Applied Mathematics E-Notes, 21(2021), 253-262 © ISSN 1607-2510
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Oscillation Tests For Nonlinear Differential Equations With Several Nonmonotone Advanced Arguments*

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Received 30 April 2020

Abstract

The objective of this paper is to examine oscillatory behaviour of all solutions of first order nonlinear advanced differential equations with several nonmonotone arguments and to establish new oscillation criteria. Examples are also given to illustrate the main results.

1 Introduction

The oscillation theory is a significant research area for modern applied mathematics. Also, substantial concern has been devoted to the oscillatory and nonoscillatory solutions of some classes of differential equations (delay, advanced, mixed type). In particular, advanced differential equations have attracted a lot of researchers in recent years. Advanced differential equations are differential equations where derivative functions rely on not only present value, but also on the future value.

Suppose that a first order nonlinear advanced differential equation is given by

$$u'(t) - \sum_{i=1}^{m} p_i(t) f_i(u(\sigma_i(t))) = 0, \ t \geq t_0, \tag{1}$$

where $m \in \mathbb{N}$, $p_i(t)$ and $\sigma_i(t)$ are the functions of nonnegative real numbers and $\sigma_i(t)$ are not necessarily monotone for $1 \leq i \leq m$, such that

$$\sigma_i(t) \geq t, \quad t \geq t_0, \quad \lim_{t\to\infty} \sigma_i(t) = \infty, \text{ for } 1 \leq i \leq m \tag{2}$$

and

$$f_i \in C(\mathbb{R}, \mathbb{R}) \text{ and } u f_i(u) > 0 \text{ for } u \neq 0, \quad 1 \leq i \leq m. \tag{3}$$

By a solution of (1), we mean continuously differentiable function defined on $[\sigma(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) satisfied for $t \geq T_0$. A solution of (1) is called *oscillatory* if it is neither eventually positive nor eventually negative. If there exists an eventually positive or an eventually negative solution, (1) is called *nonoscillatory*. An equation is oscillatory if its all solutions oscillate.

If $f(x) = x$ then, we have the linear form of (1). The question of establishing sufficient conditions for the oscillation of all solutions of linear form of (1) has been the subject field of many examinations. See, for example, [1–12, 14–19, 22].

For $m = 1$, (1) reduces to the following equation:

$$u'(t) - p(t) f(u(\sigma(t))) = 0, \quad t \geq t_0. \tag{4}$$

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*Mathematics Subject Classifications: 34C10, 34G20, 34K11.

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In 1984, Fukagai and Kusano [15] established the following result for the following type of (4).

$$u'(t) + p(t)f(u(\sigma(t))) = 0, \quad t \geq t_0. \tag{5}$$

Assume that $p(t) \leq 0$, $\sigma(t) \geq t$ is nondecreasing and

$$A = \limsup_{|u|\to\infty} \frac{|u|}{|f(u)|} < \infty. \tag{6}$$

If

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} [-p(s)]\, ds > \frac{A}{e},$$

then all solutions of (5) are oscillatory.

In 2019, Öcalan et al. [21] found out the following criteria for the oscillation of (4), under the assumptions that $p(t) \geq 0$, $\sigma(t) \geq t$ is not necessarily monotone and $B = \limsup_{|u|\to\infty} \frac{u}{f(u)}$. If

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} p(s)ds > \frac{B}{e}, \qquad 0 \leq B < \infty$$

or

$$\limsup_{t\to\infty} \int_t^{\delta(t)} p(s)ds > B, \qquad 0 < B < \infty,$$

then all solutions of (4) are oscillatory, where $\delta(t) := \inf_{s\geq t}\{\sigma(s)\}$, $t \geq 0$.

Now, let us deal with (1) again. In 1987, Ladde et al. [19] obtained the following result. Assume that (2), (3) and following conditions for $1 \leq i \leq m$ hold.

