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A Note On Periodic Solutions Of Matrix Riccati Differential Equations*

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Abstract

In this note, we show that under certain assumptions the matrix Riccati differential equation $X' = A(t)X + XB(t)X + C(t)$ with periodic coefficients admits at least one periodic solution. Also, we give an illustrative example in order to indicate the validity of the assumptions and the novelty of our result.

1 Introduction

Let us start this note by considering the second order differential equation of the form

$$y'' + p(t)y' + q(t)y = r(t),$$

where p, q and r are real functions on $\mathbb{R}$. This equation describes a large class of dynamical systems appearing throughout the field of engineering and applied mathematics. In the homogeneous case, by making the change of variable $x = -y'/y$, we are led to a first order differential equation of the form

$$x' = -p(t)x + x^2 + q(t).$$

This latter equation is a special case of a more general one, so-called *scalar Riccati differential equation*, namely

$$x' = a(t)x + b(t)x^2 + c(t),$$

where a, b and c are real functions on $\mathbb{R}$. A generalization of the scalar Riccati differential equation to the matrix case gives us *matrix Riccati differential equation*, namely

$$X' = A(t)X + XB(t)X + C(t), \tag{1}$$

where A, B and C are $(n \times n)$-real-matrix valued functions on $\mathbb{R}$.

Matrix Riccati differential equations are central objects of present-day control theory. In fact, in the theory of control systems, the qualitative control problem has received considerable research interests. This problem is regarded as an extension of the classical result of Kalman et al. [12] on controllability and stability of linear systems which is relevant to matrix Riccati differential equations (see [5, 17, 4, 3, 7, 8]). These equations also play predominant roles in other control theory problems such as dynamic games, linear systems with Markovian jumps, and stochastic control. The study of such differential equations, which also appear in a number of other areas such as biomathematics and multidimensional transport processes, is an

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interesting area of current research (see, for example, [11, 9]). There exists a rather extensive literature on matrix Riccati differential equations, mainly developed within the automatic control literature. We refer the reader to [3] for an extensive survey as well as to [5, 17, 4, 10] as fundamental papers on this area.

The analysis of periodic systems has long been a topic of interest. In this direction, an important question, which has been studied extensively by a number of authors (see, for example, [1, 13, 15, 19, 18, 14]), is whether scalar or matrix Riccati differential equations can support periodic solutions. For example, in theoretical aspects, knowledge of the periodic solutions is important for understanding the phase portrait of scalar or matrix Riccati differential equations and, in particular, the qualitative behavior of solutions (see, for example, [17]). On the other hand, on the applied side, in the problem of quadratic periodic optimization, arising for instance in the design of solar heating systems where the ambient temperature represents a periodic input, there occurs the need to compute the periodic solutions, if any, of a scalar or matrix Riccati differential equation with periodic coefficients. Another application is found in Kalman filtering of periodic systems such as orbiting satellites, seasonal phenomena like river flows, and econometric models, etc. We refer the reader to [2] for an overview on the structural properties of periodic systems, to [3] for the properties of periodic solutions to periodic scalar or matrix Riccati differential equations, and to [6] for the study of the periodic Lyapunov differential equations. Also, the book by Reid [16] covers many areas in scalar or matrix Riccati differential equations and is concerned with applications of these differential equations such as transmission line phenomena, theory of random processes, variational theory and optimal control theory, diffusion problems, and invariant imbedding.

In this note, we show that under certain assumptions the matrix Riccati differential equation (1) with periodic coefficients admits at least one periodic solution. Also, we give an illustrative example in order to indicate the validity of the assumptions and the novelty of our result.

2 Statement of the Main Result

In this section, we state the main result of this note, that is, Theorem 1, and we then treat an illustrative example in order to indicate the validity of the assumptions and the novelty of the result. We prove the main result in the next section.

In the sequel, ω is a positive real number, $\mathrm{M}_n(\mathbb{R})$ is the linear space of $n \times n$ matrices with real entries equipped with the operator norm $\| \ \|$, and $\mathbf{V} = \mathcal{C}(\mathbb{R}, \mathrm{M}_n(\mathbb{R}))$ is the linear space of continuous functions from $\mathbb{R}$ to $\mathrm{M}_n(\mathbb{R})$.

