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A Generalized Refinement of Young Inequality And Applications*

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Abstract

In this paper, we prove that if $a, b > 0$ and $0 \leq \nu \leq 1$, then for all positive intger m, we have

$$\begin{aligned} & (a^\nu b^{1-\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 + r_m\Big(((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2\chi_{(0,\frac{1}{2}]}(\nu) + ((ab)^{\frac{m}{4}} - a^{\frac{m}{2}})^2\chi_{(\frac{1}{2},1]}(\nu)\Big) \\ \leq \quad & (\nu a + (1-\nu)b)^m, \end{aligned} \tag{A-I}$$

where $r_0 = \min\{\nu, 1-\nu\}$, $r_m = \min\{2r_0^m, (1-r_0)^m - r_0^m\}$ and $\chi_I(\nu)$ is the characteristic function of the set I. The inequality (A-I) provides a generalization of an important refinement of the Young inequality obtained in 2015 by Hirzallah and Kittaneh. The inequality (A-I) extends also another important refinement of the Young inequality obtained in 2015 by J. Zhao and J. Wu which corresponds to the case $m = 1$.

As applications of the inequality (A-I), we give some refined Young type inequalities for the traces, determinants, norms of positive definite matrices, and some inequalities concerning the generalized Euclidean operator radius.

1 Introduction

The well-known Young's inequality for scalars asserts that for all positive real numbers a, b and $0 \leq \nu \leq 1$, we have

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b. \tag{1}$$

The inequality (1) implies that for $m = 1, 2, 3, ...$,

$$(a^\nu b^{1-\nu})^m \leq (\nu a + (1-\nu)b)^m. \tag{2}$$

Hirzallah and Kittaneh [3] refined Young's inequality (1) to

$$(a^\nu b^{1-\nu})^2 + r_0^2(a-b)^2 \leq (\nu a + (1-\nu)b)^2, \tag{3}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Kittaneh and Manasrah [7] refined Young's inequality so that

$$a^\nu b^{1-\nu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \nu a + (1-\nu)b, \tag{4}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

In 2015, Manasrah and Kittaneh [8] refind the inequality (2) by adding the quantity $r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2$ to its first part, and obtained the following inequality:

$$(a^\nu b^{1-\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\nu a + (1-\nu)b)^m, \tag{5}$$

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where $m = 1, 2, 3, \ldots$.

The inequality (5) is a common generalization of the inequalities (3) and (4).

In 2015, J. Zhao and J. Wu refined inequality (2) as follows:

if $0 < \nu \le \frac{1}{2}$, then

$$a^{\nu}b^{1-\nu} + \nu(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{b})^2 \le \nu a + (1-\nu)b, \tag{6}$$

and if $\frac{1}{2} < \nu \le 1$, then

$$a^{\nu}b^{1-\nu} + (1-\nu)(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{a})^2 \le \nu a + (1-\nu)b. \tag{7}$$

where $r_0 = \min\{\nu, 1-\nu\}$, and $r_1 = \min\{2r_0, 1-2r_0\}$.

We can gather the above inequalities (6) and (7) in the following form:

$$\begin{aligned} & a^{\nu}b^{1-\nu} + r_0(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 + r_1\Big(((ab)^{\frac{1}{4}} - b^{\frac{1}{2}})^2\chi_{(0,\frac{1}{2}]}(\nu) + ((ab)^{\frac{1}{4}} - a^{\frac{1}{2}})^2\chi_{(\frac{1}{2},1]}(\nu)\Big) \\ \le\; & \nu a + (1-\nu)b, \end{aligned}$$

valid for all positive numbers a, b and for all $\nu \in [0,1]$, where for any set I, we denote χ_I its characteristic function defined by

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{if } x \notin I. \end{cases}$$

One of the aims of this paper is to extend the inequality above to the following one:

$$\begin{aligned} & (a^{\nu}b^{1-\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 + r_m\Big(((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2\chi_{(0,\frac{1}{2}]}(\nu) + ((ab)^{\frac{m}{4}} - a^{\frac{m}{2}})^2\chi_{(\frac{1}{2},1]}(\nu)\Big) \\ \le\; & (\nu a + (1-\nu)b)^m, \end{aligned} \tag{A-I}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $r_m = \min\{2r_0^m, (1-r_0)^m - r_0^m\}$, which will be valid for all positive integer m, for all positive numbers a, b and all $\nu \in [0,1]$.

