

Some New Inertial Projection Methods For Quasi Variational Inequalities*

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Abstract

The purpose of the current study is to introduce some new inertial-type algorithms for finding the approximate solution of quasi-variational inequalities. We also estimate the convergence criteria of the proposed methods under a few appropriate conditions. As a special case, our results include the results of Shehu et al. [17] for quasi variational inequalities and Noor et al. [14] for solving variational inequalities as special cases. The idea of the current study may stimulate prospective research.

1 Introduction

Variational inequalities approach, as a very efficient and important origin of the modern mathematical technology, has been extensively applied to physics, mechanics, economics and transportation equilibrium, optimization and control, and engineering sciences, and so forth. Variational inequality has been extended in various directions. A significant and useful generalization of variational inequality is quasi-variational inequality. If the convex set, which was involved in the variational inequalities, also depends on the solution explicitly or implicitly, then this inequality is called the quasi-variational inequality, which was studied and introduced by Bensoussan et al. [4]. See also [5, 9] and the references therein.

Antipin et al. [3] proposed gradient projection and extra gradient methods for finding the solution of quasi-variational inequality, when the involved operator is strongly monotone and Lipschitz continuous. Mijajlovic et al. [7] introduced a more general gradient projection method with strong convergence for solving this inequality in real Hilbert space. This technique works well for various useful purposes, so it has immense potential. It is essential to consider an iterative scheme with a fast accelerated rate of convergence. Recently, the inertial method is obtained from the oscillator equation with damping and conservative restoring force. It has become an indispensable source for refining the performance of the method and has nice convergence characteristics. The general foremost aspect of inertial-type alternatives is that we use previous iterations to construct the next. Since constructing inertial methods, many authors have combined the inertial term $\{\Theta_n(\mu_n - \mu_{n-1})\}$ into various kinds of algorithms, such as Halpern, Kranselski, Mann, Viscosity, etc for finding the solution optimization problems and fixed point problems. Here Θ_n is an extrapolating factor that stimulates the convergence rate of the method. Polyak [15] was the first author to suggest the heavy ball method. Alvarez et al. [1] used it to set up a proximal point algorithm. Recently Shehu et al. [17] suggested and investigated an inertial projection methods for solving quasi variational inequalities, which include the modified double projection method Noor [11] for solving classical variational inequalities.

Noor et al. [14] have suggested and analyzed a wide class of projection type methods for solving variational inequalities. Motivated by the work of Noor et al. [14] and Shehu et al. [17], we suggest some new inertial projection methods for solving quasi variational inequalities. As applications of our results, we obtain several new and known algorithms as special cases. We analyze the convergence criteria of the proposed techniques with appropriate conditions. The ideas and techniques of this paper may be starting point for further research.

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2 Preliminaries

Let \mathcal{K} be a nonempty, closed and convex set in a real Hilbert space \mathcal{H} with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let a nonlinear operator $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$ in \mathcal{H} . Let $\mathcal{K} : \mathcal{H} \rightarrow \mathcal{H}$ be a set-valued mapping which, for any element $x \in \mathcal{H}$, associates a convex-valued and closed set $\mathcal{K}(x) \subset \mathcal{H}$.

We consider the *quasi-variational inequality* [4], which consists of finding $x \in \mathcal{K}(x)$, such that

$$\langle \mathcal{T}x, v - x \rangle \geq 0, \quad \forall v \in \mathcal{K}(x), \tag{1}$$

where $\rho > 0$ is a constant.

Clearly, if we take $\mathcal{K}(x) = \mathcal{K}$, then the above problem reduces to the *variational-inequality* [16], that is, finding $x \in \mathcal{K}$, such that

$$\langle \mathcal{T}x, v - x \rangle \geq 0, \quad \forall v \in \mathcal{K}. \tag{2}$$

Definition 1 A mapping $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$ is called *strongly monotone* ($\xi \geq 0$), if

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq \xi \|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}. \tag{3}$$

Definition 2 A mapping $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$ is called *Lipschitz continuous* ($\eta > 0$), if

$$\|\mathcal{T}\mu - \mathcal{T}\nu\| \leq \eta \|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}. \tag{4}$$

From (3) and (4), it is noted that $\xi \leq \eta$. The following projection lemma plays an indispensable role in obtaining our results.

