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On The Asymptotic Expansion Of A Generalized Smith's Determinant*

Mehdi Hassani†

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Abstract

In this paper we study the generalized Smith's determinant $\Delta_s(n) := \det\left[(\gcd(i,j))^s\right]_{1\leqslant i,j\leqslant n}$, where $s \neq 0$ is fixed real. For large values of n we obtain asymptotic expansions of $\log|\Delta_s(n)|$, and for $s > 1$ we obtain Stirling type approximations for $\Delta_s(n)$. Furthermore, we prove that for $s < 0$ the sign of $\Delta_s(n)$ is independent of s, and is same as the sign of $(-1)^{\eta_n}$, where η_n denotes the number of integers $m \in [1, n]$ having odd number of distinct prime divisors.

1 Introduction

In 1875 Smith [8] considered the determinant of the matrix $[a_{ij}]_{1\leqslant i,j\leqslant n}$ with elements given by $a_{ij} = \gcd(i,j)$, greatest common divisor of i and j. He proved that

$$\det\left[\gcd(i,j)\right]_{1\leqslant i,j\leqslant n} = \prod_{m=1}^{n} \varphi(m),$$

where $\varphi(m)$ denotes the Euler function of m, counting the number of positive integers not exceeding m and coprime to m. The above determinant is known as *Smith's determinant*. Since Smith's work this field has been studied extensively. For a recent account of the theory of gcd-matrices we refer the reader to [4] and the references given there. Also, see [2, p. 123] for some classical generalizations of Smith's determinant, including the assertion that if f is an arithmetic function then

$$\det\left[f(\gcd(i,j))\right]_{1\leqslant i,j\leqslant n} = \prod_{m=1}^{n} \sum_{d|m} \mu(d)\, f\left(\frac{m}{d}\right), \tag{1}$$

where $\mu(d)$ denotes the Möbius function of d, which is 1 if $d = 1$, is $(-1)^k$ if d is equal to the product of k distinct primes, and is 0 otherwise. In this paper we are motivated by the asymptotic growth of generalized Smith's determinant (1) for $f(n) = n^s$. The exponent s is an arbitrary non-zero real. For the case $s > 0$ we prove the following result.

Theorem 1 *Let $s > 0$ be fixed real and*

$$\Delta_s^+(n) = \det\left[(\gcd(i,j))^s\right]_{1\leqslant i,j\leqslant n}. \tag{2}$$

Define the absolute constant α_s by

$$\alpha_s = \sum_p \frac{1}{p} \log\left(1 - \frac{1}{p^s}\right), \tag{3}$$

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†Department of Mathematics, University of Zanjan, University Blvd., 45371-38791, Zanjan, Iran

where p runs over all primes. Then, as $n \to \infty$,

$$\log \Delta_s^+(n) = \begin{cases} sn\log n + (\alpha_s - s)\,n + O(n^{1-s}) & (0 < s < 1), \\ n\log n + (\alpha_1 - 1)n + \frac{1}{2}\log n + O(\log\log n) & (s = 1), \\ sn\log n + (\alpha_s - s)\,n + \frac{s}{2}\log n + s\log\sqrt{2\pi} + O\big(\frac{1}{n^{s-1}}\big) & (1 < s \leqslant 2). \end{cases}$$

Also, for each $s > 2$ the following approximation holds

$$\log \Delta_s^+(n) = sn\log n + (\alpha_s - s)\,n + \frac{s}{2}\log n + s\log\sqrt{2\pi} + \sum_{1\leqslant j\leqslant \frac{s}{2}} \frac{sB_{2j}}{(2j)(2j-1)n^{2j-1}} + O\Big(\frac{1}{n^{s-1}}\Big),$$

where B_i denotes the i-th Bernoulli number.

Stirling approximation for $n!$ asserts that $n! = \left(\frac{n}{e}\right)^n\sqrt{2\pi n}\,(1+O(\frac{1}{n}))$. By taking exponent we obtain the following Stirling type approximation for $\Delta_s^+(n)$ for each $s > 1$.