This inequality extends all the previous refinements of Young's inequality, this means the inequalities: (3), (4), (5), (6) and (7).

The inequality (A-I) is proved in the second section.

In sections two and three of this paper, we give applications of the inequality (A-I) to establish new refinements to certain Young type inequalities for the traces, determinants, norms of positive definite matrices, and some inequalities concerning the generalized Euclidean operator radius.

2 A Generalized Refinement of Young's Inequality

For the proof of our first main result, we need to recall the following theorem concerning the celebrated weighted arithmetic-geometric mean (AM-GM) inequality.

Theorem 1 *Let n be a positive integer. For $i = 1, 2, \ldots, n$, let $a_i \ge 0$, and let $\nu_i \ge 0$ satisfy $\sum_{i=1}^{n}\nu_i = 1$. Then, we have*

$$\prod_{i=1}^{n} a_i^{\nu_i} \le \sum_{i=1}^{n} \nu_i a_i.$$

By taking $n := 2$ in the weighted AM-GM inequality, we recapture the classical Young's inequality.

We need also the following lemma.

Lemma 2 *Let m be a positive integer and let ν a real number, such that $0 \leq \nu \leq 1$. Then we have*

$$\sum_{k=1}^{m} \binom{m}{k} k\nu^k (1-\nu)^{m-k} = m\nu,$$

and

$$\sum_{k=0}^{m-1} \binom{m}{k} (m-k)\nu^k (1-\nu)^{m-k} = m(1-\nu).$$

where $\binom{m}{k}$ is the binomial coefficient.

For a proof of Lemma 2, one can see [1].

The first main result of this paper reads as follows.

Theorem 3 *Let a and b be two positive numbers and $0 \leq \nu \leq 1$. Then for all positive intger m, we have the inequality (A-I).*

Proof. Suppose that $0 < \nu \leq \frac{1}{2}$. We claim that

$$(\nu a + (1-\nu)b)^m - \nu^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 - r_m((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2 \geq (a^\nu b^{1-\nu})^m.$$

Indeed, we have, the following identities:

$$\begin{aligned}
& (\nu a + (1-\nu)b)^m - \nu^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 - r_m((ab)^{\frac{m}{4}} - b^{\frac{m}{2}})^2 \\
= & \sum_{k=0}^{m} \binom{m}{k} \nu^k (1-\nu)^{m-k} a^k b^{m-k} - \nu^m \Big(a^m + b^m - 2(ab)^{\frac{m}{2}}\Big) - r_m \Big((ab)^{\frac{m}{2}} + b^m - 2(ab)^{\frac{m}{4}} b^{\frac{m}{2}}\Big) \\
= & \Big((1-\nu)^m - \nu^m - r_m\Big) b^m + \Big(2\nu^m - r_m\Big)(ab)^{\frac{m}{2}} + \sum_{k=1}^{m-1} \binom{m}{k} \nu^k (1-\nu)^{m-k} a^k b^{m-k} + 2r_m (ab)^{\frac{m}{4}} b^{\frac{m}{2}} \\
= & \sum_{k=0}^{m+1} \nu_k x_k
\end{aligned}$$

where x_i is given by

$$x_0 := b^m \text{ with } \nu_0 := \Big((1-\nu)^m - \nu^m - r_m\Big),$$

and for $1 \leq k \leq m-1$,

$$x_k := a^k b^{m-k} \text{ with } \nu_k := \binom{m}{k} \nu^k (1-\nu)^{m-k}$$

and

$$x_m := (ab)^{\frac{m}{2}} \text{ with } \nu_m := \Big(2\nu^m - r_m\Big),$$

and

$$x_{m+1} := (ab)^{\frac{m}{4}} b^{\frac{m}{2}} \text{ with } \nu_m := 2r_m.$$

We have

1. $x_k > 0$ for all $k \in \{0, 1, ..., m+1\}$,

2. $\nu_k \geq 0$ for all $k \in \{0, 1, ..., m+1\}$, with $\sum_{k=0}^{m+1} \nu_i = 1$.

