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Fixed Point To Fixed Ellipse In Metric Spaces And Discontinuous Activation Function*

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Abstract

We introduce a notion of a fixed ellipse to study the geometric properties of the set of nonunique fixed points of a discontinuous self map and establish fixed ellipse theorems. Further, we verify these by illustrative examples with geometric interpretations to demonstrate the authenticity of the postulates. Paper is concluded by a discussion of activation functions having fixed ellipse to exhibit the feasibility of results, thereby providing a better insight into the analogous explorations.

1 Introduction

The nonunique fixed points of a discontinuous self map perform an essential role in fixed point theory. Because if the fixed point is not unique then the set of nonunique fixed points may form some geometrical shape like a circle, disc, ellipse, or ellipsoid. In particular, an ellipse has several applications in Physics, Astronomy, Biology, Neural Networks, Economics, Artificial Intelligence, and so on. Actually, ellipses emerge naturally in numerous areas, for instance: planetary orbits. It is well known that an ellipse is the locus of a point for which the sum of the Euclidean distances from the two foci is constant. A more natural perspective on the ellipse is to define it as a flattened circle. In the present work, we introduce the notion of a fixed ellipse in metric spaces. Next, we establish that the set of nonunique fixed points of a map includes an ellipse in a metric space and verify these by illustrative examples with the geometric interpretation to demonstrate the authenticity of the postulates. Towards the end, motivated by the fact that majority of prevalent discontinuous activation functions that are being applied in neural networks are maps which have either fixed circles or fixed discs, or fixed ellipses, we discuss the discontinuous activation function thereby provide a better insight into the analogous explorations. These fixed ellipse results promote further examinations and applications in metric spaces.

Definition 1 ([5]) *A metric on a nonempty set $\mathcal{M}$ is a function $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ satisfying*

(i) $d(\omega, \upsilon) = 0$ *iff* $\omega = \upsilon$;

(ii) $d(\omega, \upsilon) = d(\upsilon, \omega)$;

(iii) $d(\omega, \upsilon) \leq d(\omega, u) + d(u, \upsilon), \omega, \upsilon, u \in \mathcal{M}$.

2 Main Results

We first discuss the shapes of ellipses in different metric spaces for different lengths of semi major axes and different foci and next familiarize a fixed ellipse in a metric space. Further, we exploit a classical Caristi type map [4] to demonstrate that the set of nonunique fixed points of a map includes an ellipse.

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Definition 2 *An ellipse having foci at $\mathfrak{c}_1$ and $\mathfrak{c}_2$ in a metric space $(\mathcal{M}, d)$ is defined as*

$$\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \{\omega \in \mathcal{M} : d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega) = 2\mathfrak{a}, \quad \mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{M}, \ \mathfrak{a} \in [0, \infty)\}.$$

The midpoint $\mathcal{C}$ of a line $\mathfrak{c}_1\mathfrak{c}_2$ is known as a centre of an ellipse. Here, the segment of length $2\mathfrak{a}$ on line $\mathfrak{c}_1\mathfrak{c}_2$ is the major axis, the line perpendicular to it through the center is the minor axis and $\mathfrak{a}$ is the length of a semi major axis of an ellipse. The distance $2f = d(\mathfrak{c}_1, \mathfrak{c}_2)$ is the linear eccentricity. The numerical eccentricity of an ellipse (ellipsoid) is $\varepsilon = \frac{sin f}{sin \mathfrak{a}}$. Visibly, the circles (spheres) are the ellipse (ellipsoid in higher dimensions) of vanishing eccentricity in which both the focal points are same.

Example 1 *Let $\mathcal{M} = \mathbb{R}$ and a metric $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be defined as $d(\omega, \upsilon) = |\omega - \upsilon|$, $\omega, \upsilon \in \mathcal{M}$ then*

$$\begin{aligned} \mathcal{E}(-2, 2, 5) &= \{\omega \in \mathcal{M} : d(-2, \omega) + d(2, \omega) = 10\} \\ &= \{\omega \in \mathcal{M} : |2 + \omega| + |2 - \omega| = 10\} \\ &= \{-5, 5\}, \end{aligned}$$

i.e., an ellipse centered at origin, having foci at -2 and 2 is $\{-5, 5\}$.

Example 2 *Let $\mathcal{M} = \mathbb{R}^2$ and a metric $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be defined as $d(\omega, \upsilon) = \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}$, where $\omega = (w_1, w_2)$, $\upsilon = (v_1, v_2) \in \mathcal{M}$, then*

$$\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, 2) = \{\omega \in \mathcal{M} : d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega) = 4\},$$

where $\mathfrak{c}_1 = (2, 0)$ and $\mathfrak{c}_2 = (0, 2) \in \mathcal{M}$, i.e., the equation of an ellipse centered at $(1, 1)$ having foci at $(2, 0)$ and $(0, 2)$ is $\sqrt{(2 - w_1)^2 + w_2^2} + \sqrt{w_1^2 + (2 - w_2)^2} = 4$ and is shown as the blue line in a Figure 1. If a metric $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ is defined as $d(\omega, \upsilon) = |w_1 - v_1| + |w_2 - v_2|$, $\omega = (w_1, w_2)$, $\upsilon = (v_1, v_2) \in \mathcal{M}$, then an ellipse having the same center and the same foci as above is $|2 - w_1| + |w_2| + |w_1| + |2 - w_2| = 4$ and is shown as the green line in a Figure 1.
If $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ is defined as $d(\omega, \upsilon) = \max\{|w_1 - v_1|, \ |w_2 - v_2|\}$, $\omega = (w_1, w_2), \upsilon = (v_1, v_2) \in \mathcal{M}$, then again an ellipse with the same center and the same foci is $\max\{|2 - w_1|, |w_2|\} + \max\{|w_1|, |2 - w_2|\} = 4$ *and is shown as the orange line in a Figure 1.*

Figure 1: *The blue line is the ellipse corresponding to a* metric $d(\omega, \upsilon) = \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}$, the green line shows the ellipse corresponding to a metric $d(\omega, \upsilon) = |w_1 - v_1| + |w_2 - v_2|$ and the red line shows the ellipse corresponding to a metric $d(\omega, \upsilon) = \max\{|w_1 - v_1|, \ |w_2 - v_2|\}$, centred at $(1, 1)$ and having foci at $(2, 0)$ and $(0, 2)$ in Example 1.