Corollary 1 *Let $\Delta_s^+(n)$ be the determinant defined by (2). Then, as $n \to \infty$,*

$$\Delta_s^+(n) = \begin{cases} \left(\frac{n}{e}\right)^{sn}\ \beta_s^n\ \sqrt{(2\pi n)^s}\,\big(1 + O\big(\frac{1}{n^{s-1}}\big)\big) & (1 < s < 2), \\ \left(\frac{n}{e}\right)^{sn}\ \beta_s^n\ \sqrt{(2\pi n)^s}\,\big(1 + O\big(\frac{1}{n}\big)\big) & (s \geqslant 2), \end{cases}$$

where β_s is an absolute constant defined by

$$\beta_s = \prod_p \left(1 - \frac{1}{p^s}\right)^{\frac{1}{p}},$$

and p runs over all primes.

A more sophisticated argument, similar to that used in our paper [3], enables us to consider the case of negative values of exponent.

Theorem 2 *Let $s > 0$ be fixed real and*

$$\Delta_s^-(n) = \det\left[(\gcd(i,j))^{-s}\right]_{1\leqslant i,j\leqslant n}.$$

Then, for any positive integer r there exist computable constants $c_1, \ldots, c_r$ such that as $n \to \infty$,

$$\log\left|\Delta_s^-(n)\right| = (\alpha_s + \gamma + E)\,n + s\sum_{j=1}^{r}\frac{c_j n}{\log^j n} + O\left(\frac{n}{\log^{r+1} n}\right),$$

where α_s is defined by (3), γ is Euler's constant, and E is the constant in Mertens' approximation given by

$$E = \lim_{x\to\infty}\sum_{p\leqslant x}\frac{\log p}{p} - \log x. \tag{4}$$

Furthermore, the sign of $\Delta_s^-(n)$ is independent of s, and is same as the sign of $(-1)^{\eta_n}$, where η_n denotes the number of integers $m \in [1,n]$ having odd number of distinct prime divisors.

2 Proof of Theorem 1

Proof. For $f(n) = n^s$, we conclude from (1) that

$$\Delta_s^+(n) = \prod_{m=1}^{n} m^s\, g_s(m) = n!^s \prod_{m=1}^{n} g_s(m),$$

where $g_s(m) = \sum_{d|m} \mu(d)d^{-s}$. Since g_s is multiplicative, we get

$$g_s(m) = \prod_{p^a \| m} g_s(p^a) = \prod_{p^a \| m} \left(1 - \frac{1}{p^s}\right) = \prod_{p|m} \left(1 - \frac{1}{p^s}\right).$$

Thus,

$$\Delta_s^+(n) = n!^s \prod_{m=1}^{n} \prod_{p|m} \left(1 - \frac{1}{p^s}\right),$$

and

$$\log \Delta_s^+(n) = s \log n! + \sum_{m=1}^{n} \sum_{p|m} \log \left(1 - \frac{1}{p^s}\right). \tag{5}$$

Stirling's approximation [7, p. 294] for $\log n!$ asserts that given any positive integer r, as $n \to \infty$,

$$\log n! = n \log n - n + \frac{1}{2} \log n + \log \sqrt{2\pi} + \sum_{j=1}^{r} \frac{B_{2j}}{(2j)(2j-1)n^{2j-1}} + O\Big(\frac{1}{n^{2r+1}}\Big). \tag{6}$$

To approximate the double sum in (5), we change the order of summations. Thus,

$$\begin{aligned}
\sum_{m=1}^{n} \sum_{p|m} \log \left(1 - \frac{1}{p^s}\right) &= \sum_{p \leqslant n} \log \left(1 - \frac{1}{p^s}\right) \sum_{\substack{m \leqslant n \\ p|m}} 1 = \sum_{p \leqslant n} \log \left(1 - \frac{1}{p^s}\right) \left[\frac{n}{p}\right] \\
&= \sum_{p \leqslant n} \log \left(1 - \frac{1}{p^s}\right) \left(\frac{n}{p} + O(1)\right) \\
&= n \sum_{p \leqslant n} \frac{1}{p} \log \left(1 - \frac{1}{p^s}\right) + O\left(\sum_{p \leqslant n} \log \left(1 - \frac{1}{p^k}\right)\right) \\
&= \alpha_s\, n + n \sum_{p > n} \frac{1}{p} \log \left(1 - \frac{1}{p^s}\right)^{-1} + O\left(\sum_{p \leqslant n} \log \left(1 - \frac{1}{p^s}\right)\right).
\end{aligned}$$

Since $-\log(1-t) \sim t$ as $t \to 0$, we get

$$\sum_{p > n} \frac{1}{p} \log \left(1 - \frac{1}{p^s}\right)^{-1} \ll \sum_{p > n} \frac{1}{p^{s+1}} \ll \int_n^\infty \frac{dx}{x^{s+1}} \ll \frac{1}{n^s}.$$