Theorem 1 *Let A, B and C be ω-periodic elements of $\mathbf{V}$ such that $I_n - M$ is nonsingular, where I_n is the $n \times n$ identity matrix and $M = \exp(\int_0^\omega A(\tau)d\tau)$. Set $M_1 = (I_n - M)^{-1}$ and $M_2 = MM_1$, and consider*

$$G(t,s) = \begin{cases} M_1 \exp\left(\int_s^t A(\tau)d\tau\right), & 0 \le s \le t \le \omega, \\ \\ M_2 \exp\left(\int_s^t A(\tau)d\tau\right), & 0 \le t < s \le \omega, \end{cases}$$

$$\mu = \sup_{0 \le t,s \le \omega} \|G(t,s)\|,$$

and

$$\nu = \sup_{0 \le t \le \omega} \left\| \int_0^\omega G(t,s)C(s)ds \right\|.$$

Suppose for all $t, s \in [0,\omega]$, $\int_0^t A(\tau)d\tau$ commutes with both of $\int_0^s A(\tau)d\tau$ and $A(t)$, and $A(t)$ commutes with M. If

$$\int_0^\omega \|B(\tau)\|d\tau \le \frac{1}{4\mu\nu},$$

then the matrix Riccati differential equation (1) admits at least one ω-periodic solution.

We now treat the following illustrative example, which shows the validity of the assumptions in Theorem 1. This example is generated by trial and error process using computer codes in Mathematica 5.2 with symbolic operations. Therefore, it seems to be quite far from a real practical problem. However, it shows the novelty of our result, since the previous results in the literature, in our knowledge, are inapplicable for proving the existence of a periodic solution of it.

Example 1 *Let A, B and C be 2π-periodic elements of $\mathbf{V} = \mathcal{C}(\mathbb{R}, \mathrm{M}_2(\mathbb{R}))$, which are defined as follows:*

$$A(t) = \frac{1}{8}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad B(t) = \frac{1}{8}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \quad \textit{and}$$

$$C(t) = \frac{1}{648}\begin{bmatrix} -18\cos t - \cos 3t - 81\sin t & 81\cos t + 18\sin t + \sin 3t \\ \\ 63\cos t - 18\sin t - \sin 3t & -18\cos t - \cos 3t - 63\sin t \end{bmatrix}.$$

We have $\mu \approx 1.83803$ and $\nu \approx 0.115645$, and so

$$\int_0^{2\pi} \|B(\tau)\| d\tau \approx 0.785398 < 1.176150 \approx \frac{1}{4\mu\nu}.$$

Therefore, Theorem 1 implies that the matrix Riccati differential equation (1) admits at least one 2π-periodic solution. This 2π-periodic solution may be given by

$$X(t) = \frac{1}{9}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

3 Proof of the Main Result

In this section, we prove Theorem 1 which will be done by proving a series of lemmas and recalling some definitions and known results. Let us start with the following lemma.

Lemma 1 *For all $t \in [0, \omega]$, the matrix $A(t)$ commutes with both of M_1 and M_2.*

Proof. By the assumption, for all $t \in [0, \omega]$, the matrix $A(t)$ commutes with M, and so it commutes with $I_n - M$. Therefore, $A(t)$ commutes with $(I_n - M)^{-1} = M_1$. By using this fact, for all $t \in [0, \omega]$, we may write

$$A(t)M_2 = A(t)(MM_1) = (MM_1)A(t) = M_2A(t).$$

This shows that $A(t)$ commutes with M_2 as well. ■

The following lemma shows that the function G in the statement of Theorem 1 is, in fact, the Green's function of the matrix Riccati differential equation (1).

Lemma 2 *Let $X \in \mathbf{V}$ be a solution of the integral equation*

$$X(t) = \int_0^{\omega} G(t, s)\Big(X(s)B(s)X(s) + C(s)\Big)ds.$$

Then X is a solution of the matrix Riccati differential equation (1).

Proof. By the assumption, for all $t, s \in [0, \omega]$, $\int_0^t A(\tau)d\tau$ commutes with $\int_0^s A(\tau)d\tau$, and so

$$\left(\int_0^t A(\tau)d\tau\right)\left(-\int_0^s A(\tau)d\tau\right) = \left(-\int_0^s A(\tau)d\tau\right)\left(\int_0^t A(\tau)d\tau\right).$$

This implies that for all $t, s \in [0, \omega]$,

$$\begin{aligned}
\exp\left(\int_s^t A(\tau)d\tau\right) &= \exp\left(\int_0^t A(\tau)d\tau - \int_0^s A(\tau)d\tau\right) \\
&= \exp\left(\int_0^t A(\tau)d\tau\right)\exp\left(-\int_0^s A(\tau)d\tau\right) \\
&= \exp\left(\int_0^t A(\tau)d\tau\right)\left(\exp\left(\int_0^s A(\tau)d\tau\right)\right)^{-1}.
\end{aligned}$$