(i) $\sigma_i(t)$ are strictly increasing on $\mathbb{R}_+$,

(ii) $p_i(t)$ are locally integrable and $p_i(t) \geq 0$,

(iii) f_i are nondecreasing in u, and
$$\lim_{|u|\to\infty} \frac{u}{f_i(u)} = C_i > 0.$$

If

$$\liminf_{t\to\infty} \int_t^{\sigma_*(t)} \sum_{i=1}^{m} p_i(s)ds > \frac{C^*}{e}$$

or

$$\limsup_{t\to\infty} \int_t^{\sigma_*(t)} \sum_{i=1}^{m} p_i(s)ds > C^*,$$

then all solutions of (1) are oscillatory, where $C^* = \max_{1\leq i\leq m}\{C_i\}$ and $\sigma_*(t) = \min_{1\leq i\leq m}\{\sigma_i(t)\}$.

As seen above, most of the papers are related to the specific case where the advanced arguments are monotone, while a small number of these articles interest the more general case where the arguments are nonmonotone. Thus, in this paper, our aim is to present new oscillation criteria, involving lim inf and lim sup, where the advanced arguments $\sigma_i(t)$ are not necessarily monotone for $1 \leq i \leq m$.

2 Main Results

In our main results, we establish new sufficient conditions for the oscillation of all solutions of (1), under the assumption that the arguments $\sigma_i(t)$ are not necessarily monotone for $1 \le i \le m$. Set

$$\delta_i(t) := \inf_{s \ge t}\{\sigma_i(s)\}, \quad \delta(t) = \min_{1\le i\le m}\{\delta_i(t)\}, \quad t \ge 0. \tag{7}$$

Obviously, $\delta_i(t)$ are nondecreasing and $\delta_i(t) \le \sigma_i(t)$ for all $t \ge 0$ and $1 \le i \le m$. Assume further that the functions f_i in equation (1) hold the following condition for $1 \le i \le m$.

$$\limsup_{|u|\to\infty} \frac{u}{f_i(u)} = N_i, \ 0 \le N_i < \infty. \tag{8}$$

The following lemmas are useful for the proof of the main theorems.

The following result can be obtained by using similar arguments in the proof of Lemma 2.2 in [20].

Lemma 1 *Assume that* (7) *holds and*

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} \sum_{i=1}^{m} p_i(s)ds = L > 0.$$

Then, we have

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} \sum_{i=1}^{m} p_i(s)ds = \liminf_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds = L, \tag{9}$$

where $\sigma(t) = \min_{1\le i\le m}\{\sigma_i(t)\}$.

Lemma 2 *Assume that* $u(t)$ *is an eventually positive solution of* (1). *If*

$$\limsup_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds > 0, \tag{10}$$

then $\lim_{t\to\infty} u(t) = \infty$, *where* $\delta(t)$ *is defined by* (7). *Also, assume that* $u(t)$ *is an eventually negative solution of* (1). *If* (10) *holds, then* $\lim_{t\to\infty} u(t) = -\infty$.

Proof. Let $u(t)$ be an eventually positive solution of (1). Then, there exists $t_1 > t_0$ such that $u(t)$, $u(\sigma_i(t)) > 0$ for all $t \ge t_1$, $1 \le i \le m$. Thus, from (1), we get

$$u'(t) = \sum_{i=1}^{m} p_i(t) f_i(u(\sigma_i(t))) \ge 0$$

for all $t \ge t_1$, which means that $u(t)$ is nondecreasing and has a limit $l > 0$ or $l = \infty$. Now, we claim that $\lim_{t\to\infty} u(t) = \infty$. Otherwise, $\lim_{t\to\infty} u(t) = l > 0$.
Then, integrating (1) from t to $\delta(t)$, we obtain

$$u(\delta(t)) - u(t) - \int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s) f_i(u(\sigma_i(s)))ds = 0. \tag{11}$$

Also, since f_i are continuous, then $\lim_{t\to\infty} f_i(u(\sigma_i(t))) = f_i(l)$ for $1 \le i \le m$, so, there exists a t_2 such that $f_i(u(\sigma_i(t))) \ge d_i > 0$ for $t \ge t_2$ and $1 \le i \le m$. By using this fact and (11), we get the following inequality

$$u(\delta(t)) - u(t) - d\int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds \ge 0, \tag{12}$$

where $d = \min_{1\le i\le m}\{d_i\}$. Then, (10) implies that there exists at least one sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \int_{t_n}^{\delta(t_n)} \sum_{i=1}^{m} p_i(s)ds > 0. \tag{13}$$

By writing $t \to t_n$ and taking limit as $n \to \infty$ in (12), we get

$$\lim_{n\to\infty} (u(\delta(t_n)) - u(t_n)) - d\lim_{n\to\infty} \int_{t_n}^{\delta(t_n)} \sum_{i=1}^{m} p_i(s)ds \ge 0$$

or

$$d\lim_{n\to\infty} \int_{t_n}^{\delta(t_n)} \sum_{i=1}^{m} p_i(s)ds \le 0,$$

but this contradicts with (13).