Lemma 1 ([4]) For a given $\omega \in \mathcal{H}$, $\mu \in \mathcal{K}(\mu)$ such that

$$\langle \mu - \omega, v - \mu \rangle \geq 0, \quad \forall v \in \mathcal{K}(\mu),$$

if and only if

$$\mu = \Pi_{\mathcal{K}(\mu)}[\omega],$$

where $\Pi_{\mathcal{K}(\mu)}$ is the implicit projection of \mathcal{H} onto the closed convex-valued set $\mathcal{K}(\mu)$ in \mathcal{H} .

The projection operator $\Pi_{\mathcal{K}(\mu)}$ satisfies the following assumption. The implicit projection operator $\mathcal{P}_{\mathcal{K}(\mu)}$, satisfies the condition

$$\|\Pi_{\mathcal{K}(\mu)}[\omega] - \Pi_{\mathcal{K}(\nu)}[\omega]\| \leq v \|\mu - \nu\| \quad \forall \mu, \nu, \omega \in \mathcal{H}, \tag{5}$$

where $v > 0$, is a constant. The next result is very helpful for analyzing our methods.

Lemma 2 ([18]) Consider a sequence of non negative real numbers $\{\varrho_n\}$, satisfying

$$\varrho_{n+1} \leq (1 - \Upsilon_n)\varrho_n + \Upsilon_n \sigma_n + \varsigma_n, \quad \forall n \geq 1,$$

where

(i) $\{\Upsilon_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \Upsilon_n = \infty$;

(ii) $\limsup \sigma_n \leq 0$;

(iii) $\varsigma_n \geq 0$ ($n \geq 1$), $\sum_{n=1}^{\infty} \varsigma_n < \infty$.

Then, $\varrho_n \rightarrow 0$ as $n \rightarrow \infty$.

3 Iterative Methods

Now, we use the fixed point formulation to suggest and investigate a new implicit algorithm for solving inequality (1).

Using Lemma 1, the inequality (1) can be written as the following equivalent form.

Lemma 3 ([12]) *Let $\mathcal{K}(x)$ be a closed and convex-valued set in \mathcal{H} . Then $x \in \mathcal{K}(x)$ is the solution of problem (1), if and only if,*

$$x = \Pi_{\mathcal{K}(x)} [x - \rho \mathcal{T} x], \quad (6)$$

where $\Pi_{\mathcal{K}(x)}$ is the implicit projection of \mathcal{H} onto the closed and convex-valued set $\mathcal{K}(x)$ in \mathcal{H} and $\rho > 0$ is a constant.

From (6), we have

$$x = (1 - \alpha_n)x + \alpha_n \Pi_{\mathcal{K}(x)} [(1 - \lambda)x + \lambda x - \rho \{(1 - \mu)\mathcal{T} x + \mu \mathcal{T} x\}],$$

where $\lambda, \mu \in [0, 1]$, and $\alpha_n \in [0, 1]$ for all $n \geq 0$.

This equivalent formulation can be used to suggest the following iteration method for solving inequality (1) as:

Algorithm 1 *For a given x_0 , compute x_{n+1} by the recurrence relation*

$$x_{n+1} = (1 - \alpha_n)x_{n+1} + \Pi_{\mathcal{K}(x_{n+1})} [(1 - \lambda)x_n + \lambda x_{n+1} - \rho \{(1 - \mu)\mathcal{T} x_n + \mu \mathcal{T} x_{n+1}\}], \quad n = 0, 1, 2, \dots$$

where $0 \leq \alpha_n \leq 1$ and $\lambda, \mu \in [0, 1]$.

In order to implement Algorithm 1, we can use the following inertial-type predictor and corrector technique.

Algorithm 2 *For a given x_0, x_1 , compute x_{n+1} by the iterative schemes*

$$y_n = x_n + \theta_n (x_n - x_{n-1}), \quad (7)$$

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n \pi_{\|(y_n)} [(1 - \lambda)x_n + \lambda y_n - \rho \{(1 - \mu)\sqcup x_n + \mu \sqcup y_n\}] \quad (8)$$

for $n = 1, 2, \dots$ where $0 \leq \Theta_n, \alpha_n \leq 1$ and $\lambda, \mu \in [0, 1]$.

For $\alpha_n = 0$, Algorithm 2 reduces to the following algorithm.

