\left|\frac{v}{1+t^{\frac{1}{2}}}\right|\right)\right)\right)e^{-t},$$

$$f_2(t,u,v)=t^{\frac{5}{2}}\ln\left(\left(\frac{v}{1+t^{\frac{1}{2}}}\right)^2+1\right)e^{-2t^2+1}+t^{\frac{3}{2}}\ln\left(\frac{u}{1+t^{\frac{1}{2}}}\right)^2e^{-t},$$

with $\delta_1=\frac{3}{2},\ \delta_2=\frac{7}{6},\ \gamma=-\frac{3}{2}$ *and* $\beta=1$.

(H1) *It is easy to show that the functions f_1 and f_2 are continuous for any $(t,u)\in(0,\infty)\times\mathbb{R}$.*

- *From the expression of the function f_1, it follows that*

$$\begin{aligned} F_1(t,u,v) &= t^{\beta(1+\gamma)-1}f_1\left(t,\left(1+t^{-\beta(1+\gamma)}\right)u,\left(1+t^{-\beta(1+\gamma)}\right)v\right)\\ &= \left(\sqrt{|u|}+\ln\left(\arctan\left(|v|\right)\right)\right)e^{-t}. \end{aligned}$$

If we choose $\omega_1(u)=\sqrt{u}$, $\omega_2(v)=\ln(\arctan(v))$, $\varphi_1(t)=\psi_1(t)=e^{-t}$, *then we obtain*

$$|F_1(t,u,v)|\le\varphi_1(t)\,\omega_1(|u|)+\psi_1(t)\omega_2(|v|),\ \text{on}\ (0,\infty)\times\mathbb{R},$$

with $\omega_1,\omega_2\in C((0,\infty),(0,\infty))$ *nondecreasing and* $\varphi_1,\psi_1\in L^1(0,\infty)$.

- *From the expression of the function f_2, it follows that*

$$\begin{aligned} F_2(t,u,v) &= t^{\beta(1+\gamma)-1}f_2\left(t,\left(1+t^{-\beta(1+\gamma)}\right)u,\left(1+t^{-\beta(1+\gamma)}\right)v\right)\\ &= \ln\left(v^2+1\right)te^{-2t^2+1}+\ln u^2e^{-t}. \end{aligned}$$

If we choose $\widetilde{\omega}_1(u)=\ln u^2$, $\widetilde{\omega}_2(v)=\ln\left(v^2+1\right)$, $\varphi_2(t)=e^{-t}$, $\psi_2(t)=te^{-2t^2+1}$, *then we obtain*

$$|F_2(t,u,v)|\le\varphi_2(t)\,\widetilde{\omega}_1(|u|)+\psi_2(t)\widetilde{\omega}_2(|v|),\ \text{on}\ (0,\infty)\times\mathbb{R},$$

with $\widetilde{\omega}_1,\widetilde{\omega}_2\in C((0,\infty),(0,\infty))$ *nondecreasing and* $\varphi_2,\psi_2\in L^1(0,\infty)$. *Then, the condition (H2) holds.*

If we choose $r=\frac{1}{2}$, *we show that*

$$\frac{2\sqrt{r}}{\sqrt{\pi}}+\frac{2\ln(\arctan(r))}{\sqrt{\pi}}+\Gamma\left(\frac{7}{6}\right)\ln\left(r^2\right)+\frac{e}{4}\Gamma\left(\frac{7}{6}\right)\ln\left(r^2+1\right)<r$$

therefore, (23) is satisfied. Hence, all the conditions of Theorem 4 hold, and problem (27) has at least one solution.

Example 2 *Consider the following boundary value problem for a coupled System:*

$$\begin{cases} \mathcal{D}_1^{-\frac{3}{2},\frac{5}{3}} u(t) + t^{\frac{3}{2}} \left(\sinh\left(\left| \frac{u}{1+t^{\frac{1}{2}}} \right| + 1 \right) - 1 \right) \frac{e^{-t}}{20} + t^{\frac{5}{2}} \ln\left(\left| \frac{v}{1+t^{\frac{1}{2}}} \right| + 1 \right) \frac{e^{-2t^2+1}}{20} = 0, \ t > 0, \\ \mathcal{D}_1^{-\frac{3}{2},\frac{3}{2}} v(t) + t^{\frac{5}{2}} \arctan\left(\left| \frac{v}{1+t^{\frac{1}{2}}} \right| + \frac{1}{3\sqrt{\pi}} \right) \frac{e^{-2t^2+1}}{20} + t^{\frac{5}{2}} \sinh\left(\left| \frac{u}{1+t^{\frac{1}{2}}} \right| \right) \frac{e^{-2t^2+1}}{20} = 0, \ t > 0, \\ \lim_{t\to 0} t^{\frac{1}{2}} \mathcal{I}_1^{\frac{1}{6},\frac{1}{3}} u(t) = 0, \ \ \lim_{t\to\infty} t^{-\frac{1}{2}} \mathcal{I}_1^{\frac{1}{6},\frac{1}{3}} u(t) = 0. \\ \lim_{t\to 0} t^{\frac{1}{2}} \mathcal{I}_1^{0,\frac{1}{2}} v(t) = 0, \ \ \lim_{t\to\infty} t^{-\frac{1}{2}} \mathcal{I}_1^{0,\frac{1}{2}} v(t) = 0, \end{cases} \tag{28}$$