Hence by applying Theorem 1, we get

$$(\nu a+(1-\nu)b)^m-\nu^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2-r_m((ab)^{\frac{m}{4}}-b^{\frac{m}{2}})^2\geq\prod_{k=0}^{m+1}x_i^{\nu_i}=a^{\alpha(m)}b^{\beta(m)},$$

where

$$\begin{aligned}\alpha(m) &= \sum_{k=1}^{m-1}\binom{m}{k}k\nu^k(1-\nu)^{m-k}+\frac{m}{2}\Big(2\nu^m-r_m\Big)+\frac{m}{2}r_m\\ &= \sum_{k=1}^{m}\binom{m}{k}k\nu^k(1-\nu)^{m-k}=m\nu,\quad\text{(by Lemma 2)}\end{aligned}$$

and

$$\begin{aligned}\beta(m) &= \frac{m}{2}\Big(2\nu^m-r_m\Big)+\frac{3m}{2}r_m\\ &+ \sum_{k=1}^{m-1}\binom{m}{k}(m-k)\nu^k(1-\nu)^{m-k}+m\Big((1-\nu)^m-\nu^m-r_m\Big)\\ &= \sum_{k=0}^{m-1}\binom{m}{k}(m-k)\nu^k(1-\nu)^{m-k}=m(1-\nu)\ \ \text{(by Lemma 2).}\end{aligned}$$

The case $\frac{1}{2}<\nu\leq 1$ is established by similar arguments.

This completes the proof. ■

Remark 1 *In Theorem 3 above, we have refined the inequality (5) by adding to its first part the quantity*

$$r_0^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2+r_m\Big(((ab)^{\frac{m}{4}}-b^{\frac{m}{2}})^2\chi_{(0,\frac{1}{2}]}(\nu)+((ab)^{\frac{m}{4}}-a^{\frac{m}{2}})^2\chi_{(\frac{1}{2},1]}(\nu)\Big)$$

where $r_0=\min\{\nu,1-\nu\}$ and $r_m=\min\{2r_0^m,(1-r_0)^m-r_0^m\}$. So this is a considerable generalization of the refinements of the Young inequality due to Manasrah and Kittaneh [8].

Theorem 3 also extends the result obtained by J. Zhao and J. Wu in [13] which correspondd to the particular case $m=1$.

3 Applications to Young Type Inequalities for the Traces, Determinants, and Norms of Positive Definite Matrices

In this section, we apply Theorem 3 to provide some improvements to certain refined Young type inequalities for the traces, determinants, and norms of positive definite matrices obtained by F. Kittaneh and Y. Manasrah in [8].

Let $\mathbf{M}_n(\mathbb{C})$ designate the space of all $n\times n$ complex matrices. A matrix $A\in\mathbf{M}_n(\mathbb{C})$ is called positive semidefinite, (denoted as $A\geq 0$) if $x^*Ax\geq 0$ for all $x\in\mathbb{C}^n$, and it is called positive definite (denoted as $A>0$) if $x^*Ax>0$ for all nonzero $x\in\mathbb{C}^n$. The singular values of a matrix $A\in\mathbf{M}_n(\mathbb{C})$ are the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{1/2}$, denoted by $s_i(A)$ for $i=1,2,3..,n$.

A norm $|||.|||$ on $\mathbf{M}_n(\mathbb{C})$ is called unitarily invariant if $|||UAV|||=|||A|||$ for all $A\in\mathbf{M}_n(\mathbb{C})$ and all unitary matrices $U,V\in\mathbf{M}_n(\mathbb{C})$.

The trace norm $||.||$ is given by $||A||_1=tr|A|=\sum_{k=1}^n s_k(A)$, where tr is the usual trace for matrices. It is well known that this norm is unitarily invariant.