Algorithm 3 *For given x_0, x_1 , compute x_{n+1} by the recurrence relation*

$$y_n = x_n + \Theta_n (x_n - x_{n-1}),$$

$$x_{n+1} = \Pi_{\mathcal{K}(y_n)} [(1 - \lambda)x_n + \lambda y_n - \rho \{(1 - \mu)\mathcal{T} x_n + \mu \mathcal{T} y_n\}],$$

for $n = 1, 2, \dots$ where $0 \leq \Theta_n \leq 1$ and $\lambda, \mu \in [0, 1]$.

For $\lambda = 0 = \mu$, Algorithm 2 reduces to the following algorithm to solve inequality (1).

Algorithm 4 *For given x_0, x_1 , compute x_{n+1} by the recurrence relation*

$$y_n = x_n + \Theta_n (x_n - x_{n-1}),$$

$$\mu_{n+1} = (1 - \alpha_n)y_n + \alpha_n \Pi_{\mathcal{K}(y_n)} [x_n - \rho \mathcal{T} x_n],$$

for $n = 1, 2, \dots$ where $0 \leq \Theta_n, \alpha_n \leq 1$.

For $\lambda = 1 = \mu$, Algorithm 2 reduces to the following algorithm.

Algorithm 5 ([17]) For given x_0, x_1 , compute x_{n+1} by the recurrence relation

$$y_n = x_n + \Theta_n (x_n - x_{n-1}),$$

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n \Pi_{\mathcal{K}(y_n)} [y_n - \rho \mathcal{T} y_n],$$

for $n = 1, 2, \dots$ where $0 \leq \Theta_n, \alpha_n \leq 1$

For $\lambda = \frac{1}{2} = \mu$, Algorithm 2 reduces to the following algorithm.

Algorithm 6 For given x_0, x_1 , compute x_{n+1} by the recurrence relation

$$y_n = x_n + \Theta_n (x_n - x_{n-1}),$$

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n \Pi_{\mathcal{K}(y_n)} \left[\frac{x_n + y_n}{2} - \rho \frac{\mathcal{T}x_n + \mathcal{T}y_n}{2} \right],$$

for $n = 1, 2, \dots$ where $0 \leq \Theta_n, \alpha_n \leq 1$. This method appears to be a new one.

If \mathcal{T} is a linear operator, then Algorithm 6 reduces to the following algorithm.

Algorithm 7 For a given x_0, x_1 , compute x_{n+1} by the following recurrence relation

$$y_n = x_n + \Theta_n (x_n - x_{n-1}),$$

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n \Pi_{\mathcal{K}(y_n)} \left[\frac{x_n + y_n}{2} - \rho \frac{\mathcal{T}(x_n + y_n)}{2} \right],$$

for $n = 1, 2, \dots$ where $0 \leq \Theta_n, \alpha_n \leq 1$.

Advantage of these methods is that only one projection operator is used.

By choosing adequate and suitable values for parameter λ spaces and operators, we can obtain several iteration variants from Algorithm 2. This shows that Algorithm 2 is quite general and contains the previous techniques as special cases.

4 Convergence Analysis

In this section, we analyze the convergence criteria for Algorithm 2 under some appropriate conditions.

Theorem 1 Let the mapping $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$ be a strongly monotone and Lipschitz continuous operator with constants $\xi > 0, \eta > 0$, respectively. Suppose that

1. Assumption P holds.
2. A constant $\rho > 0$ satisfies the following conditions

$$\left| \rho - \frac{\xi}{\eta^2} \right| < \frac{\sqrt{\xi^2 \mu^2 + \eta^2 (1 - \mu^2 - k_1(2 - k_1))}}{\eta^2 \mu}, \quad 1 - \mu^2 > k_1(2 - k_1), \quad k_1 < 1, \quad 0 < \mu \leq 1, \quad (9)$$

$$\left| \rho - \frac{\xi}{\eta^2} \right| < \frac{\sqrt{\xi^2 - \eta^2 k_2(2 - k_2)}}{\eta^2}, \quad \xi > \eta \sqrt{k_2(2 - k_2)}, \quad k_2 < 1, \quad (10)$$

where $k_1 = \lambda - \mu + \nu, k_2 = 2(\lambda - \mu) + \nu$, and $\mu \leq \lambda$.

3. Let $\Theta_n, \alpha_n \in [0, 1]$, for all $n \geq 1$ such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} \Theta_n \|x_n - x_{n-1}\| < \infty.$$

Then, the approximate x_{n+1} , obtained by the iterative scheme defined in Algorithm 2 converges to unique solution $x \in \mathcal{K}(x)$ of problem (1).