It is fascinating to see that shapes of some ellipses may change on changing the length of semi major axis (shape is changed in figure 2 but not in figure 3) or foci (refer to figure 4).

Figure 2: Ellipses corresponding to a metric $d(\omega, \upsilon) = |w_1 - v_1| + |w_2 - v_2|$, for semi major axis $a = 2,\ 2.5,\ 3$ and 4, centred at $(1,1)$ and having foci at $(2,0)$ and $(0,2)$ are shown by the green, the blue, the orange and the pink lines respectively in Example 1.

Figure 3: Ellipses corresponding to a metric $d(\omega, \upsilon) = \max\{|w_1 - v_1|,\ |w_2 - v_2|\}$, for semi major axis $a = 1,\ 2$ and 3, centred at $(1,1)$ and having foci at $(2,0)$ and $(0,2)$ are shown by the red, the blue and the green lines respectively, in Example 1.

Definition 3 *Let $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an ellipse having foci at $\mathfrak{c}_1$ and $\mathfrak{c}_2$ and length of a semi major axis as $\mathfrak{a}$ in a metric space $(\mathcal{M}, d)$, $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{M}$, $\mathfrak{a} \in [0, \infty)$. For a self-map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ in a metric space $(\mathcal{M}, d)$, if $\mathcal{A}\omega = \omega$ for all $\omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, then $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is called the fixed ellipse of $\mathcal{A}$.*

Theorem 1 *Let $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an ellipse in a metric space $(\mathcal{M}, d)$. Define $\varsigma : \mathcal{M} \longrightarrow [0, \infty)$ as*

$$\varsigma(\omega) = d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega),\ \ \mathfrak{c}_1,\ \ \mathfrak{c}_2,\ \ \omega \in \mathcal{M}. \tag{1}$$

If there exists a self map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ so that

($\mathcal{E}_1$) $d(\omega, \mathcal{A}\omega) \leq \varsigma(\omega) - \varsigma(\mathcal{A}\omega)$,

($\mathcal{E}_2$) $d(\mathfrak{c}_1, \mathcal{A}\omega) + d(\mathfrak{c}_2, \mathcal{A}\omega) \geq 2\mathfrak{a},\ \ \mathfrak{a} \in [0, \infty)$,

($\mathcal{E}_3$) $d(\mathcal{A}\omega, \mathcal{A}\upsilon) \leq \sigma d(\omega, \upsilon),\ \ \omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}),\ \ \upsilon \in \mathcal{M} \backslash \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}), \sigma \in [0, 1)$,

then $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a unique fixed ellipse of $\mathcal{A}$.

Figure 4: Ellipses corresponding to a metric $d(\omega, \upsilon) = |w_1 - v_1| + |w_2 - v_2|$, for semi major axis $a = 2$ and having foci at $(2,0)$, $(0,2)$; $(3,3)$, $(4,3)$; $(5,-2)$, $(3,0)$ and $(-1,-2)$, $(-3,-2)$ are shown by the green, the pink, the brown and the orange lines respectively in Example 1.

Proof. Let $\omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be any arbitrary point. Using (i) and equation (1)

$$\begin{aligned} d(\omega, \mathcal{A}\omega) &\leq d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega) - d(\mathfrak{c}_1, \mathcal{A}\omega) - d(\mathfrak{c}_2, \mathcal{A}\omega) \\ &= 2\mathfrak{a} - d(\mathfrak{c}_1, \mathcal{A}\omega) - d(\mathfrak{c}_2, \mathcal{A}\omega) \\ &\leq 2\mathfrak{a} - 2\mathfrak{a}, \quad \text{(using (ii))} \end{aligned}$$

i.e.,

$$d(\omega, \mathcal{A}\omega) = 0 \Longrightarrow \mathcal{A}\omega = \omega,$$

i.e., ω is fixed point of $\mathcal{A}$, $\forall\ \omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{M}$, $\mathfrak{a} \in [0, \infty)$. So, a self map $\mathcal{A}$ fixes an ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$,
i.e., the set of nonunique fixed points of a map $\mathcal{A}$ includes an ellipse.
Let there exist two fixed ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}_1', \mathfrak{c}_2', \mathfrak{a}')$ of $\mathcal{A}$, i.e., $\mathcal{A}$ satisfies both conditions $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ for each of the ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}_1', \mathfrak{c}_2', \mathfrak{a}')$, $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_1', \mathfrak{c}_2' \in \mathcal{M}$, $\mathfrak{a}, \mathfrak{a}' \in [0, \infty)$. Let $\omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\upsilon \in \mathcal{E}(\mathfrak{c}_1', \mathfrak{c}_2', \mathfrak{a}')$. Using (iii), $d(\omega, \upsilon) = d(\mathcal{A}\omega, \mathcal{A}\upsilon) \leq \sigma d(\omega, \upsilon)$, a contradiction. Hence $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a unique fixed ellipse of $\mathcal{A}$. ■

The following example illustrates Theorem 1.

Example 3 *Let $\mathcal{M} = \mathbb{R}^2$ and a metric $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be defined as $d(\omega, \upsilon) = \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}$ where $\omega = (w_1, w_2)$, $\upsilon = (v_1, v_2) \in \mathcal{M}$. Then*

$$\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, 5) = \{\omega \in \mathcal{M} : d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega) = 10\}, \tag{2}$$

where $\mathfrak{c}_1 = (-3, 0)$ and $\mathfrak{c}_2 = (3, 0) \in \mathcal{M}$, i.e., the equation of an ellipse centered at $(0,0)$ and having foci at $(-3, 0)$ and $(3, 0)$ is

$$\sqrt{(3 - w_1)^2 + w_2^2} + \sqrt{(3 + w_1)^2 + w_2^2} = 10,$$

i.e.,

$$\frac{w_1^2}{25} + \frac{w_2^2}{16} = 1.$$

Let $\varsigma : \mathcal{M} \longrightarrow [0, \infty)$ be defined as $\varsigma(\omega) = d((-3, 0), \omega) + d((3, 0), \omega)$, $\omega \in \mathcal{M}$. Define a self map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ as

$$\mathcal{A}(\mathfrak{a}, \mathfrak{b}) = \begin{cases} (\mathfrak{a}, \mathfrak{b}), & (\mathfrak{a}, \mathfrak{b}) \in (5\cos\theta, 4\sin\theta), \\ (c, d), & \textit{otherwise}, \end{cases}$$