Also, by using the approximation $\sum_{p \leqslant n} \frac{1}{p} \ll \log \log n$ we obtain

$$\sum_{p \leqslant n} \log \left(1 - \frac{1}{p^s}\right) \ll \sum_{p \leqslant n} \frac{1}{p^s} \ll \begin{cases} \int_2^n \frac{dx}{x^s} \ll \frac{1}{n^{s-1}} & (s \neq 1), \\ \log \log n & (s = 1). \end{cases}$$

Hence,

$$\sum_{m=1}^{n}\sum_{p|m}\log\left(1-\frac{1}{p^s}\right)=\alpha_s\,n+O\left(\left\{\begin{array}{ll} n^{1-s} & (s\neq 1)\\ \log\log n & (s=1)\end{array}\right.\right). \tag{7}$$

We let $r=[\frac{s}{2}]$ in (6). Note that $2[\frac{s}{2}]+1\geqslant s-1$. Therefore, by considering (5) and (7) we conclude the proof. ■

3 Proof of Theorem 2

Proof. Let $s>0$. We conclude from (1) that

$$\Delta_s^-(n)=\prod_{m=1}^{n}m^{-s}\,h_s(m)=n!^{-s}\prod_{m=1}^{n}h_s(m),$$

where $h_s(m)=\sum_{d|m}\mu(d)d^s$. Since h_s is multiplicative, we get

$$\begin{aligned} h_s(m)&=\prod_{p^a\|m}h_s(p^a)=\prod_{p^a\|m}(1-p^s)=\prod_{p|m}(1-p^s)\\ &=(-1)^{\omega(m)}\prod_{p|m}(p^s-1)=(-1)^{\omega(m)}\,\kappa(m)^s\prod_{p|m}\left(1-\frac{1}{p^s}\right), \end{aligned}$$

where $\omega(m)$ counts the number of distinct prime factors of m, and $\kappa(m)$ denotes the product of distinct prime factors of m. Thus,

$$\Delta_s^-(n)=(-1)^{\sum_{m=1}^{n}\omega(m)}\,n!^{-s}\left(\prod_{m=1}^{n}\kappa(m)\right)^s\prod_{m=1}^{n}\prod_{p|m}\left(1-\frac{1}{p^s}\right). \tag{8}$$

This relation implies that the sign of $\Delta_s^-(n)$ depends on the value of $\sum_{m=1}^{n}\omega(m)$, which is independent of s. Moreover, the sign of $\Delta_s^-(n)$ is same as the sign of $(-1)^{\eta_n}$, where

$$\eta_n=\sum_{\substack{1\leqslant m\leqslant n\\ \omega(m)\text{ is odd}}}1.$$

denoting the number of integers $m\in[1,n]$ which have odd number of distinct prime divisors. Furthermore, we conclude from (8) that

$$\log\left|\Delta_s^-(n)\right|=-s\log n!+s\sum_{m=1}^{n}\log\kappa(m)+\sum_{m=1}^{n}\sum_{p|m}\log\left(1-\frac{1}{p^s}\right).$$

To approximate $\sum_{m=1}^{n}\log\kappa(m)$ we recall the notion of *the index of composition* of n, which is defined by

$$\lambda(n)=\frac{\log n}{\log\kappa(n)},$$

for each integer $n\geqslant 2$. Note that $\lambda(n)$ "somehow" measures how much the integer $n\geqslant 2$ is composite! For n square-free it takes the value $\lambda(n)=1$, and for integers n having square factors in heart, it takes the value $\lambda(n)>1$. De Koninck and Kátai [1] proved that given any positive integer r, there exist computable constants $d_1,\ldots,d_r$ such that

$$\upsilon(x):=\sum_{k\leqslant x}\frac{1}{\lambda(k)}=x+\sum_{j=1}^{r}d_j\frac{x}{\log^j x}+O\Big(\frac{x}{\log^{r+1}x}\Big). \tag{9}$$

By using Abel summation we get

$$\sum_{k=1}^{n}\log\kappa(k)=\sum_{k=2}^{n}\frac{1}{\lambda(k)}\log k=\upsilon(n)\log n-\upsilon(2^{-})\log 2-\int_{2}^{n}\frac{\upsilon(t)}{t}\mathrm{d}t.$$