Therefore, for all $t \in [0, \omega]$, we may write

$$\begin{aligned}
X(t) &= \int_0^\omega G(t,s)\left(X(s)B(s)X(s)+C(s)\right)ds \\
&= \int_0^t G(t,s)\left(X(s)B(s)X(s)+C(s)\right)ds + \int_t^\omega G(t,s)\left(X(s)B(s)X(s)+C(s)\right)ds \\
&= M_1\int_0^t \exp(\int_s^t A(\tau)d\tau)\left(X(s)B(s)X(s)+C(s)\right)ds \\
&\quad -M_2\int_\omega^t \exp(\int_s^t A(\tau)d\tau)\left(X(s)B(s)X(s)+C(s)\right)ds \\
&= M_1\exp(\int_0^t A(\tau)d\tau)\int_0^t\left(\exp(\int_0^s A(\tau)d\tau)\right)^{-1}\left(X(s)B(s)X(s)+C(s)\right)ds \\
&\quad -M_2\exp(\int_0^t A(\tau)d\tau)\int_\omega^t\left(\exp(\int_0^s A(\tau)d\tau)\right)^{-1}\left(X(s)B(s)X(s)+C(s)\right)ds.
\end{aligned}$$

Also, for all $t \in [0, \omega]$, $\int_0^t A(\tau)d\tau$ commutes with $A(t)$, and so

$$\left(\exp\left(\int_0^t A(\tau)d\tau\right)\right)' = A(t)\exp\left(\int_0^t A(\tau)d\tau\right).$$

Therefore, for all $t \in [0, \omega]$, we may write

$$\begin{aligned}
X'(t) &= M_1A(t)\exp(\int_0^t A(\tau)d\tau)\int_0^t\left(\exp(\int_0^s A(\tau)d\tau)\right)^{-1}\left(X(s)B(s)X(s)+C(s)\right)ds \\
&\quad +M_1\left(X(t)B(t)X(t)+C(t)\right) \\
&\quad -M_2A(t)\exp(\int_0^t A(\tau)d\tau)\int_\omega^t\left(\exp(\int_0^s A(\tau)d\tau)\right)^{-1}\left(X(s)B(s)X(s)+C(s)\right)ds \\
&\quad -M_2\left(X(t)B(t)X(t)+C(t)\right).
\end{aligned}$$

Now, Lemma 1 implies that for all $t \in [0, \omega]$,

$$\begin{aligned} X'(t) &= A(t)\int_0^t M_1 \exp(\int_s^t A(\tau)d\tau)\,(X(s)B(s)X(s) + C(s))\,ds \\ &\quad + A(t)\int_t^\omega M_2 \exp(\int_s^t A(\tau)d\tau)\,(X(s)B(s)X(s) + C(s))\,ds \\ &\quad + (M_1 - M_2)\,(X(t)B(t)X(t) + C(t)) \\ &= A(t)\int_0^t G(t,s)\,(X(s)B(s)X(s) + C(s))\,ds \\ &\quad + A(t)\int_t^\omega G(t,s)\,(X(s)B(s)X(s) + C(s))\,ds + X(t)B(t)X(t) + C(t) \\ &= A(t)\int_0^\omega G(t,s)\,(X(s)B(s)X(s) + C(s))\,ds + X(t)B(t)X(t) + C(t) \\ &= A(t)X(t) + X(t)B(t)X(t) + C(t), \end{aligned}$$

which shows that X is a solution of the matrix Riccati differential equation (1), as required. ■

Let us recall the definition of a Banach space. A linear space V over $\mathbb{R}$ together with a norm is called a *normed space*. Every normed space induces a metric and so has an associated topology. All the standard topological notion such as open sets, closed sets, bounded sets, convergence, etc. may be applied to V. Also, in a normed space V, a subset Ω of V is called *convex* if $\alpha x + (1-\alpha)y \in \Omega$ for all $x, y \in \Omega$ and for all α with $0 \le \alpha \le 1$. A normed space V is called a *Banach space* if it is complete, that is, if every Cauchy sequence in V is convergent.