Here,

$$f_1(t,u,v) = t^{\frac{3}{2}} \left(\sinh\left(\left| \frac{u}{1+t^{\frac{1}{2}}} \right| + 1 \right) - 1 \right) \frac{e^{-t}}{20} + t^{\frac{5}{2}} \ln\left(\left| \frac{v}{1+t^{\frac{1}{2}}} \right| + 1 \right) \frac{e^{-2t^2+1}}{20},$$

$$f_2(t,u,v) = t^{\frac{5}{2}} \left(\arctan\left(\left| \frac{v}{1+t^{\frac{1}{2}}} \right| + \frac{1}{3\sqrt{\pi}} \right) + \sinh\left(\left| \frac{u}{1+t^{\frac{1}{2}}} \right| \right) \right) \frac{e^{-2t^2+1}}{20},$$

with $\delta_1 = \frac{5}{3}, \ \delta_2 = \frac{3}{2}, \ \ \gamma = -\frac{3}{2}$ *and* $\beta = 1$.

(H1) *It is easy to show that the functions* f_1 *and* f_2 *are continuous for any* $(t,u) \in (0,\infty) \times \mathbb{R}$.

- *From the expression of the function* f_1, *it follows that*

$$\begin{aligned} F_1(t,u,v) &= t^{\beta(1+\gamma)-1} f_1 \left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u, \left(1 + t^{-\beta(1+\gamma)}\right) v \right) \\ &= \left(\sinh\left(|u|+1\right) - 1\right) \frac{e^{-t}}{20} + \ln\left(|v|+1\right) \frac{te^{-2t^2+1}}{20}. \end{aligned}$$

  If we choose $\omega_1(u) = \sinh(u+1) - 1, \ \omega_2(v) = \ln(v+1), \ \varphi_1(t) = \frac{e^{-t}}{20}, \ \psi_1(t) = \frac{te^{-2t^2+1}}{20}$, *then we obtain*

$$|F_1(t,u,v)| \le \varphi_1(t)\,\omega_1(|u|) + \psi_1(t)\omega_2(|v|), \ \ on \ \ (0,\infty) \times \mathbb{R},$$

  with $\omega_1, \omega_2 \in C\left((0,\infty),(0,\infty)\right)$ *nondecreasing and* $\varphi_1, \psi_1 \in L^1(0,\infty)$.

- *From the expression of the function* f_2, *it follows that*

$$\begin{aligned} F_2(t,u,v) &= t^{\beta(1+\gamma)-1} f_2 \left(t, \left(1 + t^{-\beta(1+\gamma)}\right) u, \left(1 + t^{-\beta(1+\gamma)}\right) v \right) \\ &= \left(\arctan\left(|v| + \frac{1}{3\sqrt{\pi}} \right) + \sinh\left(|u|\right) \right) \frac{te^{-2t^2+1}}{20}. \end{aligned}$$

  If we choose $\widetilde{\omega}_1(u) = \sinh u, \ \widetilde{\omega}_2(v) = \arctan\left(v + \frac{1}{3\sqrt{\pi}}\right), \ \varphi_2(t) = \psi_2(t) = \frac{te^{-2t^2+1}}{20}$, *then we obtain*

$$|F_2(t,u,v)| \le \varphi_2(t)\,\widetilde{\omega}_1(|u|) + \psi_2(t)\widetilde{\omega}_2(|v|), \ \ on \ \ (0,\infty) \times \mathbb{R},$$

  with $\widetilde{\omega}_1, \widetilde{\omega}_2 \in C\left((0,\infty),(0,\infty)\right)$ *nondecreasing and* $\varphi_2, \psi_2 \in L^1(0,\infty)$. *Then, the condition (H2) holds.*

- *We have*

$$\left| t^{-\frac{3}{2}} f_1(t,u,v) - t^{-\frac{3}{2}} f_1(t,\widetilde{u},\widetilde{v}) \right| \le \frac{e^{-t}}{20\left(1+t^{\frac{1}{2}}\right)} |u - \widetilde{u}| + \frac{te^{-2t^2+1}}{20\left(1+t^{\frac{1}{2}}\right)} |v - \widetilde{v}|.$$