A matrix version of Young inequality, proved in [2], asserts that

$$s_j(A^\nu B^{1-\nu}) \le s_j(\nu A + (1-\nu)B) \ \text{ for } j = 1, \dots, n. \tag{8}$$

The above singular value inequality entails the following unitarily invariant norm inequality

$$||A^\nu B^{1-\nu}||_1 \le ||\nu A + (1-\nu)B||_1.$$

A determinant version of Young's inequalities is also known [[4]; p. 467], for positive semidefinite matrices A, B and $0 \le \nu \le 1$,

$$\det(A^\nu B^{1-\nu}) \le det(\nu A + (1-\nu)B). \tag{9}$$

The following inequality, proved in [9], asserts that for all positive semidefinite matrices (and unitarily invariant norms), we have:

$$|||A^\nu X B^{1-\nu}||| \le \nu|||A||| + (1-\nu)|||B|||.$$

By using the inequality (5), Manasrah and Kittaneh [8] gave refinements to the following inequalities:

$$\Big(tr|A^\nu B^{1-\nu}|)\Big)^m \le \Big(tr(\nu A + (1-\nu)B)\Big)^m,$$

$$\Big(\det(A^\nu B^{1-\nu})\Big)^m \le \Big(\det(\nu A + (1-\nu)B)\Big)^m,$$

and

$$\Big(|||A^\nu X B^{1-\nu}|||\Big)^m \le \Big(\nu|||A||| + (1-\nu)|||B|||\Big)^m. \tag{10}$$

Here, as application of Theorem 3, we give further and enhanced improvements to the above inequalities.

To prove the main results of this section, we need to recall the following two lemmas, the first lemma [5] is a Heinz-Kato type inequality for unitarily invariant norms, and the second lemma (see, e.g., [[4], p. 482]) is the Minkowski inequality for determinants.

Lemma 4 ([5]) *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive semidefinite matrices. Then we have*

$$|||A^\nu X B^{1-\nu}||| \le |||AX|||^\nu |||XB|||^{1-\nu}.$$

In particular, we have

$$tr\left|A^\nu X B^{1-\nu}\right| \le (trA)^\nu (trB)^{1-\nu}.$$

Lemma 5 *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices. Then we have*

$$\det(A+B)^{\frac{1}{n}} \ge \det(A)^{\frac{1}{n}} + det(B)^{\frac{1}{n}}.$$

The first result of this section concerns the determinant of positive definite matrices and reads as follows:

Theorem 6 *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices and $0 \le \nu \le 1$. Then for all positive intger m, we have*

$$\begin{aligned}&\big(\det(A^\nu B^{1-\nu})\big)^m + r_0^{nm}\big(\det(A)^{\frac{m}{2}} - \det(B)^{\frac{m}{2}}\big)^2 + r_{nm}\Big[\big([\det(A)\det(B)]^{\frac{m}{4}} - \det(B)^{\frac{m}{2}}\big)^2 \chi_{(0,\frac{1}{2}]}(\nu)\\ &+ \big([\det(A)\det(B)]^{\frac{m}{4}} - \det(A)^{\frac{m}{2}}\big)^2 \chi_{(\frac{1}{2},1]}(\nu)\Big] \le \det(\nu A + (1-\nu)B)^m,\end{aligned}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $r_{nm} = \min\{2r_0^{nm}, (1-r_0)^{nm} - r_0^{nm}\}$.

Proof. We have

$$
\begin{aligned}
& \det(\nu A+(1-\nu)B)^m \\
= & \left[\det(\nu A+(1-\nu)B)^{\frac{1}{n}}\right]^{nm} \\
\geq & \left[\det(\nu A)^{\frac{1}{n}}+\det((1-\nu)B)^{\frac{1}{n}}\right]^{nm} \text{ (by Lemma 5)} \\
= & \left[\nu\det(A)^{\frac{1}{n}}+(1-\nu)\det(B)^{\frac{1}{n}}\right]^{nm} \\
\geq & \left[\left(\det(A)^{\frac{1}{n}}\right)^{\nu}\left(\det(B)^{\frac{1}{n}}\right)^{1-\nu}\right]^{nm}+r_0^{nm}\left[\left(\det(A)^{\frac{1}{n}}\right)^{\frac{mn}{2}}-(\det(B))^{\frac{1}{n}\ \frac{mn}{2}}\right]^2 \\
& +r_{nm}\left([\det(A)^{\frac{1}{n}}\det(B)^{\frac{1}{n}}]^{\frac{nm}{4}}-(\det(B)^{\frac{1}{n}})^{\frac{nm}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\nu) \\
& +\left([\det(A)^{\frac{1}{n}}\det(B)^{\frac{1}{n}}]^{\frac{nm}{4}}-(\det(A)^{\frac{1}{n}})^{\frac{nm}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\nu) \text{ (by Theorem 3)} \\
= & \left(\det(A^{\nu}B^{1-\nu})\right)^m+r_0^{nm}\left(\det(A)^{\frac{m}{2}}-\det(B)^{\frac{m}{2}}\right)^2 \\
& +\left([\det(A)\det(B)]^{\frac{m}{4}}-\det(B)^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\nu) \\
& +\left([\det(A)\det(B)]^{\frac{m}{4}}-\det(A)^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\nu).
\end{aligned}
$$

