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Sharp Estimates For The Unique Solution Of The Hadamard-Type Two-Point Fractional Boundary Value Problems*

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Abstract

In this short note, we present sharp estimate for the existence of a unique solution for a Hadamard-type fractional differential equations with two-point boundary value conditions. The method of analysis is obtained by using the integral of the Green's function and the Banach contraction principle. Further, we will also obtain a sharper lower bound of the eigenvalues for an eigenvalue problem. Two examples are presented to clarify the applicability of the essential results.

1 Introduction and Preliminaries

In the book [1] Kelley and Peterson considered the following classical two-point boundary value problems:

$$\begin{cases} u''(x) = \mathcal{F}(x, u(x)), & a < x < b, \\ u(a) = A,\ u(b) = B, & A, B \in \mathbb{R}, \end{cases} \tag{1}$$

where $a, b \in \mathbb{R}$, and they included the following result:

Theorem 1 ([1], Theorem 7.7) *Let $\mathcal{F} : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_1) *There exists $K > 0$ such that $|\mathcal{F}(x, \omega) - \mathcal{F}(x, \varpi)| \le K |\omega - \varpi|$ for all $(x, \omega), (x, \varpi) \in [a, b] \times \mathbb{R}$.*

Then the boundary value problem (1) has a unique solution on $[a, b]$ if $b - a < 2\sqrt{2/K}$.

Ferreira in 2016 [2] discussed the existence and uniqueness of solutions for the following fractional boundary value problems with Reimman-Liouville fractional derivative:

$$\begin{cases} {}^{\mathcal{R}}\mathfrak{D}_a^\sigma u(x) = -\mathcal{F}(x, u(x)), & a < x < b, 1 < \sigma \le 2, \\ u(a) = 0,\ u(b) = B, & B \in \mathbb{R}. \end{cases} \tag{2}$$

Theorem 2 ([2]) *Let $\mathcal{F} : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption (H_1). Then the boundary value problem (2) has a unique solution on $[a, b]$ if $b - a < (\sigma^{(\sigma+1)/\sigma}\Gamma^{1/\sigma}(\sigma))/(K^{1/\sigma}(\sigma - 1)^{(\sigma-1)/\sigma})$.*

In 2019, Ferreira [3] corrected a recent uniqueness result [4] for a two-point fractional boundary value problem with Caputo derivative:

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_a^\sigma u(x) = -\mathcal{F}(x, u(x)), & a < x < b, 1 < \sigma \le 2, \\ u(a) = A,\ u(b) = B, & A, B \in \mathbb{R}. \end{cases} \tag{3}$$

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Theorem 3 ([3]) *Let $\mathcal{F} : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption (H_1). Then the boundary value problem (3) has a unique solution on $[a,b]$ if $M(\sigma,a,b) < 1/K$, where*

$$M(\sigma,a,b) = \frac{1}{\Gamma(\sigma+1)} \max_{x\in[a,b]} \Big(-2(x-\varphi(x))^{\sigma} + 2\frac{(x-a)(b-\varphi(x))^{\sigma}}{b-a} + (x-a)^{\sigma} - (x-a)(b-a)^{\sigma-1} \Big),$$

with $\varphi(x) = \Big((\frac{x-a}{b-a})^{\frac{1}{\sigma-1}} b - 1 \Big) / \Big((\frac{x-a}{b-a})^{\frac{1}{\sigma-1}} - 1 \Big)$.

See [2, 3] and references therein for more details.

In the last few decades, the differential equations involving a fractional order have witnessed a wide attention from the researchers, because they were extensively implemented in daily life and in various scientific and technological fields and in many branches including physics, biology, chemistry, economics, astronomy, control theory, viscoelastic materials, robotics, signal processing, electromagnetism, electrodynamics of complex medium, anomalous diffusion and fractured media, electromagnetism, potential theory and electro statistics, polymer rheology, and aerodynamics, etc. We refer the interested reader to paper [7], and the references contained therein.

It is well known that the existence of solution plays an important role in the theory and applications of fractional differential equations with boundary conditions. Recently, many researchers are interested in studying the Hadamard-type fractional boundary value problems, where there are several results about the existence of solutions for the differential equations with Hadamard derivative, we refer the reader to the book [8] that contains the most important works that have been published in this domain. In addition, some researchers are interested in studying the stability of solutions to fractional differential equations, including Laypunov stability, exponential stability, Mittag-Leffler stability, and Hyers-Ulam stability, have been introduced. Among these concepts, Hyers-Ulam stability analysis was recognized as a simple method of investigation. We refer the readers to [10, 11, 12, 13, 14, 15], and the references contained therein.