We now continue the proof of Theorem 1. Suppose that

$$\mathbf{V}_\omega = \{X \in \mathbf{V} \mid X \text{ is } \omega\text{-periodic}\}$$

and for $X \in \mathbf{V}_\omega$ define

$$\|X\|_\omega = \sup_{0\le t\le\omega} \|X(t)\|,$$

where $\|X(t)\|$ is the operator norm of the matrix $X(t)$. It is easy to see that $\mathbf{V}_\omega$ together with the norm $\|\ \|_\omega$ is a Banach space. Suppose that the ω-periodic real function $\Phi_0 \in \mathbf{V}_\omega$ is defined on $[0, \omega]$ by

$$\Phi_0(t) = \int_0^\omega G(t,s)C(s)ds,$$

and consider

$$\mathbf{F}_\omega = \{\Phi \in \mathbf{V}_\omega \mid \|\Phi - \Phi_0\|_\omega \le \nu\}.$$

It is easy to see that $\mathbf{F}_\omega$ is a closed, bounded and convex subset of $\mathbf{V}_\omega$.

Now, define the operator $R : \mathbf{V}_\omega \longrightarrow \mathbf{V}_\omega$ by sending Φ to $R(\Phi)$, where $R(\Phi)$ is a ω-periodic real function which is defined on $[0, \omega]$ by

$$R(\Phi)(t) = \int_0^\omega G(t,s)\,(\Phi(s)B(s)\Phi(s) + C(s))\,ds.$$

Lemma 3 *The operator R maps $\mathbf{F}_{\boldsymbol\omega}$ into $\mathbf{F}_{\boldsymbol\omega}$.*

Proof. Let $\Phi \in \mathbf{F}_\omega$ be given. Therefore, $\|\Phi\|_\omega - \|\Phi_0\|_\omega \le \|\Phi - \Phi_0\|_\omega \le \nu$, and so the assumption implies that

$$\|\Phi\|_\omega \le \nu + \|\Phi_0\|_\omega = \nu + \sup_{0\le t\le\omega} \left\| \int_0^\omega G(t,s)C(s)ds \right\| = \nu + \nu = 2\nu.$$

Hence, for all $t \in [0, \omega]$, we have $\|\Phi(t)\| \leq 2\nu$. Thus, for all $t \in [0, \omega]$, by using the submultiplicative property of the operator norm, we may write

$$\begin{aligned}
\|R(\Phi)(t) - \Phi_0(t)\| &= \left\| \int_0^\omega G(t,s)\Phi(s)B(s)\Phi(s)ds \right\| \\
&\leq \int_0^\omega \|G(t,s)\Phi(s)B(s)\Phi(s)\|ds \\
&\leq \int_0^\omega \|G(t,s)\| \; \|\Phi(s)\|^2 \|B(s)\|ds \\
&\leq \mu(2\nu)^2 \int_0^\omega \|B(s)\|ds \\
&\leq 4\mu\nu^2 \left(\frac{1}{4\mu\nu} \right) \\
&= \nu.
\end{aligned}$$

Therefore, $\|R(\Phi) - \Phi_0\|_\omega \leq \nu$, and so $R(\Phi) \in \mathbf{F}_\omega$. This shows that R maps $\mathbf{F}_\omega$ into $\mathbf{F}_\omega$, as required. ■

In the sequel, we need the following version of Ascoli–Arzelà theorem. We recall that $\mathrm{M}_n(\mathbb{R})$ is the linear space of $n \times n$ matrices with real entries equipped with the operator norm $\| \ \|$. Also, a given sequence $(\Phi_k(t))_{k\in\mathbb{N}}$ of functions from $[a, b]$ to $\mathrm{M}_n(\mathbb{R})$ is called *equicontinuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $k \in \mathbb{N}$ and for all $t_1, t_2 \in [a, b]$, $|t_1 - t_2| < \delta$ implies that $\|\Phi_k(t_1) - \Phi_k(t_2)\| < \epsilon$.

Theorem 2 (Ascoli–Arzelà) *Let $(\Phi_k(t))_{k\in\mathbb{N}}$ be a sequence of functions from $[a, b]$ to $\mathrm{M}_n(\mathbb{R})$ which is uniformly bounded and equicontinuous. Then $(\Phi_k(t))_{k\in\mathbb{N}}$ has a uniformly convergent subsequence.*

The above version of Ascoli–Arzelà theorem can be proved straightforwardly using standard version of the theorem. We give here a sketch of proof for the convenience of the reader. If the conditions are satisfied with the operator norm, then they are satisfied for every entry. Then apply the standard version of the theorem on the first entry, extract a uniformly convergent subsequence. Then repeat this process on the second entry but with the indices of the first convergent subsequence and move a further subsequence, etc. Therefore, we will have to move from one subsequence to further subsequences n^2 times or so. In this position, in fact, we have a subsequence of the original sequence. This subsequence will converge uniformly for every entry and one may prove that it converges uniformly in the operator norm.