This ends the proof. ■

The second result of this section concerns the traces of positive definite matrices and reads as follows:

Theorem 7 *Let $A, B \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices and $0 \leq \nu \leq 1$. Then for all positive integer m, we have*

$$
\begin{aligned}
& \left[tr(|A^{\nu}B^{1-\nu}|)\right]^m+r_0^m\left((trA)^{\frac{m}{2}}-(trB)^{\frac{m}{2}}\right)^2+r_m\left([(trA)(trB)]^{\frac{m}{4}}-(trB)^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\nu) \\
& +\left([(trA)(trB)]^{\frac{m}{4}}-(trA)^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\nu)\leq[tr(\nu A+(1-\nu)B)]^m,
\end{aligned}
$$

where $r_0=\min\{\nu,1-\nu\}$ and $r_m=\min\{2r_0^m,(1-r_0)^m-r_0^m\}$.

Proof. We have

$$
\begin{aligned}
& \left[tr(|A^{\nu}B^{1-\nu}|)\right]^m+r_0^m\left[(trA)^{\frac{m}{2}}-(trB)^{\frac{m}{2}}\right]^2+r_{nm}\left([(trA)(trB)]^{\frac{m}{4}}-(trB)^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\nu) \\
& +\left([(trA)(trB)]^{\frac{m}{4}}-(trA)^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\nu) \\
\leq & \left[(trA)^{\nu}(trB)^{1-\nu}\right]^m+r_0^m\left[(trA)^{\frac{m}{2}}-(trB)^{\frac{m}{2}}\right]^2+r_{nm}\left([(trA)(trB)]^{\frac{m}{4}}-(trB)^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\nu) \\
& +\left([(trA)(trB)]^{\frac{m}{4}}-(trA)^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\nu) \text{ (by Lemma 4)} \\
\leq & \left[tr(\nu A+(1-\nu)B)\right]^m \text{ (by Theorem 3).}
\end{aligned}
$$

This ends the proof. ■

The third and last result of this section provides an improvement to the inequality (10) and reads as follows:

Theorem 8 *Let $A, X, B \in \mathbf{M}_n(\mathbb{C})$ be positive semidefinite matrices and $0 \leq \nu \leq 1$. Then for all positive integer m, we have*

$$
\begin{aligned}
& |||A^{\nu}XB^{1-\nu}|||^m+r_0^m\Big(|||AX|||^{\frac{m}{2}}-|||XB|||^{\frac{m}{2}}\Big)^2+r_m\Big[\Big((|||AX||||||XB|||)^{\frac{m}{4}}-|||XB|||^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu) \\
& +\Big((|||AX||||||XB|||)^{\frac{m}{4}}-|||AX|||^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big]\leq\Big[\nu|||AX|||+(1-\nu)|||XB|||\Big]^m,
\end{aligned}
$$

where $r_0=\min\{\nu,1-\nu\}$ and $r_m=\min\{2r_0^m,(1-r_0)^m-r_0^m\}$.