Motivated by the above mentioned works and the papers [5, 6], in this paper, we investigated the sharp estimate for the unique solution of the following fractional differential equation with Hadamard derivative:

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}_a^{\sigma} u(x) = -\mathcal{F}(x,u(x)), & 0 < a < x < b, 1 < \sigma \le 2, \\ u(a) = 0, \quad u(b) = B, & B \in \mathbb{R}, \end{cases} \tag{4}$$

where $\mathcal{F}$ is a given function, ${}^{\mathcal{H}}\mathfrak{D}_a^{\sigma}$ denotes the Hadamard fractional derivative of order σ, and B is real constant. Further, we will also obtain a sharp estimate for the lower bound of the eigenvalues of the follwing eigenvalue problem

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}_a^{\sigma} u(x) = \lambda u(x), & 0 < a < x < b, 1 < \sigma \le 2, \\ u(a) = 0 = u(b). \end{cases} \tag{5}$$

We start now to present some fundamental definitions and lemmas which will be used in this work.

Definition 1 ([8, 9]) *Let $0 < a \le b$ and $\sigma \in \mathbb{R}^+$ where $n-1 < \sigma \le n$ with $n \in \mathbb{N}$. The Hadamard fractional integral of ordre σ for a function g is defined by: ${}^{\mathcal{H}}\mathcal{I}_a^0 g(x) = g(x)$ and*

$${}^{\mathcal{H}}\mathcal{I}_a^{\sigma} g(x) = \frac{1}{\Gamma(\sigma)} \int_a^x \left(\ln\frac{x}{\tau}\right)^{\sigma-1} g(\tau)\frac{d\tau}{\tau} \text{ for } \sigma > 0. \tag{6}$$

Definition 2 ([8, 9]) *Let $0 < a < b$; $\delta = x\frac{d}{dx}$ and let $AC[a,b]$ be the space of functions g which are absolutely continuous on $[a,b]$, and $AC_{\delta}^{n}[a,b] = \{g : [a,b] \times \mathbb{R} \to \mathbb{R} \ s.t. \ \delta^{n-1}[g(x)] \in AC[a,b]\}$. The Hadamard fractional derivative of order $\sigma \ge 0$ for a function $g \in AC_{\delta}^{n}[a,b]$ is defined by: ${}^{\mathcal{H}}\mathfrak{D}_a^0 g(x) = g(x)$, and*

$${}^{\mathcal{H}}\mathfrak{D}_a^{\sigma} g(x) = \frac{1}{\Gamma(n-\sigma)} \left(t\frac{d}{dx}\right)^n \int_a^x \left(\ln\frac{x}{\tau}\right)^{n-\sigma-1} g(\tau)\frac{d\tau}{\tau} \text{ for } \sigma > 0, \tag{7}$$

where $n-1 < \sigma \le n$, $n \in \mathbb{N}$.

Lemma 1 **([8, 9])** *Let $0 < a \leq b$, and $\sigma > 0$ where $n-1 < \sigma \leq n$, $n \in \mathbb{N}$. The differential equation ${}^{\mathcal{H}}\mathfrak{D}_a^\sigma u(x) = 0$ has the general solution:*

$$u(x) = \sum_{i=1}^{i=n} c_i \left(\ln \frac{x}{a}\right)^{\sigma-i}, \; x \in [a,b], \tag{8}$$

where $c_i \in \mathbb{R}$ $(i = 1, ..., n)$ are arbitrary constants. And moreover

$$ {}^{\mathcal{H}}\mathcal{I}_a^{\sigma}\, {}^{\mathcal{H}}\mathfrak{D}_a^\sigma u(x) = u(x) + \sum_{i=1}^{i=n} c_i \left(\ln \frac{x}{a}\right)^{\sigma-i}. \tag{9}$$

Lemma 2 *Let $y \in C([a,b],\mathbb{R}) \cap L^1([a,b],\mathbb{R})$, the solution of the following linear fractional boundary value problem*