By using same process, it is easy to see that when $u(t)$ is an eventually negative solution of (1) under assumption that (10) holds, $\lim_{t\to\infty} u(t) = -\infty$. ■

Theorem 1 *Assume that* (2), (3), (7) *and* (8) *hold. If*

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} \sum_{i=1}^{m} p_i(s)ds > \frac{N^*}{e}, \tag{14}$$

then all solutions of (1) *oscillate, where* $N^* = \max_{1\le i\le m}\{N_i\}$ *and* $\sigma(t) = \min_{1\le i\le m}\{\sigma_i(t)\}$.

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution $u(t)$ of (1). If there exists an eventually negative solution $u(t)$ of (1), then the proof can be done similarly as below. Then, there exists $t_1 > t_0$ such that $u(t)$, $u(\sigma_i(t)) > 0$ for all $t \ge t_1$ and $1 \le i \le m$. Thus, from (1) we have

$$u'(t) = \sum_{i=1}^{m} p_i(t) f_i(u(\sigma_i(t))) \ge 0$$

for all $t \ge t_1$, which means that $u(t)$ is eventually nondecreasing function. Condition (14) and Lemma 2 imply that $\lim_{t\to\infty} u(t) = \infty$.

Case I: Suppose that $N_i > 0$ for $1 \le i \le m$. Then by (8), we can choose $t_2 > t_1$ so large that

$$f_i(u(t)) \ge \frac{1}{2N_i} u(t) \ge \frac{1}{2N^*} u(t) \tag{15}$$

for $t \ge t_2$. Since $\sigma_i(t) \ge \delta(t)$ for $1 \le i \le m$ and $u(t)$ is nondecreasing by using (15), (1) turns into

$$u'(t) - \frac{1}{2N^*} \sum_{i=1}^{m} p_i(t) u(\delta(t)) \ge 0, \ t \ge t_3. \tag{16}$$

Also, from (14) and Lemma 1, it follows that there exists a constant $c > 0$ such that

$$\int_{t}^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds \geq c > \frac{N^*}{e}, \ t \geq t_3 \geq t_2. \tag{17}$$

So, from (17), there exists a real number $t^* \in (t, \delta(t))$ for all $t \geq t_3$ such that

$$\int_{t}^{t^*} \sum_{i=1}^{m} p_i(s)ds > \frac{N^*}{2e} \text{ and } \int_{t^*}^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds > \frac{N^*}{2e}. \tag{18}$$

Integrating (16) from t to t^* and using $u(t)$ and $\delta(t)$ are nondecreasing, we get

$$u(t^*) - u(t) - \frac{1}{2N^*} \int_{t}^{t^*} \sum_{i=1}^{m} p_i(s)u(\delta(s))ds \geq 0$$

or

$$u(t^*) - u(t) - \frac{1}{2N^*} u(\delta(t)) \int_{t}^{t^*} \sum_{i=1}^{m} p_i(s)ds \geq 0.$$

Thus, by (18), we have

$$u(t^*) - \frac{1}{2N^*} u(\delta(t)) \frac{N^*}{2e} > 0. \tag{19}$$

Integrating (16) from t^* to $\delta(t)$, using the same facts, we get

$$u(\delta(t)) - u(t^*) - \frac{1}{2N^*} \int_{t^*}^{\delta(t)} \sum_{i=1}^{m} p_i(s)u(\delta(s))ds \geq 0$$

or

$$u(\delta(t)) - u(t^*) - \frac{1}{2N^*} u(\delta(t^*)) \int_{t^*}^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds \geq 0$$

and

$$u(\delta(t)) - \frac{1}{2N^*} u(\delta(t^*)) \frac{N^*}{2e} > 0. \tag{20}$$

Combining inequalities (19) and (20), we get

$$u(t^*) > u(\delta(t)) \frac{1}{4e} > u(\delta(t^*)) \left(\frac{1}{4e} \right)^2$$

and hence, we have

$$\frac{u(\delta(t^*))}{u(t^*)} < (4e)^2, \ t \geq t_4.$$

Let

$$z = \liminf_{t \to \infty} \frac{u(\delta(t))}{u(t)} \geq 1 \tag{21}$$

and because of $1 \leq z \leq (4e)^2$, z is finite.