Figure 5: The blue lines demonstrate the ellipse (2) which is fixed by the function A.

where (c, d) *is so that* $d((a, b), (c, d)) < \frac{1}{10} d(\omega, \upsilon)$, $\omega, \upsilon \in \mathcal{M}$. *Then a self map* $\mathcal{A}$ *verifies all the postulates of Theorem 1 and fixes the unique ellipse* $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, 5)$, *i.e., the set of nonunique fixed points of* $\mathcal{A}$ *contains a unique ellipse* $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, 5)$. *It is obvious that geometrically the condition* $(\mathcal{E}_1)$ *implies that* $\mathcal{A}\omega$ *is in the exterior of an ellipse and the condition* $(\mathcal{E}_2)$ *implies that* $\mathcal{A}\omega$ *is in the interior of an ellipse.*

The following examples depict the significance of conditions $(\mathcal{E}_1)$, $(\mathcal{E}_2)$ and $(\mathcal{E}_3)$ in the existence of a fixed ellipse or a unique fixed ellipse in Theorem 1.

Example 4 *Let* $\mathcal{M} = \mathbb{R}$ *and a metric* $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ *be defined as:*

$$d(\omega, \upsilon) = |\omega - \upsilon|, \quad \omega, \upsilon \in \mathcal{M}.$$

The ellipse

$$\begin{aligned} \mathcal{E}(1, 2, 2.5) &= \{\omega \in \mathcal{M} : d(1, \omega) + d(2, \omega) = 5\} \\ &= \{\omega \in \mathcal{M} : |1 - \omega| + |2 - \omega| = 5\} \\ &= \{-1, 4\}. \end{aligned}$$

Let $\varsigma : \mathcal{M} \longrightarrow [0, \infty)$ *be defined as* $\varsigma(\omega) = d(1, \omega) + d(2, \omega)$, $\omega \in \mathcal{M}$. *Define a self map* $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ *as*

$$\mathcal{A}\omega = \begin{cases} 1, & \omega \in \mathcal{E}(1, 2, 2.5), \\ -1, & \textit{otherwise}. \end{cases}$$

Then a self map $\mathcal{A}$ *verifies only the condition* $(\mathcal{E}_1)$ *of Theorem 1 and does not satisfy the conditions* $(\mathcal{E}_2)$ *and* $(\mathcal{E}_3)$. *One may verify that a self map* $\mathcal{A}$ *does not fix the ellipse* $\mathcal{E}(1, 2, 2.5)$ *and consequently, no ellipse.*

Example 5 *Let* $\mathcal{M} = \mathbb{R}^2$ *and a metric* $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ *be defined as*

$$d(\omega, \upsilon) = \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2},$$

where $\omega = (w_1, w_2), \upsilon = (v_1, v_2) \in \mathcal{M}$. *The ellipse*

$$\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, 13) = \{\omega \in \mathcal{M} : d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega) = 26\},$$

where $\mathfrak{c}_1 = (-5,0)$ *and* $\mathfrak{c}_2 = (5,0) \in \mathcal{M}$, *i.e., the equation of an ellipse centered at* $(0,0)$ *having foci at* $(-5,0)$ *and* $(5,0)$ *is*

$$\sqrt{(5-w_1)^2 + w_2^2} + \sqrt{(5+w_1)^2 + w_2^2} = 26,$$

i.e., $\frac{w_1^2}{169} + \frac{w_2^2}{144} = 1$. *Let* $\varsigma : \mathcal{M} \longrightarrow [0,\infty)$ *be difined as* $\varsigma(\omega) = d((-5,0),\omega) + d((5,0),\omega),\ \omega \in \mathcal{M}$. *Define a self map* $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ *as*

$$\mathcal{A}(\omega,\upsilon) = \begin{cases} (\omega,0), & \omega > 0, \\ (0,\upsilon), & \omega \leq 0. \end{cases}$$

Then a self map $\mathcal{A}$ *satifies the condition* $(\mathcal{E}_2)$ *and does not satisfy the conditions* $(\mathcal{E}_1)$ *and* $(\mathcal{E}_3)$ *of Theorem 1. Clearly,* $\mathcal{A}$ *does not fix the ellipse* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,13)$ *but fixes some points* $(13,0)$, $(0,12)$ *and* $(0,-12)$ *of an ellipse.*

Example 6 *Let* $\mathcal{M} = \mathbb{R}$ *and a metric* $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ *be defined as:* $d(\omega,\upsilon) = |\omega - \upsilon|,\ \omega,\upsilon \in \mathcal{M}$. *The ellipse*

$$\begin{aligned} \mathcal{E}(2,4,5) &= \{\omega \in \mathcal{M} : d(2,\omega) + d(4,\omega) = 10\} \\ &= \{\omega \in \mathcal{M} : |2-\omega| + |4-\omega| = 10\} \\ &= \{-2\ ,\ 8\}. \end{aligned}$$

Let $\varsigma : \mathcal{M} \longrightarrow [0,\infty)$ *be difined as* $\varsigma(\omega) = d(2,\omega) + d(4,\omega),\ \omega \in \mathcal{M}$. *Define a self map* $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ *as*

$$\mathcal{A}\omega = \begin{cases} -2, & \omega \in (-\infty,-2], \\ 8, & \omega \in (-2,\infty). \end{cases}$$

Then a self map $\mathcal{A}$ *verifies all the postulates of Theorem 1 except* $(\mathcal{E}_3)$ *and fixes an ellipse* $\mathcal{E}(2,4,5)$, *i.e., the set of nonunique fixed points of a self map* $\mathcal{A}$ *contains at least one ellipse* $\mathcal{E}(2,4,5)$. *However, there may exist infinitely many ellipses which are fixed by a self map* $\mathcal{A}$.

Using the equation (1), we give one more result for the existence of a unique fixed ellipse on a metric space.