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}_a^\sigma u(x) = -y(x), & 0 < a < x < b, 1 < \sigma \leq 2, \\ u(a) = 0, \; u(b) = B, & B \in \mathbb{R}, \end{cases} \tag{10}$$

is given by

$$u(x) = \int_a^b G(x,\tau) y(\tau) d\tau + B \left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1},$$

where

$$G(x,\tau) = \frac{1}{\Gamma(\sigma)} \begin{cases} \left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1} \left(\ln \frac{b}{\tau}\right)^{\sigma-1} \frac{1}{\tau} - \left(\ln \frac{x}{\tau}\right)^{\sigma-1} \frac{1}{\tau}, & a \leq \tau \leq x \leq b, \\ \left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1} \left(\ln \frac{b}{\tau}\right)^{\sigma-1} \frac{1}{\tau}, & a \leq x \leq \tau \leq b. \end{cases} \tag{11}$$

Proof. Applying the operator ${}^{\mathcal{H}}\mathcal{I}_a^\sigma$ on the equation ${}^{\mathcal{H}}\mathfrak{D}_a^\sigma u(x) = -y(x)$, we get

$$u(x) = -\frac{1}{\Gamma(\sigma)} \int_a^x \left(\ln \frac{x}{\tau}\right)^{\sigma-1} y(\tau) \frac{d\tau}{\tau} + c_1 \left(\ln \frac{x}{a}\right)^{\sigma-1} + c_2 \left(\ln \frac{x}{a}\right)^{\sigma-2}, \tag{12}$$

where $c_1, c_2 \in \mathbb{R}$. Using the boundary conditions $u(a) = 0$ and $u(b) = B$, we get $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\sigma)} \left(\ln \frac{b}{a}\right)^{1-\sigma} \int_a^b \left(\ln \frac{b}{\tau}\right)^{\sigma-1} y(\tau) \frac{d\tau}{\tau} + B \left(\ln \frac{b}{a}\right)^{1-\sigma}.$$

Substituting the values of c_1 and c_2 in (12), we obtain

$$\begin{aligned} u(x) &= \frac{1}{\Gamma(\sigma)} \left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1} \int_a^b \left(\ln \frac{b}{\tau}\right)^{\sigma-1} y(\tau) \frac{d\tau}{\tau} - \frac{1}{\Gamma(\sigma)} \int_a^x \left(\ln \frac{t}{\tau}\right)^{\sigma-1} y(\tau) \frac{d\tau}{\tau} + B \left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1} \\ &= \frac{1}{\Gamma(\sigma)} \int_a^x \left[\left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1} \left(\ln \frac{b}{\tau}\right)^{\sigma-1} - \left(\ln \frac{x}{\tau}\right)^{\sigma-1}\right] y(\tau) \frac{d\tau}{\tau} + \frac{1}{\Gamma(\sigma)} \int_x^b \left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1} \\ &\times \left(\ln \frac{b}{\tau}\right)^{\sigma-1} y(\tau) \frac{d\tau}{\tau} + B \left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1} \\ &= \int_a^b G(x,\tau) y(\tau) d\tau + B \left(\frac{\ln \frac{x}{a}}{\ln \frac{b}{a}}\right)^{\sigma-1}. \end{aligned}$$

Hence, the proof is completed. ■

2 Main Results

This section is devoted to prove the main results of the problem (4), and present a lower bound for the eigenvalues of the eigenvalue problem (5).

Lemma 3 *The Green's function G defined in Lemma 2 has the following property:*

$$\max_{x\in[a,b]}\int_a^b |G(x,\tau)|\,d\tau = \frac{(\sigma-1)^{\sigma-1}\left(\ln\frac{b}{a}\right)^{\sigma}}{\sigma^{\sigma+1}\Gamma(\sigma)}. \tag{13}$$

Proof. From Lemma 4 of [5], we have $G(x,\tau)\geq 0$ for all $(x,\tau)\in[a,b]\times[a,b]$. Therefore,