Proof. We have

$$\begin{aligned}
&|||A^\nu XB^{1-\nu}|||^m + r_0^m\Big(|||AX|||^{\frac{m}{2}} - |||XB|||^{\frac{m}{2}}\Big)^2 + r_m\Big[\Big((|||AX||||||XB|||)^{\frac{m}{4}} - |||XB|||^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu)\\
&+\Big((|||AX||||||XB|||)^{\frac{m}{4}} - |||AX|||^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big]\\
\le\ & \Big[|||AX|||^\nu|||XB|||^{1-\nu}\Big]^m + r_0^m\Big(|||AX|||^{\frac{m}{2}} - |||XB|||^{\frac{m}{2}}\Big)^2\\
&+r_m\Big[\Big((|||AX||||||XB|||)^{\frac{m}{4}} - |||XB|||^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu)\\
&+\Big((|||AX||||||XB|||)^{\frac{m}{4}} - |||AX|||^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big] \text{ (by Lemma 4)}\\
\le\ & \Big[\nu|||AX||| + (1-\nu)|||XB|||\Big]^m \text{ (by Theorem 3).}
\end{aligned}$$

This completes the proof. ■

4 Improvements of Some Inequalities for Generalized Euclidean Operator Radius

Let $\mathcal{H}$ be a real or complex Hilbert space. The generalized Euclidean operator radius ω_p of operators $T_1, ..., T_n \in \mathcal{H}$ is defined for $p \ge 1$ as follows (see, [11]):

$$\omega_p(T_1, ..., T_n) := \sup_{||x||=1}\Big(\sum_{i=1}^n |\langle T_i x, x\rangle|^p\Big)^{\frac{1}{p}}.$$

Concerning the generalized Euclidean operator radius ω_p, A. Sheikhhosseini, M. S. Moslehian and K. Shebrawi established in [12] the following result:

Theorem 9 *Let $T_i \in B(\mathcal{H})$ for $i = 1, 2, 3..,$ and $p \ge 2m$ for some $m = 1, 2, 3, ..$ Then for $0 \le \nu \le 1$,*

$$\omega_p^p(T_1, .., T_n) \le \Big|\Big|\sum_{i=1}^n\Big(\nu|T_i|^p + (1-\nu)|T^*|_i^p\Big)^m\Big|\Big| - \inf_{||x||=1}\zeta(x),$$

where

$$\zeta(x) := \min\{\nu, 1-\nu\}^m\sum_{i=1}^n\Big(\langle|T_i|^{\frac{p}{m}}x, x\rangle^{\frac{m}{2}} - \langle|T_i^*|^{\frac{p}{m}}x, x\rangle^{\frac{m}{2}}\Big)^2.$$

The first aim of this section is to provide some improvements to Theorem 9.

Before giving our results, we recall the following lemmas. The first lemma is known as the generalized mixed Schwarz inequality, this lemma is proved by F. Kittaneh in [6].

Lemma 10 *Let $T \in \mathcal{B}(\mathcal{H})$ and $\nu \in (0, 1)$. Then*

$$|\langle Tx, y\rangle|^2 \le \langle |T|^{2\nu}x, x\rangle\langle|T^*|^{2(1-\nu)}y, y\rangle, \quad \forall x, y \in \mathcal{H}.$$

The second Lemma follows from the spectral theorem for positive operators and Jensen's inequality, this lemma is proved in [10].

Lemma 11 (McCarthy inequality) *Let $T \in \mathcal{B}(\mathcal{H})$ such that $T \ge 0$ and let $x \in \mathcal{H}$ be any unit vector. Then we have the following assertions:*

(a) $\langle Tx, x\rangle^r \le \langle T^r x, x\rangle$ *for* $r \ge 1$,

(b) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r$ *for* $0 < r \leq 1$.

Now, by using Theorem 3, we give the following refinement of Theorem 9.

Theorem 12 *Let* $T_i \in B(\mathcal{H})$ *for* $i = 1, 2, 3..$, *and* $p \geq 2m$ *for some* $m = 1, 2, 3, ..$ *Then for all real number* ν *satisfying* $0 \leq \nu \leq 1$, *we have*

$$\omega_p^p(T_1, .., T_n) \leq \Big|\Big| \sum_{i=1}^{n} \Big(\nu |T_i|^{\frac{p}{m}} + (1-\nu)|T_i^*|^{\frac{p}{m}}\Big)^m \Big|\Big| - \inf_{||x||=1} \zeta_1(x) - \inf_{||x||=1} \zeta_2(x),$$