Theorem 2 *Conclusion of Theorem 1 remains true even if we replace* $(\mathcal{E}_2)$ *by*

$(\mathcal{E}_2')$ $\eta d(\omega,\mathcal{A}\omega) + d(\mathfrak{c}_1,\mathcal{A}\omega) + d(\mathfrak{c}_2,\mathcal{A}\omega) \geq 2\mathfrak{a}, \quad \eta \in [0,1).$

Proof. Let $\omega \in \mathcal{C}(\omega_0,\mathfrak{r})$ be any arbitrary point. Using (i) and equation (1)

$$\begin{aligned} d(\omega,\mathcal{A}\omega) &\leq d(\omega,\mathfrak{c}_1) + d(\omega,\mathfrak{c}_2) - d(\mathcal{A}\omega,\mathfrak{c}_1) - d(\mathcal{A}\omega,\mathfrak{c}_2) \\ &= 2\mathfrak{a} - d(\mathcal{A}\omega,\mathfrak{c}_1) - d(\mathcal{A}\omega,\mathfrak{c}_2 \\ &\leq \eta d(\omega,\mathcal{A}\omega), \quad (\text{using } (\mathcal{E}_2')), \end{aligned}$$

which is a contradiction, i.e., $d(\omega,\mathcal{A}\omega) = 0 \Longrightarrow \mathcal{A}\omega = \omega$. Uniqueness of a fixed ellipse may be proved as in Theorem 1. ∎

The following example illustrates Theorem 2.

Example 7 *Let* $\mathcal{M} = \mathbb{R}^2$ *and a metric* $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ *be defined as:* $d(\omega,\upsilon) = |w_1 - v_1| + |w_2 - v_2|$, $\omega = (w_1,w_2)$, $\upsilon = (v_1,v_2) \in \mathcal{M}$. *The ellipse*

$$\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2) = \{\omega \in \mathcal{M} : d(\mathfrak{c}_1,\omega) + d(\mathfrak{c}_2,\omega) = 4\}, \tag{3}$$

where $\mathfrak{c}_1 = (1,2)$ *and* $\mathfrak{c}_2 = (0,2) \in \mathcal{M}$, *i.e., the equation of an ellipse centered at* $(0.5,2)$ *with foci at* $(1,2)$ *and* $(0,2)$ *is*

$$|1-w_1| + |2-w_2| + |w_1| + |2-w_2| = 4.$$

Figure 6: The blue lines demonstrate the ellipse (3) which is fixed by the function A.

Let $\varsigma : \mathcal{M} \longrightarrow [0,\infty)$ *be defined as* $\varsigma(\omega) = d((1,2),\omega) + d((0,2),\omega),\ \omega \in \mathcal{M}$. *Define a self map* $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ *as*

$$\mathcal{A}(\mathfrak{a},\mathfrak{b}) = \begin{cases} (\mathfrak{a},\mathfrak{b}), & (\mathfrak{a},\mathfrak{b}) \in \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2), \\ \frac{1}{15}(0,0.5), & \textit{otherwise}. \end{cases}$$

Then a self map $\mathcal{A}$ *verifies all the postulates of Theorem 1 and fixes the unique ellipse* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2)$*, i.e., the set of nonunique fixed points of* $\mathcal{A}$ *contains a unique ellipse* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2)$.

It is clear that geometrically the condition $(\mathcal{E}_1)$ implies that $\mathcal{A}\omega$ is in the exterior of an ellipse and the condition $(\mathcal{E}_2')$ implies that $\mathcal{A}\omega$ is in the interior of an ellipse. The following examples depict the significance of conditions $(\mathcal{E}_1)$, $(\mathcal{E}_2')$ and $(\mathcal{E}_3)$ in the existence of a fixed ellipse or a unique fixed ellipse in Theorem 2.

Example 8 *Let* $\mathcal{M} = \mathbb{R}$ *and a metric* $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ *be defined as:* $d(\omega,\upsilon) = |\omega - \upsilon|,\ \omega,\upsilon \in \mathcal{M}$. *The ellipse*

$$\begin{aligned} \mathcal{E}(-2,2,3) &= \{\omega \in \mathcal{M} : d(-2,\omega) + d(2,\omega) = 6\} \\ &= \{\omega \in \mathcal{M} : |2-\omega| + |2+\omega| = 6 \\ &= \{-3\ ,\ 3\}. \end{aligned}$$

Let $\varsigma : \mathcal{M} \longrightarrow [0,\infty)$ *be defined as* $\varsigma(\omega) = d(-2,\omega) + d(2,\omega),\ \omega \in \mathcal{M}$. *Define a self map* $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ *as*

$$\mathcal{A}\omega = \begin{cases} 2, & \omega \in \mathcal{E}(-2,2,3), \\ 3, & \textit{otherwise}. \end{cases}$$

Then a self map $\mathcal{A}$ *verifies only the condition* $(\mathcal{E}_1)$ *of Theorem 2 and does not satisfy the conditions* $(\mathcal{E}_2')$ *and* $(\mathcal{E}_3)$. *One may verify that a self map* $\mathcal{A}$ *does not fix the ellipse* $\mathcal{E}(-2,2,3)$ *and consequently, no ellipse.*

Example 9 *Let* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *be any arbitrary ellipse on any metric space* $(\mathcal{M},d)$, $\mathfrak{c}_1,\mathfrak{c}_2 \in \mathcal{M},\ \mathfrak{a} \in [0,\infty)$. *Let* ξ *be chosen such that* $d(\mathfrak{c}_1,\xi) + d(\mathfrak{c}_2,\xi) = \mu > 2\mathfrak{a}$. *Let* $\varsigma : \mathcal{M} \longrightarrow [0,\infty)$ *be defined as* $\varsigma(\omega) = d(\mathfrak{c}_1,\omega) + d(\mathfrak{c}_2,\omega),\ \omega \in \mathcal{M}$. *Define a self-map* $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ *as* $\mathcal{A}\omega = \xi,\ \omega \in \mathcal{M}$. *Then a self-map* $\mathcal{A}$ *verifies the postulate* $(\mathcal{E}_2')$ *but does not verify the postulate* $(\mathcal{E}_1)$. *Obviously,* $\mathcal{A}$ *does not fix the ellipse* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$.