Now, dividing (1) with $u(t)$ and integrating from t to $\delta(t)$, we get

$$\int_t^{\delta(t)} \frac{u'(s)}{u(s)}ds - \int_t^{\delta(t)} \sum_{i=1}^m p_i(s)\frac{f_i(u(\sigma_i(s)))}{u(s)}ds = 0,$$

$$\ln\frac{u(\delta(t))}{u(t)} - \int_t^{\delta(t)} \sum_{i=1}^m p_i(s)\frac{f_i(u(\sigma_i(s)))}{u(\sigma_i(s))}\frac{u(\sigma_i(s))}{u(s)}ds = 0.$$

By using $u(t)$ is nondecreasing and $\delta(t) \le \sigma_i(t)$ for $1 \le i \le m$, we have

$$\ln\frac{u(\delta(t))}{u(t)} - \int_t^{\delta(t)} \sum_{i=1}^m p_i(s)\frac{f_i(u(\sigma_i(s)))}{u(\sigma_i(s))}\frac{u(\delta(s))}{u(s)}ds \ge 0$$

and also, there exists a η such that $t \le \eta \le \delta(t)$. Then, we have

$$\ln\frac{u(\delta(t))}{u(t)} \ge \sum_{i=1}^m \frac{f_i(u(\sigma_i(\eta)))}{u(\sigma_i(\eta))}\frac{u(\delta(\eta))}{u(\eta)}\int_t^{\delta(t)} p_i(s)ds. \tag{22}$$

Then, taking lower limit on both side of (22), we find $\ln z > \frac{z}{e}$. But, this is impossible since $\ln x \le \frac{x}{e}$ for all $x > 0$.

Case II: Suppose that $N^* = 0$. It is explicit that $\frac{u}{f_i(u)} > 0$ for $1 \le i \le m$ and from (8)

$$\lim_{u\to\infty}\frac{u}{f_i(u)} = 0 \text{ for } 1 \le i \le m. \tag{23}$$

By (23), we have

$$\frac{u}{f_i(u)} < \epsilon_i < \epsilon^* \text{ for } 1 \le i \le m$$

or

$$\frac{f_i(u)}{u} > \frac{1}{\epsilon^*}, \text{ for } 1 \le i \le m, \tag{24}$$

where $0 < \epsilon^* = \max_{1\le i\le m}\{\epsilon_i\}$ is an arbitrary real number. Because $\delta(t) \le \sigma_i(t)$ for $1 \le i \le m$, $u(t)$ and $\delta(t)$ are nondecreasing, using these facts and (24), (1) converts to following inequality

$$u'(t) - \frac{1}{\epsilon^*}\sum_{i=1}^m p_i(t)u(\delta(t)) > 0. \tag{25}$$

Now, integrating (25) from t to $\delta(t)$, we have

$$u(\delta(t)) - u(t) - \frac{1}{\epsilon^*}\int_t^{\delta(t)} \sum_{i=1}^m p_i(s)u(\delta(s))ds > 0$$

or

$$u(\delta(t)) - \frac{1}{\epsilon^*}u(\delta(t))\int_t^{\delta(t)} \sum_{i=1}^m p_i(s)ds > 0$$

and

$$u(\delta(t))\left[1 - \frac{1}{\epsilon^*}\int_t^{\delta(t)} \sum_{i=1}^m p_i(s)ds\right] > 0.$$

Then, using (17), we get

$$1 > \frac{c}{\epsilon^*}$$

or

$$\epsilon^* > c.$$

But, this contradicts with $\lim\limits_{u\to\infty} \frac{u}{f_i(u)} = 0$ for $1 \le i \le m$. The proof of the theorem is completed. ■

Theorem 2 *Assume that* (2), (3), (7) *and* (8) *hold with* $0 < N^* < \infty$. *If*

$$\limsup_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^m p_i(s)ds > N^*, \tag{26}$$

then all solutions of (1) *oscillate, where* $N^* = \max\limits_{1\le i\le m} \{N_i\}$.