Proof. Let $x \in \mathcal{K}(x)$ be a solution of (1). Then

$$\begin{aligned}
\|x_{n+1} - x\| &= \|(1 - \alpha_n)y_n + \alpha_n \Pi_{\mathcal{K}(y_n)} [(1 - \lambda)x_n + \lambda y_n - \rho \{(1 - \mu)\mathcal{T}x_n + \mu\mathcal{T}y_n\}] \\
&\quad - (1 - \alpha_n)x - \alpha_n \Pi_{\mathcal{K}(x)} [(1 - \lambda)x + \lambda x - \rho \{(1 - \mu)\mathcal{T}x + \mu\mathcal{T}x\}]\| \\
&\leq (1 - \alpha_n)\|y_n - x\| + \alpha_n \|\Pi_{\mathcal{K}(y_n)} [(1 - \lambda)x_n + \lambda y_n - \rho \{(1 - \mu)\mathcal{T}x_n + \mu\mathcal{T}y_n\}] \\
&\quad - \Pi_{\mathcal{K}(x)} [(1 - \lambda)x + \lambda x - \rho \{(1 - \mu)\mathcal{T}x + \mu\mathcal{T}x\}]\| \\
&\leq (1 - \alpha_n)\|y_n - x\| + \alpha_n \|\Pi_{\mathcal{K}(y_n)} [(1 - \lambda)x_n + \lambda y_n - \rho \{(1 - \mu)\mathcal{T}x_n + \mu\mathcal{T}y_n\}] \\
&\quad - \Pi_{\mathcal{K}(y_n)} [(1 - \lambda)x + \lambda x - \rho \{(1 - \mu)\mathcal{T}x + \mu\mathcal{T}x\}]\| \\
&\quad + \alpha_n \|\Pi_{\mathcal{K}(y_n)} [(1 - \lambda)x + \lambda x - \rho \{(1 - \mu)\mathcal{T}x + \mu\mathcal{T}x\}] \\
&\quad - \Pi_{\mathcal{K}(x)} [(1 - \lambda)x + \lambda x - \rho \{(1 - \mu)\mathcal{T}x + \mu\mathcal{T}x\}]\| \\
&\leq (1 - \alpha_n)\|y_n - x\| + \alpha_n \left\{ \|(1 - \lambda)(x_n - x) + \lambda(y_n - x) - (1 - \mu)\rho(\mathcal{T}x_n - \mathcal{T}x) \right. \\
&\quad \left. - \mu\rho(\mathcal{T}y_n - \mathcal{T}x)\| + \nu\|y_n - x\| \right\} \\
&= (1 - \alpha_n)\|y_n - x\| + \alpha_n \left\{ \|(1 - \lambda)(x_n - x) + (\lambda - \mu)(y_n - x) \right. \\
&\quad \left. + (1 - \mu)(x_n - x - \rho(\mathcal{T}x_n - \mathcal{T}x)) + \mu(y_n - x + \rho(\mathcal{T}y_n - \mathcal{T}x))\| + \nu\|y_n - x\| \right\} \\
&\leq (1 - \alpha_n)\|y_n - x\| + \alpha_n \left\{ (\lambda - \mu)\|x_n - x\| + (\lambda - \mu)\|y_n - x\| \right. \\
&\quad \left. + (1 - \mu)\|x_n - x - \rho(\mathcal{T}x_n - \mathcal{T}x)\| + \mu\|y_n - x + \rho(\mathcal{T}y_n - \mathcal{T}x)\| \right. \\
&\quad \left. + \nu\|y_n - x\| \right\}, \tag{11}
\end{aligned}$$

where we have used the Assumption P.

From the strong monotonicity and Lipschitz continuity of operator \mathcal{T} , we have

$$\begin{aligned}
&\|x_n - x - \rho[\mathcal{T}x_n - \mathcal{T}x]\|^2 \\
&= \|x_n - x\|^2 - 2\rho \langle \mathcal{T}x_n - \mathcal{T}x, x_n - x \rangle + \rho^2 \|\mathcal{T}x_n - \mathcal{T}x\|^2 \\
&\leq (1 - 2\rho\xi + \rho^2\eta^2) \|x_n - x\|^2. \tag{12}
\end{aligned}$$

In a similar way, we have

$$\|y_n - x + \rho(\mathcal{T}y_n - \mathcal{T}x)\| \leq (1 - 2\rho\xi + \rho^2\eta^2) \|y_n - x\|^2.$$