Example 10 *Let* $\mathcal{M} = (0,\infty)$ *and a metric* $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ *be defined as:* $d(\omega,\upsilon) = |\ln\omega - \ln\upsilon|,\ \omega,\upsilon \in \mathcal{M}$. *The ellipse*

$$\begin{aligned} \mathcal{E}(1,e,3.5) &= \{\omega \in \mathcal{M} : d(1,\omega) + d(e,\omega) = 7\} \\ &= \{\omega \in \mathcal{M} : |\ln 1 - \ln\omega| + |\ln e - \ln\omega| = 7 \\ &= \{e^{-3},\ e^{4}\}. \end{aligned}$$

Let $\varsigma : \mathcal{M} \longrightarrow [0, \infty)$ *be defined as* $\varsigma(\omega) = d(1, \omega) + d(e, \omega)$, $\omega \in \mathcal{M}$. *Define a self map* $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ *as*

$$\mathcal{A}\omega = \begin{cases} e^{-3}, & \omega \in (0, e], \\ e^{4}, & \omega \in (e, \infty). \end{cases}$$

Then a self map $\mathcal{A}$ *verifies all the postulates of Theorem 2 except* $(\mathcal{E}_3)$ *and fixes the ellipse* $\mathcal{E}(1, 2, 3.5)$, *i.e., the set of nonunique fixed points of* $\mathcal{A}$ *contains at least one ellipse* $\mathcal{E}(1, 2, 3.5)$. *However, there may exist infinitely many ellipses which are fixed by a self map* $\mathcal{A}$.

Example 11 *Let a discrete metric* $d : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ *on* $\mathcal{M}$ *be defined as:*

$$d(\omega, \upsilon) = \begin{cases} 0, & \omega = \upsilon, \\ 1, & \omega \neq \upsilon, \end{cases}$$

and

$$\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \{\omega \in \mathcal{M} : d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega) = 2\mathfrak{a}\}, \quad \mathfrak{c}_1, \mathfrak{c}_2, \omega, \upsilon \in \mathcal{M}, \ \mathfrak{a} \in [0, \infty). \tag{4}$$

So

(i) if $\omega = \mathfrak{c}_1 \neq \mathfrak{c}_2$, $\mathfrak{a} = \frac{1}{2}$, *then* $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathcal{M} \backslash \{\mathfrak{c}_2\}$,

(ii) if $\omega = \mathfrak{c}_1 \neq \mathfrak{c}_2$, $\mathfrak{a} \neq \frac{1}{2}$, *then* $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \phi$,

(iii) if $\omega \neq \mathfrak{c}_1 \neq \mathfrak{c}_2$, $\mathfrak{a} = 1$, *then* $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathcal{M} \backslash \{\mathfrak{c}_1, \mathfrak{c}_2\}$,

(iv) if $\omega \neq \mathfrak{c}_1 \neq \mathfrak{c}_2$, $\mathfrak{a} \neq 1$, *then* $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \phi$.

Let $\varsigma : \mathcal{M} \longrightarrow [0, \infty)$ *be defined as* $\varsigma(\omega) = d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega)$, $\omega \in \mathcal{M}$. *Define a self map* $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ *as*

$$\mathcal{A}\omega = \begin{cases} \mathfrak{c}_1, & \omega = \mathfrak{c}_2, \\ \mathfrak{c}_2, & \omega = \mathfrak{c}_1, \\ \omega, & \textit{otherwise}. \end{cases}$$

Then a self map $\mathcal{A}$ *verifies all the postulates of Theorem 2 except the condition* $(\mathcal{E}_3)$, *i.e., the set of nonunique fixed points of* $\mathcal{A}$ *contains at least one ellipse. However, one may verify that an ellipse is not unique. There may exist infinitely many ellipses on varying the values of foci* $\mathfrak{c}_1, \mathfrak{c}_2$ *and lengths of semi major axis* $\mathfrak{a}$.

Remark 1 *(i) It is not necessary that an ellipse defined in a metric space is same as an ellipse in a Euclidean space. Noticeably, ellipses discussed in Examples 1, 2, 4, 6, 7, 10 and 11 are different from the ellipses in a Euclidean space. The shape of the ellipse may also change on changing the center, the semi major axis, foci, or the metric.*

(ii) Noticeably, the semi major axis $\mathfrak{a}$ *of the fixed ellipse does not depend on the center and may not be maximal.*

(iii) $\mathcal{A}\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ *does not imply that* $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ *is a fixed ellipse of* $\mathcal{A}$.

(iv) For work on the set of nonunique fixed points forming a circle or a disc, one may refer to Aydi et. al [1], Mlaiki et. al [9], Özgür at. al [10], Özgür and Taş [11]-[12], Pant et. al [13], Taş et. al [14], and references therein.

(v) It is worth mentioning here that the Banach contraction principle [2] and its generalizations give the existence of a unique fixed point for a self map (for instance, Baradol et al. [3], Gopal et al. [6]-[7], Lakzian et al. [8], Tomar and Joshi [18], Tomar and Sharma[16], Tomar et al. [15]-[17] and so on) however, Theorems 1 and 2 (see, supporting examples also) establish the significant fact that a discontinuous self-map fix a unique ellipse (i.e., a set of nonunique fixed points of a discontinuous self map includes a unique fixed ellipse) which naturally arise in numerous real-world problems.

It is interesting to see that the fixed ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is not essentially unique (see, for instance: Examples 6, 10, 11) unless some additional contraction condition is assumed. In Theorems 1 and 2, we have assumed Banach contraction [2] to prove the uniqueness of a fixed ellipse. So it is significant to establish the uniqueness of a fixed ellipse using different contractive conditions. In the following, we establish uniqueness using a more general contractive condition.

Theorem 3 *Let $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an ellipse on a metric space $(\mathcal{M}, d)$. Let $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ be a self map satisfying conditions $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ of Theorems 1 and 2 along with the contraction condition*

$$\begin{aligned} & d(\mathcal{A}\omega, \mathcal{A}\upsilon) \\ \leq\; & \eta \max\left\{d(\omega, \upsilon),\ d(\omega, \mathcal{A}\upsilon),\ d(\upsilon, \mathcal{A}\omega), \frac{1}{2}(d(\omega, \mathcal{A}\omega) + d(\upsilon, \mathcal{A}\upsilon)), \frac{1}{2}(d(\omega, \mathcal{A}\upsilon) + d(\upsilon, \mathcal{A}\omega))\right\}, \end{aligned} \tag{5}$$

$\omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}), \upsilon \in \mathcal{M} \setminus \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, where $\eta \in [0, 1)$, then $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{M}$, $\mathfrak{a} \in [0, \infty)$, is a unique fixed ellipse of $\mathcal{A}$.