If we put $q_1(t) = \frac{e^{-t}}{20\left(1+t^{\frac{1}{2}}\right)}$, $\widetilde{q}_1(t) = \frac{te^{-2t^2+1}}{20\left(1+t^{\frac{1}{2}}\right)}$ *then we obtain*

$$\begin{aligned} q_1^* &= \int_0^{+\infty} \left(1+t^{\frac{1}{2}}\right) \frac{\exp(-t)}{20\left(1+t^{\frac{1}{2}}\right)} dt = \frac{1}{20} < \infty. \\ \widetilde{q}_1^* &= \int_0^{+\infty} \left(1+t^{\frac{1}{2}}\right) \frac{t\exp\left(-2t^2+1\right)}{20\left(1+t^{\frac{1}{2}}\right)} dt = \frac{e}{280} < \infty. \end{aligned}$$

- *We have*

$$\left|t^{-\frac{3}{2}} f_2(t,u,v) - t^{-\frac{3}{2}} f_2(t,\widetilde{u},\widetilde{v})\right| \leq \frac{te^{-2t^2+1}}{20\left(1+t^{\frac{1}{2}}\right)} |u-\widetilde{u}| + \frac{te^{-2t^2+1}}{20\left(1+t^{\frac{1}{2}}\right)} |v-\widetilde{v}|.$$

If we put $q_2(t) = \widetilde{q}_2(t) = \frac{te^{-2t^2+1}}{20\left(1+t^{\frac{1}{2}}\right)}$, *then we obtain*

$$q_2^* = \widetilde{q}_2^* = \int_0^{+\infty} \left(1+t^{\frac{1}{2}}\right) \frac{t\exp\left(-2t^2+1\right)}{20\left(1+t^{\frac{1}{2}}\right)} dt = \frac{e}{280} < \infty.$$

Hence, the condition (H3) is satisfied.
Moreover, we have

$$\frac{\beta\left(q_1^* + \widetilde{q}_1^*\right)}{\Gamma(\delta_1)} + \frac{\beta\left(q_2^* + \widetilde{q}_2^*\right)}{\Gamma(\delta_2)} \simeq 6.6579 \times 10^{-2} < 1,$$

the condition (24) is satisfied. It follows from Theorem 5 that the boundary value problem (28) has a unique solution $(u,v) \in E$.

5 Conclusion

In this work, the existence and uniqueness of solution for the (SFDEs) with initial conditions comprising the Erdélyi–Kober fractional derivatives have been discussed in a special Banach space. For our discussion, we have used the Leray–Schauder nonlinear alternative fixed point Theorem, as well as the Banach contraction principle. Finally, several indispensable examples are presented to illustrate the usefulness of our main results. Future work will be directed toward the Caputo version of the Erdélyi–Kober fractional differential equation and fractional coupled systems.

Acknowledgment. The authors are deeply grateful to the anonymous referees for their kind comments.

References

[1] B. Ahmad, A. Alsaedi, S. K. Ntouyas and J. Tariboon, Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities, Springer International Publishing, 2017.

[2] B. Ahmad, S. K. Ntouyas, J. Tariboonc and A. Alsaedi, A study of nonlinear fractional-order boundary value problem with nonlocal Erdélyi-Kober and generalized Riemann-Liouville type integral boundary conditions, Math. Model. Anal., 22(2017), 121–139.

[3] B. Ahmad, S. K. Ntouyas, J. Tariboonc and A. Alsaedi, Caputo type fractional differential equations with nonlocal Riemann-Liouville and Erdélyi-Kober type integral boundary conditions, Filomat, 31(2017), 4515–4529.

[4] R. P. Agarwal and D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic, Dordrecht, 2001.

[5] Y. Arioua and M. Titraoui, New class of boundary value problem for nonlinear fractional differential equations involving Erdélyi-Kober derivative, Com. Math., 27(2019), 121–139.

[6] R. G. Bartle, A Modern Theory of Integration, Graduate Studies in Mathematics, 32. American Mathematical Society, Providence, RI, 2001.

[7] R. F. Brown, A Topological Introduction to Nonlinear Analysis, Springer-Switzerland, 2014.

[8] C. Corduneanu, Integral Equations and Stability of Feedback Systems, Academic Press, New York, 1973.

[9] S. Das, Functional Fractional Calculus for System Identification and Controls, Springer-Verlag Berlin Heidelberg, 2008.

[10] K. Diethelm, The Analysis of Fractional Differential Equations, Springer Berlin, 2010.

[11] S. L. Kalla and V. S. Kiryakova, An H-function generalized fractional calculus based upon compositions of Erdélyi-Kober operators in L_p, Math. Japon., 35(1990), 1–21.