From (7), we have

$$\begin{aligned}
\|y_n - x\| &= \|x_n - x + \Theta_n(x_n - x_{n-1})\| \\
&\leq \|x_n - x\| + \Theta_n \|x_n - x_{n-1}\|. \tag{13}
\end{aligned}$$

From condition (9), (11), (12) and (13), we have

$$\begin{aligned}
\|x_{n+1} - x\| &\leq [1 - \alpha_n(1 - \vartheta_1)] [\|x_n - x\| + \Theta_n \|x_n - x_{n-1}\|] + \alpha_n \vartheta_2 \|x_n - x\| \\
&\leq [1 - \alpha_n(1 - \vartheta_1)] \|x_n - x\| + \Theta_n \|x_n - x_{n-1}\| + \alpha_n \vartheta_2 \|x_n - x\| \\
&= [1 - \alpha_n \{1 - (\vartheta_1 + \vartheta_2)\}] \|x_n - x\| + \Theta_n \|x_n - x_{n-1}\|,
\end{aligned}$$

where

$$\vartheta_1 := \lambda - \mu + \mu \sqrt{1 - 2\rho\xi + \rho^2\eta^2} + \nu,$$

$$\begin{aligned} \vartheta_2 &:= \lambda - \mu + (1 - \mu) \sqrt{1 - 2\rho\xi + \rho^2\eta^2}, \\ \vartheta_1 + \vartheta_1 &:= 2(\lambda - \mu) + \sqrt{1 - 2\rho\xi + \rho^2\eta^2} + \nu. \end{aligned}$$

Let $\mathbf{w} = \vartheta_1 + \vartheta_1$. Then, from condition (10), we have $\mathbf{w} \in (0, 1)$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, setting $\sigma_n = 0$ and $\varsigma = \sum_{n=1}^{\infty} \Theta_n \|x_n - x_{n-1}\| < \infty$, using Lemma 2, we have $x_n \rightarrow x, n \rightarrow \infty$. Hence the sequence $\{x_n\}$ obtained from Algorithm 2 converges to a unique solution $x \in \mathcal{K}(x)$ satisfying the inequality (1), the desired result. ■

Similar convergence analysis for other iterative schemes can be estimated.

For $\mathcal{K}(x) = \mathcal{K}$, following result can be obtained from Theorem 1.

Theorem 2 *Let a mapping $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$ be strongly monotone and Lipschitz continuous operator with constants $\xi > 0, \eta > 0$ respectively. We assume that the constant $\rho > 0$ satisfies the following conditions*

$$\begin{aligned} \left| \rho - \frac{\xi}{\eta^2} \right| &< \frac{\sqrt{\xi^2\mu^2 + \eta^2(1 - \mu^2 - k_1(2 - k_1))}}{\eta^2\mu}, \quad 1 - \mu^2 > k_1(2 - k_1), \quad k_1 < 1, \quad 0 < \mu \leq 1, \\ \left| \rho - \frac{\xi}{\eta^2} \right| &< \frac{\sqrt{\xi^2 - \eta^2 k_2(2 - k_2)}}{\eta^2}, \quad \xi > \eta\sqrt{k_2(2 - k_2)}, \quad k_2 < 1, \end{aligned}$$

where $k_1 = \lambda - \mu, k_2 = 2(\lambda - \mu)$ and $\mu \leq \lambda$. Then x_{n+1} , obtained from

$$y_n = x_n + \Theta_n(x_n - x_{n-1}),$$

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n \Pi_{\mathcal{K}}[(1 - \lambda)x_n + \lambda y_n - \rho\{(1 - \mu)\mathcal{T}x_n + \mu\mathcal{T}y_n\}],$$

for $n = 1, 2, \dots$, converges to a unique solution $x \in \mathcal{K}$ satisfying the quasi variational inequalities (2).

Open Problem. For $\lambda = 0, \mu = 1$, Algorithm 1 reduces to the extragradient method of Korpelevich method [6] for quasi-variational inequalities, which is still an interesting problem for finding the solution of quasi variational inequalities.

Conclusion. In the paper, we have proposed and investigate some new inertial methods to solve quasi variational inequalities using the equivalence relation between quasi-variational inequality and fixed point problem. These proposed projection methods include several new and known inertial methods for solving variational, quasi-variational inequalities and related optimization problems as special cases. We have studied the convergence criteria of the proposed method. It is an interesting problem to implement these methods and compare their efficiency with other methods. Results obtained from this study may encourage future research in this area.

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