To deal with the last integral, we study the functions $\mathrm{L}_j(t)$ defined for each integer $j\geqslant 1$ by the following anti-derivative

$$\mathrm{L}_j(t):=\int\frac{\mathrm{d}t}{\log^j t},$$

Note that $\mathrm{L}_1(t)$ is the logarithmic integral function, which admits the following expansion

$$\mathrm{L}_1(t)=\mathrm{li}(t)=\sum_{i=1}^{r}(i-1)!\,\frac{t}{\log^i t}+O\Big(\frac{t}{\log^{r+1}t}\Big). \tag{10}$$

Integrating by parts gives

$$\mathrm{L}_{j-1}(t)=\int\Big(\frac{1}{\log^{j-1}t}\Big)\,(\mathrm{d}t)=\frac{t}{\log^{j-1}t}+(j-1)\int\frac{\mathrm{d}t}{\log^j t}.$$

Hence, for $j\geqslant 2$ the functions $\mathrm{L}_j(t)$ satisfy the recurrence

$$\mathrm{L}_j(t)=\frac{1}{j-1}\mathrm{L}_{j-1}(t)-\frac{t}{(j-1)\log^{j-1}t}.$$

By repeated using this recurrence we deduce that

$$(j-1)!\,\mathrm{L}_j(t)=\mathrm{li}(t)-\sum_{i=1}^{j-1}(i-1)!\,\frac{t}{\log^i t}.$$

Hence, by using the expansion (10), for $1\leqslant j\leqslant r$ we obtain

$$\mathrm{L}_j(t)=\sum_{i=j}^{r}\frac{(i-1)!}{(j-1)!}\frac{t}{\log^i t}+O\Big(\frac{t}{\log^{r+1}t}\Big). \tag{11}$$

We deduce from the expansion (9) that

$$\begin{aligned}\int_2^n\frac{\upsilon(t)}{t}\mathrm{d}t&=\int_2^n\left(1+\sum_{j=1}^{r}d_j\frac{1}{\log^j t}+O\Big(\frac{1}{\log^{r+1}t}\Big)\right)\mathrm{d}t\\&=n+\sum_{j=1}^{r}d_j\mathrm{L}_j(n)-\left(2+\sum_{j=1}^{r}d_j\mathrm{L}_j(2)\right)+O\Big(\frac{n}{\log^{r+1}n}\Big).\end{aligned}$$

With r replaced by $r+1$ in (9), we obtain

$$\upsilon(n)\log n=n\log n+d_1n+\sum_{j=1}^{r}d_{j+1}\frac{n}{\log^j n}+O\Big(\frac{n}{\log^{r+1}n}\Big).$$

Combining the above expansions, we obtain

$$\sum_{k=1}^{n}\log\kappa(k)=n\log n+(d_1-1)n+\sum_{j=1}^{r}\Big(d_{j+1}\frac{n}{\log^j n}-d_j\mathrm{L}_j(n)\Big)-C_r+O\Big(\frac{n}{\log^{r+1}n}\Big),$$

where

$$C_r = 2 + \upsilon(2^-)\log 2 + \sum_{j=1}^{r} d_j \mathrm{L}_j(2)$$

is a constant depending only on r. Thus, $C_r = O_r(1)$. Moreover, we deduce from the expansion (11) that

$$\sum_{j=1}^{r}\left(d_{j+1}\frac{n}{\log^j n} - d_j\mathrm{L}_j(n)\right) = \sum_{j=1}^{r}\left(d_{j+1}\frac{n}{\log^j n} - \sum_{i=j}^{r} d_j\frac{(i-1)!}{(j-1)!}\frac{n}{\log^i n}\right) + O\Big(\frac{n}{\log^{r+1} n}\Big).$$

Note that

$$\sum_{j=1}^{r}\left(d_{j+1}\frac{n}{\log^j n} - \sum_{i=j}^{r} d_j\frac{(i-1)!}{(j-1)!}\frac{n}{\log^i n}\right) = \sum_{j=1}^{r} c_j\frac{n}{\log^j n} + O\Big(\frac{n}{\log^{r+1} n}\Big),$$

where c_js are computable constants in terms of d_js. Thus, letting $c_0 = d_1 - 1$, we obtain

$$\sum_{m=1}^{n}\log\kappa(m) = n\log n + c_0\, n + \sum_{j=1}^{r} c_j\frac{n}{\log^j n} + O\Big(\frac{n}{\log^{r+1} n}\Big).$$

