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A Note On The Stability Of Nonnegative Solutions To Classes Of Fractional Laplacian Problems With Concave Nonlinearity*

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Abstract

In this paper, we study the stability of nonnegative stationary solutions of fractional Laplacian equations with concave nonlinearity condition. In particular, we employ a principle of linearized stability to this class of problems to prove sufficient conditions for the stability of such solutions. The main results of the present paper are new and extend the previously known results.

1 Introduction

In this note, we consider the stability of nonnegative stationary solutions of the elliptic problem with fractional Laplacian

$$\begin{cases} -(\Delta)^s u = f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in R^n \setminus \Omega, \end{cases} \tag{1}$$

where $-(\Delta)^s u$ is the fractional Laplacian operator of u, Ω is a bounded domain in $\mathbb{R}^N$, $n > 2s$ with $s \in (0,1)$, with sufficiently smooth boundary and f is a strictly concave C^2 function on $[0,\infty)$.

The motivation for our study is the local case ($s = 1$). This was first studied by Shivaji and his co-authors for convex nonlinearity. They have shown that every non-trivial solution of

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

is unstable if $f'' > 0$ and $f(0) \le 0$. They first considered the monotone case, i.e. $f' > 0$ in [4]. The statement in the non-monotone case was first proved by Tertikas [14] using sub- and supersolutions. The first simplification was given by Maya and Shivaji in [10] by reducing the problem to the monotone case via decomposition of f to a monotone and a linear function. Karátson and Simon gave a direct proof of the result in [8]. Moreover, this proof showed the stability of the concave counterpart at the same time, and could be easily extended to the general elliptic operator $\mathrm{div}(A\nabla u)$, where $A : \Omega \to \mathbb{R}^{n\times n}$. In [9] the corresponding equation with p-Laplacian is studied. Also see [1] for stability properties of non-negative solutions to a non-autonomous p-Laplacian problems. In [2] this study was extended to p-Laplacian systems.

For reaction-diffusion systems, both cooperative and competitive, Castro, Chhetri and Shivaji established sufficient conditions on the nonlinearity for the solutions to be stable and unstable (see [6]). Also in [17] stability of non-negative stationary solutions of symmetric cooperative semilinear systems with some convex (resp. concave) nonlinearity condition was studied. The focus of this paper is to extend the studies in [8] to fractional operator. Due to the appearance of fractional operator in (1), the extensions are challenging and nontrivial. To the best of our knowledge, this is an interesting and new research topic for fractional operator.

Recently, a great deal of attention has been focused on studying problems involving fractional Sobolev spaces and corresponding nonlocal equations, both from a pure mathematical point of view and for concrete

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applications, since they naturally arise in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultrarelativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. For more details, we can see [5, 15, 16].

The natural space to look for solutions of the problem (1) is the usual fractional Sobolev space $W^{s,2}(\mathbb{R}^n) = \Big\{u \in L^2(\mathbb{R}^n) : \Big(\frac{u(x)-u(y)}{|x-y|^{\frac{n}{2}+s}}\Big) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\Big\}$ endowed with the norm:

$$\|u\|_{W^{s,2}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} |u|^2\,dx\Big)^{1/2}, \tag{2}$$

where the term

$$[u]_{W^{s,2}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}$$

is the so-called Gagliardo (semi) norm of u. To study fractional Sobolev space in detail, we refer to [11, 13]. We define

$$X_0^{s,2}(\Omega) = \{u \in W^{s,2}(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

The space $X_0^{s,2}(\Omega)$ is a normed linear space endowed with the norm $\|.\|_{X_0^{s,2}(\Omega)}$ defined as

$$\|u\|_{X_0^{s,2}(\Omega)} = \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

We recall that, $X_0^{s,2}(\Omega)$ is a closed subspace of $W^{s,2}(\mathbb{R}^n)$, and its norm is equivalent to the usual one defined in (2).