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution $u(t)$ of (1). Then, there exists $t_1 > t_0$ such that $u(t), u(\sigma_i(t)) > 0$ for all $t \ge t_1$ and $1 \le i \le m$. From Theorem 1, $u(t)$ is eventually nondecreasing and also, from (26) and Lemma 2, $\lim\limits_{t\to\infty} u(t) = \infty$.

By taking into (8) for $\theta > 1$, we get the following inequality

$$f_i(u(t)) \ge \frac{1}{\theta N_i} u(t) \ge \frac{1}{\theta N^*} u(t) \text{ for } 1 \le i \le m. \tag{27}$$

From (26), there exists a constant $K > 0$ such that

$$\limsup_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^m p_i(s)ds = K > N^*. \tag{28}$$

Since $K > N^*$, we have $N^* < \frac{K+N^*}{2} < K$. Also, with the help of (27) and (1), we get

$$u'(t) - \frac{1}{\theta N^*} \sum_{i=1}^m p_i(t) u(\sigma_i(t)) \ge 0.$$

As $\delta(t) \le \sigma_i(t)$ for $1 \le i \le m$ and $u(t)$ is nondecreasing, we have

$$u'(t) - \frac{1}{\theta N^*} \sum_{i=1}^m p_i(t) u(\delta(t)) \ge 0. \tag{29}$$

Integrating (29) from t to $\delta(t)$ and using the fact that $u(t)$ and $\delta(t)$ are nondecreasing, we obtain

$$u(\delta(t)) - u(t) - \frac{1}{\theta N^*} \int_t^{\delta(t)} \sum_{i=1}^m p_i(s) u(\delta(s)) ds \ge 0$$

or

$$u(\delta(t)) - \frac{1}{\theta N^*} u(\delta(t)) \int_t^{\delta(t)} \sum_{i=1}^m p_i(s)ds \ge 0$$

and

$$u(\delta(t)) \left[1 - \frac{1}{\theta N^*} \int_t^{\delta(t)} \sum_{i=1}^m p_i(s)ds \right] \ge 0$$

and hence

$$\int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds < \theta N^*$$

for sufficiently large t. Therefore,

$$\limsup_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds \le \theta N^*.$$

Since $\theta > 1$ and $\frac{K+N^*}{2N^*} > 1$, we can choose this term instead of θ. If the term $\theta = \frac{K+N^*}{2N^*} > 1$ is replaced in the last inequality, we get

$$\limsup_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^{m} p_i(s)ds = K \le \frac{K+N^*}{2}.$$

But, this contradicts with $K > \frac{K+N^*}{2}$, then the proof of the theorem is completed. ∎

Example 1 *Consider the following advanced differential equation:*

$$u'(t) - \frac{2}{e}u(\sigma_1(t))\log(5+|u(\sigma_1(t))|) - \frac{4}{e}u(\sigma_2(t))\log(7+|u(\sigma_2(t))|) = 0,\ t \ge 1 \tag{30}$$

where

$$\sigma_1(t) = \begin{cases} 4t-6k-2, & t \in [2k+1, 2k+2], \\ -2t+6k+10, & t \in [2k+2, 2k+3], \end{cases}$$

and

$$\sigma_2(t) = \sigma_1(t) + 2,$$

also, by (7), we see that

$$\delta_1(t) := \inf_{s\ge t}\{\sigma_1(s)\} = \begin{cases} 4t-6k-2, & t \in [2k+1, 2k+1,5], \\ 2k+4, & t \in [2k+1,5, 2k+3], \end{cases}$$

and

$$\delta_2(t) := \inf_{s\ge t}\{\sigma_2(s)\} = \delta_1(t) + 2,$$

$k \in \mathbb{N}_0$, $\mathbb{N}_0$ is the set of nonnegative integers. Therefore,

$$\delta(t) = \min_{1\le i\le 2}\{\delta_i(t)\} = \delta_1(t).$$

If we put $p_1(t) = \frac{2}{e}$, $p_2(t) = \frac{4}{e}$ and $f_1(u) = u\log(5+|u|)$, $f_2(u) = u\log(7+|u|)$, then we have