[12] A. A. Kilbas, H. H. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V, Amsterdam, 2006.

[13] V. Kiryakova, A brief story about the operators of the generalized fractional calculus, Frac. Calc. Appl. Anal., 11(2008), 203–220.

[14] V. Kiryakova, Generalized Fractional Calculus and Applications, Longman and John Wiley, New York, 1994.

[15] V. Kiryakova and Y. Luchko, Riemann-Liouville and Caputo type multiple Erdélyi-Kober operators, Cent. Eur. J. Phys., 11(2013), 1314–1336.

[16] X. Liu and M. Jia, Multiple solutions of nonlocal boundary value problems for fractional differential equations on half-line, Electron. J. Qual. Theory Differ. Equ., 56(2011), 1–14.

[17] Y. Luchko, Operational rules for a mixed operator of the Erdélyi-Kober type, Fract. Calc. Appl. Anal., 7(2007), 339–364.

[18] Y. Luchko and J. Trujillo, Caputo-type modification of the Erdélyi-Kober fractional derivative. Fract. Calc. Appl. Anal., 10(2007), 249–267.

[19] H. Maagli and A. Dhifli, Positive solutions to a nonlinear fractional Dirichlet problem on the half-space, Electron. J. Differ. Equ., 50(2014), 1–7.

[20] A. M. Mathai and H. J. Haubold, Erdélyi-Kober Fractional Calculus, Springer Nature Singapore Pte Ltd, 2018.

[21] S. K. Ntouyas, Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions, Opuscula Math., 33(2013), 117–138.

[22] J. J. Nieto and B. Ahmad, Existence results for a coupled system of nonlinear fractional differential equations with three point boundary conditions, Comput. Math. Appl., 58(2009) 1838–1843, .

[23] G. Pagnini, Erdélyi-Kober fractional diffusion, Fract. Calc. Appl. Anal., 15(2012), 117–127.

[24] L. Plociniczak and S. Sobieszek, Numerical schemes for integro-differential equations with Erdélyi-Kober fractional operator, Numer Algor., 76(2017), 125–150.

[25] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, New York, 1999.

[26] J. Sabatier, O. P. Agrawal and J. A. Tenreiro Machado, Advances in Fractional Calculus Theoretical Developments and Applicationsin Physics and Engineering, Springer, 2007.

[27] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integral and Derivatives Theory and Applications, Gordon and Breach, Switzerland, 1993.

[28] B. A1-Saqabi and V. S. Kiryakova, Explicit solutions of fractional integral and differential equations involving Erdé1yi-Kober operators, Appl. Math. Comput., 95(1998), 1–13.

[29] I. N. Sneddon, Mixed Boundary Value Problems in Potential Theory, North-Holland Publ, Amsterdam, 1966.

[30] I. N. Sneddon, The use in mathematical analysis of the Erdélyi-Kober operators and some of their applications, In Lect Notes Math., 457, Springer–Verlag, New York (1975) 37–79.

[31] I. N. Sneddon, The Use of Operators of Fractional Integration in Applied Mathematics. RWN Polish Sci. Publ, Warszawa-Poznan, 1979.

[32] Q. Sun, S. Meng and Y. Cu, Existence results for fractional order differential equation with nonlocal Erdélyi-Kober and generalized Riemann-Liouville type integral boundary conditions at resonance, Adv. Difference Equ., 2018, 16 pp.

[33] N. Thongsalee, S. Laoprasittichok, S. K. Ntouyas and J. Tariboon, System of fractional differential equations with Erdélyi-Kober fractional integral conditions, Open Math., 13(2015) 847–859.

[34] B. Yan and Y. Liu, Unbounded solutions of the singular boundary value problems for second order differential equations on the half-line, Appl. Math. Comput., 147(2004), 629–644.

[35] B. Yan, D. O'Regan and R. P. Agarwal, Unbounded solutions for singular boundary value problems on the semi-infinite interval Upper and lower solutions and multiplicity, Int. J. Comput. Appl. Math., 197(2006), 365–386.

[36] R. Jiang and C. Zhai. Unique solutions for a new coupled system of fractional differential equations, Adv. Difference Equ., 2018, 12 pp.

[37] Z. Zhao, Positive solutions of nonlinear second order ordinary differential equations, Proc. Amer. Math. Soc., 121(1994), 465–469.

[38] X. Zhao and W. Ge, Unbounded solutions for a fractional boundary value problems on the infinite interval, Acta Appl. Math., 109(2010), 495–505.

[39] X. Zhao and W. Ge, Existence of at least three positive solutions for multi-point boundary value problem on infinite intervals with p-Laplacian operator, J. Appl. Math. Comput., 28(2008), 391–403.