On the other hand, the spaces $W^{s,2}(\mathbb{R}^n)$ and $X_0^{s,2}(\Omega)$ are strictly related to the fractional Laplacian operator. The fractional Laplacian is the pseudo-differential operator with Fourier symbol $\mathcal{F}$ satisfying

$$\mathcal{F}\Big((-\Delta)^s u\Big)(\xi) = |\xi|^{2s}\widehat{u}(\xi),\ 0 < s < 1,$$

where $\widehat{u}$ denotes the Fourier transform of u, (see [3]). Using Fourier transforms, it can be shown that (see [3]) an equivalent characterization of the fractional Laplacian is given by

$$(-\Delta)^s u(x) = C(n,s)P.V.\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy = C(n,s)\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy.$$

Here $P.V.$ is a commonly used abbreviation for "in the principal value sense"(as defined by the latter equation) and $C(n,s)$ is a dimensional constant that depends on n and s, precisely given by

$$C(n,s) = \Big(\int_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}}\Big)^{-1},\ \zeta = (\zeta_1, \zeta_2, ..., \zeta_n) \in \mathbb{R}^n.$$

In [11, Proposition 3.6], the author proved the relation between the fractional Laplacian operator $(-\Delta)^s$and the fractional Sobolev space $W^{s,2}(\mathbb{R}^n)$. They established

$$[u]_{W^{s,2}(\mathbb{R}^n)} = 2C(n,s)^{-1}\|(-\Delta)^{\frac{s}{2}}u\|^2_{L^{\mathbb{R}^n}}. \tag{3}$$

2 Main Results

In this section, we shall prove the stability of positive solutions of (1).

Definition 1 *A function $u \in X_0^{s,2}(\Omega)$ is said to be a (weak) solution of (1), if for any $\varphi \in X_0^{s,2}(\Omega)$, we have*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u.(-\Delta)^{\frac{s}{2}} \varphi \, dx - \int_{\Omega} f(u)\varphi \, dx = 0. \tag{4}$$

We recall that $u \in X_0^{s,2}(\Omega)$ is stable (see [7]) if

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx - \int_{\Omega} f'^2 \, dx > 0, \ \forall v \in C_c^1(\Omega).$$

Now we are ready to state our main result.

Theorem 1 *If $f'' < 0$ and $f(0) \geq 0$, then every nontrivial nonnegative stationary solution of (1) is stable.*

Proof. From the convexity of the map $t \longmapsto t^2$, it is easy to see that

$$(a-b)^2 - (c-d)\Big(\frac{a^2}{c} - \frac{b^2}{d}\Big) \geq 0,$$

for all $a, b, c, d \in \mathbb{R}$, with $c > 0$, and $d > 0$. Let $l(u) = uf'(u) - f(u)$, for $u \in \mathbb{R}^+$. Since $l(0) \leq 0$ and $l'(u) < 0$, we have $l(u) < 0$. Now, let u be a nontrivial nonnegative stationary solution of (1). Take $v \in C_c^1(\Omega)$, and let $\varphi = \frac{v^2}{u}$ in (4), (we note that, in this case $\varphi \in X_0^{s,2}(\Omega)$). Then, from (3), we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u.(-\Delta)^{\frac{s}{2}} (\frac{v^2}{u}) \, dx - \int_{\Omega} f(u)(\frac{v^2}{u}) \, dx \\
&= \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u(x)-u(y)]\Big(\frac{v^2(x)}{u(x)} - \frac{v(y)}{u(y)}\Big)}{|x-y|^{n+2s}} dxdy - \int_{\Omega} f(u)(\frac{v^2}{u}) \, dx \\
&\leq \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x)-v(y))^2}{|x-y|^{n+2s}} dxdy - \int_{\Omega} \frac{f(u)}{u} v^2 \, dx \\
&< \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x)-v(y))^2}{|x-y|^{n+2s}} dxdy - \int_{\Omega} f'^2 \, dx \\
&= \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx - \int_{\Omega} f'^2 \, dx.
\end{aligned}$$

Hence u is stable. This completes the proof of Theorem 1 ■

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