$$N_1 = \limsup_{|u|\to\infty} \frac{u}{f_1(u)} = \limsup_{|u|\to\infty} \frac{u}{u\log(5+|u|)} = 0,$$

$$N_2 = \limsup_{|u|\to\infty} \frac{u}{f_2(u)} = \limsup_{|u|\to\infty} \frac{u}{u\log(7+|u|)} = 0,$$

$$\max\{N_1, N_2\} = N^* = 0.$$

Now, at $t = 2k+3$, $k \in \mathbb{N}_0$, we get

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} \sum_{i=1}^{2} p_i(s)ds = \liminf_{t\to\infty} \int_t^{\delta(t)} \sum_{i=1}^{2} p_i(s)ds = \liminf_{t\to\infty} \int_{2k+3}^{2k+4} \frac{6}{e}ds = \frac{6}{e} > \frac{N^*}{e},$$

that is all conditions of Theorem 1 satisfied and therefore all solutions of (30) oscillate.

Example 2 *Consider the following advanced differential equation:*

$$u'(t) - \frac{1}{e}u(\sigma_1(t))\ln(e^{-|u(\sigma_1(t))|}+2) - \frac{2}{e}u(\sigma_2(t))\ln(e^{-|u(\sigma_2(t))|}+3) = 0,\ t \geq 0 \tag{31}$$

where

$$\sigma_1(t) = \begin{cases} t+1, & t \in [3.5k, 3.5k+1], \\ 3t-7k-1, & t \in [3.5k+1, 3.5k+1.5], \\ -t+7k+5, & t \in [3.5k+1.5, 3.5k+2], \\ t+1, & t \in [3.5k+2, 3.5k+2.5], \\ 3t-7k-4, & t \in [3.5k+2.5.5, 3.5k+3], \\ -t+7k+8, & t \in [3.5k+3, 3.5k+3.5], \end{cases}$$

$$\sigma_2(t) = \sigma_1(t)+1,$$

and

$$\delta_1(t) := \inf_{s \geq t}\{\sigma_1(s)\} = \begin{cases} t+1, & t \in [3.5k, 3.5k+1], \\ 3t-7k-1, & t \in [3.5k+1, 3.5k+4/3], \\ 3.5k+3, & t \in [3.5k+4/3, 3.5k+2], \\ t+1, & t \in [3.5k+2, 3.5k+2.5], \\ 3t-7k-4, & t \in [3.5k+2.5.5, 3.5k+17/6], \\ 3.5k+4.5, & t \in [3.5k+17/6, 3.5k+3.5], \end{cases}$$

and

$$\delta_2(t) := \inf_{s \geq t}\{\sigma_2(s)\} = \delta_1(t)+1,$$

$k \in \mathbb{N}_0$*,* $\mathbb{N}_0$ *is the set of nonnegative integers. Then*

$$\delta(t) = \min_{1 \leq i \leq 2}\{\delta_i(t)\} = \delta_1(t).$$

If we take $p_1(t) = \frac{1}{e}$*,* $p_2(t) = \frac{2}{e}$ *and* $f_1(u) = u\ln(e^{-|u|}+2)$*,* $f_2(u) = u\ln(e^{-|u|}+3)$*, then we have*

$$N_1 = \limsup_{|u|\to\infty}\frac{u}{f_1(u)} = \limsup_{|u|\to\infty}\frac{u}{u\ln(e^{-|u|}+2)} = \frac{1}{\ln 2} \cong 1.44269,$$

$$N_2 = \limsup_{|u|\to\infty}\frac{u}{f_2(u)} = \limsup_{|u|\to\infty}\frac{u}{u\ln(e^{-|u|}+3)} = \frac{1}{\ln 3} \cong 0.91023$$

and

$$N^* = \max_{1 \leq i \leq 2}\{N_1, N_2\} = N_1.$$

Then, we obtain

$$\limsup_{t\to\infty}\int_t^{\delta(t)}\sum_{i=1}^{2}p_i(s)ds = \limsup_{t\to\infty}\int_{3.5k+4/3}^{3.5k+3}\frac{3}{e}ds \cong 1.83939 > N^* \cong 1.44269,$$

that is all conditions of Theorem 2 satisfied and therefore all solutions of (31) *oscillate.*

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