Proof. Let there exist two fixed ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$ of $\mathcal{A}$, $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}'_1, \mathfrak{c}'_2 \in \mathcal{M}$, $\mathfrak{a}, \mathfrak{a}' \in [0, \infty)$, i.e., $\mathcal{A}$ satisfies conditions $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ for both the ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$. Let $\omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\upsilon \in \mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$. Using Inequality (5),

$$\begin{aligned} d(\omega, \upsilon) &= d(\mathcal{A}\omega, \mathcal{A}\upsilon) \\ &\leq \eta \max\left\{d(\omega, \upsilon), d(\omega, \upsilon), d(\upsilon, \omega), \frac{1}{2}(d(\omega, \omega) + d(\upsilon, \upsilon)), \frac{1}{2}(d(\omega, \upsilon) + d(\upsilon, \omega)\right\} \\ &= \eta d(\omega, \upsilon) \\ &< d(\omega, \upsilon), \end{aligned}$$

which is a contradiction. Hence $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a unique fixed ellipse of $\mathcal{A}$. ■

Next, we give propositions for the existence of a self map which fixes the given ellipses.

Proposition 4 *Let $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$ be any two ellipses in a metric space $(\mathcal{M}, d)$, $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}'_1, \mathfrak{c}'_2 \in \mathcal{M}$, $\mathfrak{a}, \mathfrak{a}' \in [0, \infty)$, then there exists at least one self map $\mathcal{A}$ on $\mathcal{M}$ such that a self map $\mathcal{A}$ fixes the ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$.*

Proof. Define $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ as

$$\mathcal{A}\omega = \begin{cases} \omega, & \omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) \cup \mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}'), \\ \mu, & \text{otherwise}, \end{cases}$$

where $\mathfrak{c}_1, \mathfrak{c}_2, \omega \in \mathcal{M}$, $\mathfrak{a} \in [0, \infty)$, μ is some constant such that $d(\mathfrak{c}_1, \mu) + d(\mathfrak{c}_2, \mu) \neq 2\mathfrak{a}$ and $d(\mathfrak{c}'_1, \mu) + d(\mathfrak{c}'_2, \mu) \neq 2\mathfrak{a}'$. Now define $\varsigma_1, \varsigma_2 : \mathcal{M} \longrightarrow [0, \infty)$ as $\varsigma_1(\omega) = d(\mathfrak{c}_1, \omega) + d(\mathfrak{c}_2, \omega)$ and $\varsigma_2(\omega) = d(\mathfrak{c}'_1, \omega) + d(\mathfrak{c}'_2, \omega)$, $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}'_1, \mathfrak{c}'_2$, $\omega \in \mathcal{M}$. Then a self map $\mathcal{A}$ verifies all the hypotheses of Theorems 1 and 2 (except $(\mathcal{E}_3)$) for the ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$. Hence $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$ are fixed ellipses of $\mathcal{A}$. ■

Following the similar pattern, above proposition may be extended for n ellipses.

Proposition 5 *Let $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}_1)$, $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}'_1)$, $\ldots, \mathcal{E}(\mathfrak{c}^{\mathfrak{n}}_1, \mathfrak{c}^{\mathfrak{n}}_2, \mathfrak{a}^{\mathfrak{n}}_1)$, $\mathfrak{c}_1, \mathfrak{c}'_1, \ldots, \mathfrak{c}^{\mathfrak{n}}_1, \mathfrak{c}_2, \mathfrak{c}'_2, \ldots, \mathfrak{c}^{\mathfrak{n}}_2 \in \mathcal{M}$, $\mathfrak{a}_1, \mathfrak{a}'_1, \ldots, \mathfrak{a}^{\mathfrak{n}}_1 \in [0, \infty)$ be any n ellipses in a metric space $(\mathcal{M}, d)$. Then there exists at least one self map $\mathcal{A}$ on $\mathcal{M}$ such that a self map $\mathcal{A}$ fixes ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}_1)$, $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}'_1)$, $\ldots, \mathcal{E}(\mathfrak{c}^{\mathfrak{n}}_1, \mathfrak{c}^{\mathfrak{n}}_2, \mathfrak{a}^{\mathfrak{n}}_1)$.*

One may observe that the ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}_1)$, $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}'_1)$, $\ldots, \mathcal{E}(\mathfrak{c}^{\mathfrak{n}}_1, \mathfrak{c}^{\mathfrak{n}}_2, \mathfrak{a}^{\mathfrak{n}}_1)$, in above proposition, need not be disjoint.

Next, we give a proposition on a metric space in which an ellipse contains all the points of a space except its foci and validate it by giving an example.

Proposition 6 *For $\mathfrak{a} \in \mathcal{M} = \mathbb{R}^+$, define the function $d_{\mathfrak{a}} : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ as*

$$d_{\mathfrak{a}}(\omega, \upsilon) = \begin{cases} 0, & \omega = \upsilon, \\ \mathfrak{a}, & \omega \neq \upsilon, \end{cases}$$

where $\omega,\ \upsilon \in \mathcal{M}$ and $\mathfrak{a} \in [0, \infty)$. Then the ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ in $(\mathcal{M}, d_{\mathfrak{a}})$ contains all the points $\omega \in \mathcal{M}$ except the foci $\mathfrak{c}_1,\ \mathfrak{c}_2 \in \mathcal{M}$.

Proof. Obviously, the function $d_{\mathfrak{a}}$ is a metric on $\mathcal{M}$ and consequently, $(\mathcal{M}, d_{\mathfrak{a}})$ is a metric space. Let, the ellipse

$$\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \{\omega \in \mathcal{M} : d_{\mathfrak{a}}(\mathfrak{c}_1, \omega) + d_{\mathfrak{a}}(\mathfrak{c}_2, \omega) = 2\mathfrak{a},\ \mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{M},\ \mathfrak{a} \in [0, \infty)\}.$$

Clearly, $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ consists of all of the points $\omega \in \mathcal{M}$ so that $\omega \neq \{\mathfrak{c}_1, \mathfrak{c}_2\}$. ■

Example 12 *Let $(\mathcal{M}, d_{\mathfrak{a}})$ be a metric space so that the metric $d_{\mathfrak{a}}$ be as in Proposition 2.3. Consider a set $\mathcal{J} = \{\omega_i : 1 \leq i \leq n\},\ n \in \mathbb{N}$. Obviously, there exists an ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ consisting of the elements of $\mathcal{J}$ as follows:*

$$\begin{aligned} \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) &= \{\omega \in \mathcal{M} : d_{\mathfrak{a}}(\mathfrak{c}_1, \omega) + d_{\mathfrak{a}}(\mathfrak{c}_2, \omega) = 2\mathfrak{a}, \mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{M},\ \mathfrak{a} \in [0, \infty)\} \\ &= \{\omega_1, \omega_2, \ldots, \omega_n\}, \end{aligned}$$

where $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{M} \setminus \mathcal{J}$.