$$\begin{aligned}
\Gamma(\sigma)\int_a^b |G(x,\tau)|d\tau &= \int_a^x \left[\left(\frac{\ln\frac{x}{a}}{\ln\frac{b}{a}}\right)^{\sigma-1}\left(\ln\frac{b}{\tau}\right)^{\sigma-1} - \left(\ln\frac{x}{\tau}\right)^{\sigma-1}\right]\frac{d\tau}{\tau} \\
&\quad + \int_x^b \left(\frac{\ln\frac{x}{a}}{\ln\frac{b}{a}}\right)^{\sigma-1}\left(\ln\frac{b}{\tau}\right)^{\sigma-1}\frac{d\tau}{\tau} \\
&= \left(\frac{\ln\frac{x}{a}}{\ln\frac{b}{a}}\right)^{\sigma-1}\int_a^b \left(\ln\frac{b}{\tau}\right)^{\sigma-1}\frac{d\tau}{\tau} - \int_a^x\left(\ln\frac{x}{\tau}\right)^{\sigma-1}\frac{d\tau}{\tau} \\
&= \frac{1}{\sigma}\left(\ln\frac{b}{a}\right)^{1-\sigma}\left(\ln\frac{x}{a}\right)^{\sigma-1}\left(\ln\frac{b}{a}\right)^{\sigma} - \frac{1}{\sigma}\left(\ln\frac{x}{a}\right)^{\sigma},
\end{aligned}$$

which yields

$$\Gamma(\sigma+1)\int_a^b G(x,\tau)\,d\tau = \left(\ln\frac{b}{a}\right)\left(\ln\frac{x}{a}\right)^{\sigma-1} - \left(\ln\frac{x}{a}\right)^{\sigma}. \tag{14}$$

It follows that we need to get the maximum value of the function

$$g(x) = \left(\ln\frac{b}{a}\right)\left(\ln\frac{x}{a}\right)^{\sigma-1} - \left(\ln\frac{x}{a}\right)^{\sigma},\quad x\in[a,b]. \tag{15}$$

Observe that $g(x)\geq 0$ for all $x\in[a,b]$, and $g(a)=g(b)=0$. Now we differentiate $g(x)$ on (a,b) to get

$$g'(x) = \frac{(\sigma-1)}{x}\left(\ln\frac{b}{a}\right)\left(\ln\frac{x}{a}\right)^{\sigma-2} - \frac{\sigma}{x}\left(\ln\frac{x}{a}\right)^{\sigma-1},$$

from which follows that $g'(x^*)=0$ has a unique zero, attained at the point

$$x^* = a\left(\frac{b}{a}\right)^{(\sigma-1)/\sigma}.$$

It is easily seen that $x^*\in(a,b)$. Because $g(x)$ is continuous function and $x^*\in(a,b)$, we conclude that

$$\begin{aligned}
\max_{x\in[a,b]} g(x) &= g(x^*) \\
&= \left(\ln\frac{b}{a}\right)\left(\ln\left(\frac{b}{a}\right)^{(\sigma-1)/\sigma}\right)^{\sigma-1} - \left(\ln\left(\frac{b}{a}\right)^{(\sigma-1)/\sigma}\right)^{\sigma} \\
&= \frac{1}{\sigma-1}\left(\frac{\sigma-1}{\sigma}\ln\frac{b}{a}\right)^{\sigma} \\
&= \frac{(\sigma-1)^{\sigma-1}\left(\ln\frac{b}{a}\right)^{\sigma}}{\sigma^{\sigma}}.
\end{aligned} \tag{16}$$

By (14), (15) and (16) we get the formula (13). The proof is completed. ■

Theorem 4 *Let $\mathcal{F} : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption (H_1). If*

$$\frac{b}{a} < \exp\left(\frac{\sigma^{(\sigma+1)/\sigma}\Gamma^{1/\sigma}(\sigma)}{(\sigma-1)^{(\sigma+1)/\sigma} K^{1/\sigma}}\right), \tag{17}$$

then the fractional boundary value problem (4) has a unique solution on $[a,b]$.

Proof. *Let $E = C([a,b],\mathbb{R})$ be the Banach space endowed with the norm $\|u\| = \sup_{x\in[a,b]}|u(x)|$ (see Proposition 2.18 in [16]), and we define the operator $\mathfrak{R} : E \to E$ by*

$$\mathfrak{R}u(x) = \int_a^b G(x,\tau)\mathfrak{R}(\tau,u(\tau))d\tau + B\left(\frac{\ln\frac{x}{a}}{\ln\frac{b}{a}}\right)^{\sigma-1},$$

where the function G is given by (11). Notice that the prolem (4) has a solution u if only if u is fixed point of the operator $\mathfrak{R}$. For all $(x,u),(x,v) \in [a,b]\times E$, we have