where

$$\zeta_1(x) := r_0^m \sum_{i=1}^{n} \Big(\langle |T_i|^{\frac{p}{m}} x, x\rangle^{\frac{m}{2}} - \langle |T_i^*|^{\frac{p}{m}} x, x\rangle^{\frac{m}{2}}\Big)^2,$$

and

$$\begin{aligned} \zeta_2(x) \quad : \quad &= r_m \sum_{i=1}^{n} \Big[\Big([\langle |T_i|^{\frac{p}{m}} x, x\rangle \langle |T_i^*|^{\frac{p}{m}} x, x\rangle]^{\frac{m}{4}} - \langle |T_i^*|^{\frac{p}{m}} x, x\rangle^{\frac{m}{2}} \Big)^2 \chi_{(0,\frac{1}{2}]}(\nu) \\ &+ \Big([\langle |T_i|^{\frac{p}{m}} x, x\rangle \langle |T_i^*|^{\frac{p}{m}} x, x\rangle]^{\frac{m}{4}} - \langle |T_i|^{\frac{p}{m}} x, x\rangle^{\frac{m}{2}} \Big)^2 \chi_{(\frac{1}{2},1]}(\nu) \Big]. \end{aligned}$$

Proof. For all $x \in \mathcal{H}$, we have the following inequalities:

$$\begin{aligned} \sum_{i=1}^{n} |\langle T_i x, x\rangle|^p &= \sum_{i=1}^{n} \Big(|\langle T_i x, x\rangle|^2 \Big)^{\frac{p}{2}} \\ &\leq \sum_{i=1}^{n} \Big(\langle |T_i|^{2\nu} x, x\rangle \langle |T_i^*|^{2(1-\nu)} x, x\rangle \Big)^{\frac{p}{2}} \text{ (by Lemma 10)} \\ &\leq \sum_{i=1}^{n} \Big(\langle |T_i|^{\frac{p\nu}{m}} x, x\rangle \langle |T_i^*|^{\frac{p(1-\nu)}{m}} x, x\rangle \Big)^{m} \text{ (by Lemma 11(a))} \\ &\leq \sum_{i=1}^{n} \Big(\langle |T_i|^{\frac{p}{m}} x, x\rangle^{\nu} \langle |T_i^*|^{\frac{p}{m}} x, x\rangle^{(1-\nu)} \Big)^{m} \text{ (by Lemma 11(b))} \\ &\leq \sum_{i=1}^{n} \Big(\nu \langle |T_i|^{\frac{p}{m}} x, x\rangle + (1-\nu) \langle |T_i^*|^{\frac{p}{m}} x, x\rangle \Big)^{m} - \zeta_1(x) - \zeta_2(x) \text{ (by Theorem 3)} \\ &\leq \sum_{i=1}^{n} \Big(\langle \Big(\nu |T_i|^{\frac{p}{m}} + (1-\nu) |T_i^*|^{\frac{p}{m}} \Big) x, x\rangle \Big)^{m} - \inf_{||x||=1} \zeta_1(x) - \inf_{||x||=1} \zeta_2(x) \\ &\leq \sum_{i=1}^{n} \langle \Big(\nu |T_i|^{\frac{p}{m}} + (1-\nu) |T_i^*|^{\frac{p}{m}} \Big)^{m} x, x\rangle - \inf_{||x||=1} \zeta_1(x) - \inf_{||x||=1} \zeta_2(x) \text{ (by Lemma 11(a))} \\ &\leq \langle \sum_{i=1}^{n} \Big(\nu |T_i|^{\frac{p}{m}} + (1-\nu) |T_i^*|^{\frac{p}{m}} \Big)^{m} x, x\rangle - \inf_{||x||=1} \zeta_1(x) - \inf_{||x||=1} \zeta_2(x). \end{aligned}$$

By taking the supremum over all $x \in \mathcal{H}$ with $||x|| = 1$, we deduce the result. This completes the proof. ■

Next, we provide some improvements to the results stated in Theorem 4.1 of [12].