3 Discontinuous Maps as Activation Functions in Neural Networks

First, we discuss continuity of a self map on a fixed ellipse. It is interesting to mention here that a discontinuous map has a strong fascination midst scientist, as most of the phenomenon emerging in the real-world is discontinuous in nature.

Theorem 7 *Let $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}),\ \mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{M},\ \mathfrak{a} \in [0, \infty)$ be a fixed ellipse of a self-map $\mathcal{A}$ in metric space $(\mathcal{M}, d)$ satisfying*

(i) $d(\mathcal{A}\omega, \mathcal{A}\upsilon) \leq \eta \mathbf{M}(\omega, \upsilon)$, *where*

$$\mathbf{M}(\omega, \upsilon) = \max\{d(\omega, \upsilon), d(\omega, \mathcal{A}\omega), d(\upsilon, \mathcal{A}\omega), \frac{1}{2}(d(\omega, \mathcal{A}\upsilon) + d(\upsilon, \mathcal{A}\omega))\},$$

$\eta \in [0, 1),\ \omega, \upsilon \in \mathcal{M}$.

(ii) For $\varepsilon > 0$, there exists a $\delta > 0$ satisfying $\varepsilon < \mathbf{M}(\omega, \upsilon) < \varepsilon + \delta \Longrightarrow d(\mathcal{A}\omega, \mathcal{A}\upsilon) < \varepsilon$.

If $\mathcal{A}$ has a unique fixed point, say $u \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\lim_{\omega_n \longrightarrow u} \mathcal{A}\omega_n = u \in \mathcal{M}$. Then a self map $\mathcal{A}$ is continuous at an ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ iff $\lim_{\omega_n \longrightarrow u} \mathbf{M}(\omega_n, u) = 0$ or in other words, $\mathcal{A}$ is discontinuous at an ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ iff $\lim_{\omega_n \longrightarrow u} \mathbf{M}(\omega_n, u) \neq 0$.

Proof. Let $\mathcal{A}$ be continuous at $u \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\omega_n \longrightarrow u$. So $\mathcal{A}\omega_n \longrightarrow \mathcal{A}u = u$.

$$\begin{aligned} \lim_{\omega_n \longrightarrow u} \mathbf{M}(\omega_n, u) &= \lim_{\omega_n \longrightarrow u} \max\{d(\omega_n, u), d(\omega_n, \mathcal{A}\omega_n), d(u, \mathcal{A}u), \frac{1}{2}(d(\omega_n, \mathcal{A}u) + d(u, \mathcal{A}\omega_n))\} \\ &= \max\{d(u, u), d(u, \mathcal{A}u), d(u, \mathcal{A}u), \frac{1}{2}(d(u, \mathcal{A}u) + d(u, \mathcal{A}u))\} \\ &= 0. \end{aligned}$$

Conversely, if $\lim_{\omega_n \longrightarrow u} \mathbf{M}(\omega_n, u) = 0$, i.e., $\lim_{\omega_n \longrightarrow u} d(\omega_n, \mathcal{A}\omega_n) = 0$. Hence, $\mathcal{A}\omega_n \longrightarrow u = \mathcal{A}u$, i.e., $\mathcal{A}$ is continuous at $u \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and hence continuous at an ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$. ■

Also motivated by the fact that the storage capacity of a discontinuous activation function is higher than continuous activation function, we discuss Mexican-hat-type non-monotonic discontinuous activation function which is used to introduce non-linear properties to the neural network. Let

$$\mathcal{A}_i\omega = \begin{cases} r_i, & -\infty < \omega < a_i \\ m_{i,1}\omega + n_{i,1}, & a_i \leq \omega < b_i \\ m_{i,2}\omega + n_{i,2}, & b_i \leq \omega < c_i \\ s_i, & c_i < \omega < \infty \end{cases}, \quad i = 1, 2, \ldots, n, \tag{6}$$

where $a_i, b_i, c_i, r_i, m_{i,1}, m_{i,2}, n_{i,1}, n_{i,2}$ are constants satisfying $-\infty < a_i < b_i < c_i < \infty$, $m_{i,1} > 0$, $r_i = m_{i,1}a_i + n_{i,1} = m_{i,2}c_i + n_{i,2}$, $m_{i,1}b_i + n_{i,1} = m_{i,2}b_i + n_{i,2}$, $s_i > \mathcal{A}b_i$, $i = 1, 2, \ldots, n$. Let $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be defined by $d(\omega, \upsilon) = |\omega - \upsilon|$, $\omega, \upsilon \in \mathcal{M}$.

Example 13 *Taking $m_{i,1} = \frac{1}{4}$, $m_{i,2} = -\frac{1}{4}$, $n_{i,1} = -1$, $n_{i,2} = -\frac{5}{4}$, $r_i = -\frac{3}{2}$, $s_i = \frac{9}{2}$, $a_i = -2$, $b_i = -\frac{1}{2}$, $c_i = 1$ in (6), we get the following discontinuous activation function*

$$\mathcal{A}\omega = \begin{cases} -\frac{3}{2}, & -\infty < \omega < -2 \\ \frac{1}{4}\omega - 1, & -2 \leq \omega < -\frac{1}{2} \\ -\frac{1}{4}\omega - \frac{5}{4}, & -\frac{1}{2} \leq \omega < 1 \\ \frac{9}{2}, & 1 \leq \omega < \infty \end{cases}. \tag{7}$$

Here $\mathcal{A}$ verifies all the postulates of Theorems 1 and 2 except (iii) for the ellipse $\mathcal{E}(1, 2, 3) = \{-\frac{3}{2}, \frac{9}{2}\}$ having

Figure 7: Maxican-hat-type discontinuous nonmonotonic activation function (6).

foci at $1, 2$, center $\omega = 3$ and semi major axis $\mathfrak{a} = 3$. Hence, $\mathcal{A}$ fixes the ellipse $\mathcal{E}(1, 2, 3)$. Since we have $\lim_{\omega \longrightarrow -\frac{3}{2}} \mathbf{M}(\omega, -\frac{3}{2}) = 0$ and $\lim_{\omega \longrightarrow \frac{9}{2}} \mathbf{M}(\omega, \frac{9}{2}) = 0$, $\mathcal{A}$ is continuous at the fixed points $u = -\frac{3}{2} \in \mathcal{E}(1, 2, 3)$ and $u = \frac{9}{2} \in \mathcal{E}(1, 2, 3)$. Thus, $\mathcal{A}$ is continuous at an ellipse $\mathcal{E}(1, 2, 3)$.