$$\begin{aligned}
|\mathfrak{R}u(x) - \mathfrak{R}v(x)| &\leq \int_a^b G(x,\tau)\,|\mathcal{F}(\tau,u(\tau)) - \mathcal{F}(\tau,v(\tau))|\,d\tau \\
&\leq \int_a^b KG(x,\tau)\,|u(\tau) - v(\tau)|\,d\tau \\
&\leq K\int_a^b G(x,\tau)\,d\tau\,\|u-v\|,
\end{aligned}$$

using the formula (13) yields

$$\|\mathfrak{R}u - \mathfrak{R}v\| \leq \frac{K(\sigma-1)^{\sigma-1}\left(\ln\frac{b}{a}\right)^{\sigma}}{\sigma^{\sigma+1}\Gamma(\sigma)}\|u-v\|.$$

It can be easily checked that the assumption (17) leads to principle of contraction mapping. Hence, the operator $\mathfrak{R}$ is contraction mapping, we conclude that the problem (4) has a unique solution. ∎

Now we present a lower bound for the eigenvalues of the eigenvalue problem (5).

Theorem 5 *If the eigenvalue problem (5) has a non-trivial continuous solution, then*

$$|\lambda| \geq \frac{\sigma^{\sigma+1}\Gamma(\sigma)}{(\sigma-1)^{\sigma-1}\left(\ln\frac{b}{a}\right)^{\sigma}}, \tag{18}$$

Proof. From Lemma 2, the solution of the problem (5) can be written as follows

$$u(x) = \int_a^b \lambda G(x,\tau)\,u(\tau)\,d\tau.$$

which yields

$$\|u\| \leq |\lambda|\,\|u\| \max_{x\in[a,b]}\int_a^b |G(x,\tau)|\,d\tau$$

Since u is non-trivial, then $\|u\| \neq 0$. So, using now to the formula of the Green function G proved in Lemma 3, we get

$$1 \leq |\lambda| \max_{x\in[a,b]}\int_a^b |G(x,\tau)|\,d\tau = |\lambda|\frac{(\sigma-1)^{\sigma-1}\left(\ln\frac{b}{a}\right)^{\sigma}}{\sigma^{\sigma+1}\Gamma(\sigma)},$$

from which the inequality (18) follows. The proof is completed. ∎

Example 1 *We consider the following Hadamard frational boundary value problem*

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}_a^{3/2}u(x) = (x-1)^2 + \sqrt{x-1+u^2(x)},\ 1 < x < e, \\ u(1) = 0,\ u(e) = 1, \end{cases} \tag{19}$$

where e *is an irrational number and it's defined by the infinite series* $e = \sum_{k=0}^{+\infty} \frac{1}{k!}$ *and approximately equal to* 2.718281828459. *Here* $\sigma = \frac{3}{2}$ *and* $\mathcal{F}(x,u) = (x-1)^2 + \sqrt{x-1+u^2}$. *For all* $(x,u) \in (1,e] \times \mathbb{R}$, *we have*

$$|\partial_u \mathcal{F}(x,u)| = \frac{|u|}{\sqrt{x-1+u^2}} \leq 1.$$

Choose $K = 1$. *So, by using the given values, we get*

$$\exp\left(\frac{\sigma^{(\sigma+1)/\sigma}\Gamma^{1/\sigma}(\sigma)}{(\sigma-1)^{(\sigma+1)/\sigma} K^{1/\sigma}}\right) = \exp\left(\frac{3}{4}(9\pi)^{1/3}\right) > e.$$

Then the inequality (17) is satisfied. Hence, by Theorem 4, we conclude that the Hadamard fractional boundary value problem (19) has a unique solution on the interval $[1,e]$.

Example 2 *Consider the following eigenvalue problem*

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}_a^{3/2}u(x) = \lambda u(x),\ 1 < x < e, \\ u(1) = 0 = u(e). \end{cases} \tag{20}$$

Here $\sigma = \frac{3}{2}$, *and* $[a,b] = [1,e]$. *So, we obtain*

$$\frac{\sigma^{\sigma+1}\Gamma(\sigma)}{(\sigma-1)^{\sigma-1}\left(\ln\frac{b}{a}\right)^{\sigma}} = \frac{9\sqrt{3\pi}}{8},$$

By Theorem 5, we conclude that: If λ *is an eigenvalue of the problem (20), we must have* $|\lambda| \geq 9\sqrt{3\pi}/8$.

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