Let us also recall that for a given Banach space V, a continuous operator $S : V \longrightarrow V$ is called *compact* if it maps every bounded subset of V into a set with compact closure, that is, if every bounded sequence $(\Phi_k)_{k\in\mathbb{N}}$ on V has a subsequence $(\Phi_{k_i})_{i\in\mathbb{N}}$ such that $(S(\Phi_{k_i}))_{i\in\mathbb{N}}$ is convergent on V.

We continue the proof of Theorem 1 by proving the following lemma.

Lemma 4 *The operator R is compact.*

Proof. Let $(\Phi_k)_{k\in\mathbb{N}}$ be a bounded sequence on $\mathbf{V}_\omega$. In order to show the compactness of R, by the observation just before the statement of the lemma, it is enough to show that $(\Phi_k)_{k\in\mathbb{N}}$ has a subsequence, say $(\Phi_{k_i})_{i\in\mathbb{N}}$, such that $(R(\Phi_{k_i}))_{i\in\mathbb{N}}$ is convergent on $\mathbf{V}_\omega$. To do this, by the boundedness of $(\Phi_k)_{k\in\mathbb{N}}$, there exists $L > 0$ such that for all $k \in \mathbb{N}$, $\|\Phi_k\|_\omega \leq L$. Therefore, for all $k \in \mathbb{N}$ and for all $t \in [0, \omega]$, we have $\|\Phi_k(t)\| \leq L$. Note that, by the proof of Lemma 2, for all $k \in \mathbb{N}$, the function $R(\Phi_k)$ is, in fact, differentiable and for all $t \in [0, \omega]$, we have

$$R(\Phi_k)'(t) = A(t)\Phi_k(t) + \Phi_k(t)B(t)\Phi_k(t) + C(t).$$

This implies that for all $k \in \mathbb{N}$ and for all $t \in [0, \omega]$,

$$\|R(\Phi_k)'(t)\| \leq \|A\|_\omega L + \|B\|_\omega L^2 + \|C\|_\omega,$$

and so if for a given $\epsilon > 0$, we consider

$$\delta = \epsilon/(\|A\|_\omega L + \|B\|_\omega L^2 + \|C\|_\omega),$$

then for all $k \in \mathbb{N}$ and for all $t_1, t_2 \in [0, \omega]$, $|t_1 - t_2| < \delta$ implies that

$$\|R(\Phi_k)(t_1) - R(\Phi_k)(t_2)\| \le (\|A\|_\omega L + \|B\|_\omega L^2 + \|C\|_\omega)|t_1 - t_2| < \epsilon.$$

Thus, $(R(\Phi_k)(t))_{k\in\mathbb{N}}$ as a sequence of matrix functions on $[0, \omega]$ is equicontinuous and is also uniformly bounded. Now, Theorem 2 implies that there exists a subsequence of $(R(\Phi_k)(t))_{k\in\mathbb{N}}$, say $(R(\Phi_{k_i})(t))_{i\in\mathbb{N}}$, which is uniformly convergent on $[0, \omega]$. This means that $(R(\Phi_{k_i}))_{i\in\mathbb{N}}$ is convergent on $\mathbf{V}_\omega$, and so R is compact, as required. ■

Finally, the following fixed point theorem which is originally due to Schauder completes the proof of Theorem 1.

Theorem 3 (Schauder) *Let V be a Banach space and Ω be a closed, bounded and convex subset of V. If S maps Ω into Ω and it is a compact operator, then S has at least one fixed point on Ω.*

Now, Lemmas 3 and 4, together with Theorem 3, prove that there exists $\Phi \in \mathbf{F}_\omega$ such that $R(\Phi) = \Phi$. Thus, for all $t \in [0, \omega]$, we have

$$\Phi(t) = \int_0^\omega G(t,s)\Big(\Phi(s)B(s)\Phi(s) + C(s)\Big)ds.$$

Hence, the ω-periodic real function Φ is a solution of the integral equation

$$X(t) = \int_0^\omega G(t,s)\Big(X(s)B(s)X(s) + C(s)\Big)ds,$$

and so, by Lemma 2, it is a solution of the matrix Riccati differential equation (1). This completes the proof of Theorem 1.

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