Example 14 *Now taking $m_{i,1} = 2$, $m_{i,2} = -1$, $n_{i,1} = 2$, $n_{i,2} = 8$, $r_i = 4$, $s_i = 9$, $a_i = 1$, $b_i = 2$, $c_i = 4$ in (6), we get the following discontinuous activation function*

$$\mathcal{A}\omega = \begin{cases} 4, & -\infty < \omega \leq 1 \\ 2\omega + 2, & 1 < \omega \leq 2 \\ -\omega + 8, & 2 < \omega \leq 4 \\ 9, & 4 < \omega < \infty \end{cases}.$$

The function $\mathcal{A}$ verifies all the postulates of Theorem 1 and 2 except (iii) for the ellipse $\mathcal{E}(6,7,2.5) = \{4,9\}$

Figure 8: Maxican-hat-type discontinuous nonmonotonic activation function (7).

having foci at $6,7$ center $\omega = \frac{13}{2}$ and semi major axis $\mathfrak{a} = 2.5$. Hence, $\mathcal{A}$ fixes the ellipse $\mathcal{E}(6,7,2.5)$. Since we have $\lim_{\omega \longrightarrow 4} \mathbf{M}(\omega, 4) \neq 0$ and $\lim_{\omega \longrightarrow 9} \mathbf{M}(\omega, 9) = 0$, $\mathcal{A}$ is discontinuous at the fixed point $u = 4 \in \mathcal{E}(6,7,2.5)$ and continuous at the fixed point $u = 9 \in \mathcal{E}(6,7,2.5)$. Thus, $\mathcal{A}$ is discontinuous at an ellipse $\mathcal{E}(6,7,2.5)$.

Remark 2 *(i) Using $\mathbf{M}(\omega, \upsilon)$, we may decide the ellipse at which the activation function is continuous or discontinuous.*

(ii) It is worth mentioning here that maps having nonunique fixed points forming fixed circle, fixed disc, or fixed ellipse have been utilized in neural networks as activation functions. Hence, our results may also be applicable under suitable conditions.

(iii) The fixed points of described real valued activation function are on ellipses which increase the quantity of fixed points of a neural network with a geometrical significance. Consequently, the underlying activation function may be beneficial to create numerous neural nets and lead to fascinating applications.

4 Conclusion

Motivated by the reflecting property of an ellipse which is useful in Medical science, Optics, Astronomy, Whispering Gallery, and so on, we explored a new direction to the geometric properties of the set of nonunique fixed point of a discontinuous map. It is interesting to mention here that ellipses in metric spaces, change their shapes on changing the length of semi major axis, foci or metric under consideration. Further, we discussed continuity at fixed-ellipses of discontinuous activation functions on metric spaces to demonstrate the significance of novel fixed-ellipse results in the neural networks. This permits us to select the suitable activation function according to the underlying problem on a neural networks which may have feasible applications in numerous neural networks.

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References

[1] H. Aydi, N. Y. Taş and N. Mlaiki, Fixed-discs in rectangular metric spaces, Symmetry, 11(2019), 294, 13pp.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équation intégrales, Fund. Math., 3(1922), 133–181.

[3] P. Baradol, D. Gopal and S. Radenović, Computational fixed points in graphical rectangular metric spaces with application, J. Comput. Appl. Math., 375(2020), 112805, 12 pp.

[4] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Am. Math. Soc., 215(1976), 241–251.

[5] M. Fréchet, Sur quelques points du calcul fonctionnel, Monatsh. Math. Phys., 19(1908), 247–257.

[6] D. Gopal, M. Abbas and C. Vetro, Some new fixed point theorems in Menger PM-spaces with application to Volterra type integral equation, Appl. Math. Comput., 232(2014), 955–967.

[7] D. Gopal, Fixed points of α-type F-contractive mappings with an application to nonlinear fractional differential equation, Acta Math. Sinica, 36B(2016), 957–970.

[8] H. Lakzian, D. Gopal and W. Sintunavarat, New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations, J. Fixed Point Theory Appl., 18(2016), 251–266.

[9] N. Mlaiki, N. Taş and N. Y. Özgür, On the fixed-circle problem and Khan type contractions, Axioms, 7(2018), 80, 10pp.

[10] N. Y. Özgür, N. Taş and U. Çelik, New fixed-circle results on S-metric spaces, Bull. Math. Anal. Appl., 9(2017), 10–23.

[11] N. Y. Özgür and N. Taş, Some fixed-circle theorems and discontinuity at fixed circle, AIP Conference Proceedings, 1926, 020048 (2018)

[12] N. Y. Özgür and N. Taş, Some fixed-circle theorems on metric spaces, Bull. Malays. Math. Sci. Soc., 42(2017), 1433–1449.

[13] R. P. Pant, N. Y. Özgür and N. Taş, On discontinuity problem at fixed point, Bull. Malays. Math. Sci. Soc., 43(2020), 499–517.

[14] N. Taş, N. Y. Özgür and N. Mlaiki, New Types of F_C-contractions and the fixed-circle problem, Mathematics, 6(2018), 188, 9 pp.

[15] A. Tomar, M. Joshi, S. K. Padaliya, B. Joshi and A. Dwivedi, Fixed point under set-valued relation-theoretic nonlinear contractions and application, Filomat, 33(2019), 4655–4664.

[16] A. Tomar and R. Sharma, Almost alpha-Hardy-Rogers-F-contraction and applications, Armenian J. Math., 11(2019), 1–19.

[17] A. Tomar, M. Joshi and Amar Deep, Fixed points and its applications in C^*-Algebra valued partial metric space, TWMS J. App. and Eng. Math., 11(2021), 329–340.

[18] A. Tomar and M. Joshi, Note on results in C^*-algebra-valued metric spaces, 9(2021), 262–264.