Theorem 13 *Let* $T_i \in B(\mathcal{H})$ *for* $i = 1, 2, 3..$, $r \geq 1$, *and* $p \geq q \geq 1$ *such that* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. *Then for* $m = 1, 2, 3...$,

$$\omega_p^{mr}(|T_1|, .., |T_n|)\omega_q^{mr}(|T_1^*|, .., |T_n^*|) \leq \left(\frac{r}{p} \Big|\Big| \sum_{i=1}^{n} |T_i|^p \Big|\Big| + \frac{r}{q} \Big|\Big| \sum_{j=1}^{n} |T_j^*|^q \Big|\Big| \right)^m - \inf_{||x||=1} \delta_1(x) - \inf_{||x||=1} \delta_2(x),$$

where

$$\delta_1(x) := \left(\frac{r}{p}\right)^m \left(\Big(\sum_{i=1}^{n} \langle |T_i|x, x\rangle^p \Big)^{\frac{m}{2}} - \Big(\sum_{j=1}^{n} \langle |T_j^*|x, x\rangle^q \Big)^{\frac{m}{2}} \right)^2,$$

and

$$\begin{aligned}\delta_2(x) \quad : \quad &= r_m \Big(\Big[\Big(\Big(\sum_{i=1}^{n} \langle |T_i|x, x\rangle^p \Big) \Big(\sum_{i=1}^{n} \langle |T_i^*|x, x\rangle^q \Big) \Big)^{\frac{m}{4}} - \Big(\sum_{i=1}^{n} \langle |T_i^*|x, x\rangle^q \Big)^{\frac{m}{2}} \Big]^2 \chi_{(2r,+\infty)}(p) \\ &+ \Big[\Big(\Big(\sum_{i=1}^{n} \langle |T_i|x, x\rangle^p \Big) \Big(\sum_{i=1}^{n} \langle |T_i^*|x, x\rangle^q \Big) \Big)^{\frac{m}{4}} - \Big(\sum_{i=1}^{n} \langle |T_i|x, y\rangle^p \Big)^{\frac{m}{2}} \Big]^2 \chi_{(r,2r)}(p) \Big),\end{aligned}$$

and $r_m = \min\{2(\frac{r}{p})^m, (1-\frac{r}{p})^m - (\frac{r}{p})^m\}$.

Proof. We set $\nu := \frac{r}{p}$, then $1-\nu := \frac{r}{q}$. Therefore, we have $r_0 = \frac{r}{p}$ and $r_m = \min\{2(\frac{r}{p})^m, (1-\frac{r}{p})^m - (\frac{r}{p})^m\}$. We put $a := \sum_{i=1}^{n} \langle |T_i|x, x\rangle^p$ and $b := \sum_{i=1}^{n} \langle |T_i^*|x, x\rangle^q$. We apply Theorem 3 to a and b.

So, according to Theorem 3 and Lemma 11(a), we have the following successive inequalities:

$$\begin{aligned} & \left[\left(\sum_{i=1}^{n} \langle |T_i|x, x\rangle^p \right)^{\frac{r}{p}} \left(\sum_{j=1}^{n} \langle |T_j^*|x, x\rangle^q \right)^{\frac{r}{q}} \right]^m \\ \leq \quad & \left(\frac{r}{p} \sum_{i=1}^{n} \langle |T_i|x, x\rangle^p + \frac{r}{q} \sum_{j=1}^{n} \langle |T_j^*|x, x\rangle^q \right)^m - \delta_1(x) - \delta_2(x) \\ \leq \quad & \left(\frac{r}{p} \sum_{i=1}^{n} \langle |T_i|^p x, x\rangle + \frac{r}{q} \sum_{j=1}^{n} \langle |T_j^*|^q x, x\rangle \right)^m - \inf_{||x||=1} \delta_1(x) - \inf_{||x||=1} \delta_2(x) \\ \leq \quad & \left(\langle \Big(\frac{r}{p} \sum_{i=1}^{n} |T_i|^p \Big) x, x\rangle + \langle \Big(\frac{r}{q} \sum_{j=1}^{n} |T_j^*|^q \Big) x, x\rangle \right)^m - \inf_{||x||=1} \delta_1(x) - \inf_{||x||=1} \delta_2(x), \end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$, $||x|| = 1$, we get the result. This completes the proof. ∎

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