To compute the precise value of c_0 we write

$$\sum_{m=1}^{n}\log\kappa(m) = \sum_{m=1}^{n}\log\prod_{p|m} p = \sum_{m=1}^{n}\sum_{p|m}\log p = \sum_{p\leqslant n}\left[\frac{n}{p}\right]\log p = n\,\mathcal{M}(n) - \mathcal{R}(n), \tag{12}$$

where

$$\mathcal{M}(n) := \sum_{p\leqslant n}\frac{\log p}{p},$$

and

$$\mathcal{R}(n) := \sum_{p\leqslant n}\left\{\frac{n}{p}\right\}\log p.$$

It is known due to Landau [5, p. 198] that

$$\mathcal{M}(n) = \log n + E + O\Big(\frac{1}{\log n}\Big), \tag{13}$$

where E is the constant given by (4). To estimate $\mathcal{R}(n)$ we let

$$\mathcal{S}(n) = \sum_{p\leqslant n}\left\{\frac{n}{p}\right\},$$

and

$$\mathcal{L}(n) = \sum_{p^\alpha\leqslant n}\left\{\frac{n}{p^\alpha}\right\}.$$

It is known due to Lee [6] that

$$\mathcal{L}(n) = (1-\gamma)\frac{n}{\log n} + O\Big(\frac{n}{\log^2 n}\Big).$$

We observe that although the summation $\mathcal{L}(n)$ has the summation $\mathcal{S}(n)$ in heart, but their difference is not too large in comparison the true size of $\mathcal{L}(n)$. More precisely,

$$\mathcal{L}(n) - \mathcal{S}(n) = \sum_{\substack{p^\alpha\leqslant n\\ \alpha\geqslant 2}}\left\{\frac{n}{p^\alpha}\right\} < \sum_{\substack{p^\alpha\leqslant n\\ \alpha\geqslant 2}} 1 = \sum_{\substack{p\leqslant n^{\frac{1}{\alpha}}\\ \alpha\geqslant 2}} 1 = \sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}}\pi(n^{\frac{1}{\alpha}})$$

$$\ll \sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}}\frac{n^{\frac{1}{\alpha}}}{\log n^{\frac{1}{\alpha}}} \leqslant \frac{n^{\frac{1}{2}}}{\log n}\sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}}\alpha \ll \sqrt{n}\log n,$$

where $\pi(t)$ denotes the number of primes p not exceeding t, and we use the simple estimate $\pi(t) \ll \frac{t}{\log t}$ in the above argument. Hence,

$$\mathcal{S}(n) = (1-\gamma)\frac{n}{\log n} + O\Big(\frac{n}{\log^2 n}\Big). \tag{14}$$

Let $\varpi(k)$ to be 1 when k is prime and 0 otherwise. By using Abel summation we get

$$\mathcal{R}(n) = \sum_{k=2}^{n} \Big\{\frac{n}{k}\Big\} \varpi(k) \log k = \mathcal{S}(n) \log n - \mathcal{S}(2^-) \log 2 - \int_2^n \frac{\mathcal{F}_n(t)}{t}\, \mathrm{d}t,$$

where

$$\mathcal{F}_n(t) = \sum_{p \leqslant t} \Big\{\frac{n}{p}\Big\}.$$

Since $0 \leqslant \mathcal{F}_n(t) \leqslant \pi(t) \ll \frac{t}{\log t}$, by using the approximation (14) we deduce that

$$\mathcal{R}(n) = (1-\gamma)n + O\Big(\frac{n}{\log n}\Big) - \int_2^n O\Big(\frac{t}{\log t}\Big)\frac{\mathrm{d}t}{t} = (1-\gamma)n + O\Big(\frac{n}{\log n}\Big).$$

Thus, by substituting (13) and the last approximation in (12) we obtain the truncated approximation

$$\sum_{m=1}^{n} \log \kappa(m) = n \log n + (\gamma + E - 1)\, n + O\Big(\frac{n}{\log n}\Big),$$

implying that $c_0 = \gamma + E - 1$. Hence, given any positive integer r, there exist computable constants $c_1, \ldots, c_r$ such that

$$\sum_{m=1}^{n} \log \kappa(m) = n \log n + (\gamma + E - 1)\, n + \sum_{j=1}^{r} \frac{c_j n}{\log^j n} + O\Big(\frac{n}{\log^{r+1} n}\Big).$$

By using this approximation and the relations (6) and (7) we conclude the proof. ∎

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