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Rotational Surfaces With Rotations In $\mathbf{x}_3\mathbf{x}_4$-Plane*

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Abstract

In the present study we consider generalized rotational surfaces in Euclidean 4-space $\mathbb{E}^4$. Further, we obtain some curvature properties of these surfaces. We also introduce some kind of generalized rotational surfaces in $\mathbb{E}^4$ with the choice of meridian curve. Finally, we give some examples.

1 Introduction

In the 3D differential geometry, surface of revolution has wide applications in many areas. Especially, they have been studied in kinematics and soliton theory [16]. In [17], Cole studied the general theory of rotations in 4-dimensional Euclidean space in $\mathbb{E}^4$. Later, Moore considered the *general rotational surface* M in $\mathbb{E}^4$ with the parametrization of the form

$$\begin{cases} \widetilde{x}_1(u,v) = x_1(u)\cos cv - x_2(u)\sin cv, \\ \widetilde{x}_2(u,v) = x_1(u)\sin cv + x_2(u)\cos cv, \\ \widetilde{x}_3(u,v) = x_3(u)\cos dv - x_4(u)\sin dv, \\ \widetilde{x}_3(u,v) = x_3(u)\sin dv + x_4(u)\cos dv, \end{cases} \tag{1}$$

where, $\gamma(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$ is the meridian curve, c and d are the rates of rotation in fixed planes of the rotation [13]. If c or d is zero, then the surface generated by (1) becomes a *(simple) rotational surface*. However, the rotational surfaces in $\mathbb{E}^4$ with constant curvatures are studied in [14] and [7]. If one can choose the meridian curve $\gamma(u)$ in the x_1x_3-plane as $\gamma(u) = (x_1(u), 0, x_3(u), 0)$ then the resultant rotational surface should have the parametrization

$$M_1 : W(u,v) = (x_1(u)\cos cv, x_1(u)\sin cv, x_3(u)\cos dv, x_3(u)\sin dv), \tag{2}$$

where $u \in I \subset R$, $v \in (0, 2\pi)$ and $c^2x_1^2(u) + d^2x_3^2(u) > 0$ on I [4], [8], [10]. Moreover, for the values $c = d = 1$ and $x_1(u) = r(u)\cos u$, $x_3(u) = r(u)\sin u$ in (2) the rotational surface is known as *Vranceanu rotational surface* in $\mathbb{E}^4$ [15].

Meanwhile, if we consider the meridian curve as a space curves, $\gamma(u) \subset \mathbb{R}^3$ then the rotational surface in $\mathbb{E}^4$ should have the parametrization;

$$M_2 : Z(u,v) = (x_1(u), x_2(u), x_3(u)\cos v, x_3(u)\sin v); \tag{3}$$

$u \in I$, $v \in (0, 2\pi)$, which means that, M_2 is obtained by the rotation of the curve $\gamma(u)$ around the unit circle $\delta(v) = (\cos v, \sin v)$ [9]. These surfaces are also known as *spherical product surfaces in* $\mathbb{E}^4$ [5].

Furthermore, if we rotate the planar curve $\gamma(u) = (x_1(u), x_2(u))$ around a space curve $\delta(v)$ satisfying the conditions, $\|\delta(v)\| = 1$, $\|\delta'(v)\| = 1$. Then the resultant rotational surface in $\mathbb{E}^4$ should have the parametrization

$$M_3 : Y(u,v) = x_1(u)\overrightarrow{e}_1 + x_2(u)\delta(v) \tag{4}$$

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where $u \in I \subset R$, $v \in (0, 2\pi)$ and $\overrightarrow{e}_1 = (1, 0, 0, 0)$. Actually, these surfaces are called *meridian surfaces* in $\mathbb{E}^4$ (See, [11], [2] and [1]).

This paper is organized as follows: In section 2 we give some basic concepts of the second fundamental form and curvatures of the surfaces in $\mathbb{E}^4$. In Section 3 we consider generalized rotational surface given with $c = 0$, $d = 1$ in (1). That is, after the rotation the point with coordinates x_1, x_2, x_3, x_4 passes into the point with the coordinates $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4$ by

$$\begin{cases} \widetilde{x}_1(u,v) = x_1(u,v), \\ \widetilde{x}_2(u,v) = x_2(u,v), \\ \widetilde{x}_3(u,v) = x_3(u)\cos v - x_4(u)\sin v, \\ \widetilde{x}_3(u,v) = x_3(u)\sin v + x_4(u)\cos v. \end{cases} \tag{5}$$

Further, we obtain some curvature properties of the generalized rotational surface given with the position vector (5). We also introduce some kind of generalized rotational surfaces in $\mathbb{E}^4$ with the choice of meridian curve $\gamma(u)$. Consequently, we obtained some results related with their curvatures. Finally, we give some examples of these type of surfaces.

2 Basic Concepts

Let M be a local surface in $\mathbb{E}^4$ given with position vector $X(u,v)$. The tangent space T_pM is spanned by the vector fields X_u and X_v. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \tag{6}$$

where $\langle , \rangle$ is the inner product in $\mathbb{E}^4$. We assume that $X(u,v)$ is regular i.e., $W^2 = EG - F^2 > 0$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$, and $X_1 = X_u$, $X_2 = X_v$ tangent vector fields of M then *Gauss equation* gives

$$\widetilde{\nabla}_{X_i} X_j = \sum_{k=1}^{2} \Gamma_{ij}^k X_k + \sum_{\alpha=1}^{2} L_{ij}^{\alpha} N_\alpha; \ 1 \le i,j \le 2, \tag{7}$$

where L_{ij}^{α} are the coefficients of the second fundamental form with respect to unit normal vector N_α and Γ_{ij}^k are the *Christoffel symbols* of M [12].

Consequently, the *Gaussian curvature* and mean curvature vector $\overrightarrow{H}$ of M are given by

$$K = \frac{1}{W^2} \sum_{\alpha=1}^{2} \left(L_{11}^{\alpha} L_{22}^{\alpha} - (L_{12}^{\alpha})^2 \right) \tag{8}$$

and

$$\overrightarrow{H} = \frac{1}{2W^2} \sum_{\alpha=1}^{2} \left(E L_{22}^{\alpha} - 2F L_{12}^{\alpha} + G L_{11}^{\alpha} \right) N_\alpha \tag{9}$$

respectively. The norm $\left\| \overrightarrow{H} \right\|$ of the mean curvature vector $\overrightarrow{H}$ is known as *mean curvature* of M. Recall that, a surface M is said to be *minimal* (resp. *flat*) if its mean curvature (resp. Gaussian curvature) vanishes identically [3, 6].

3 General Rotational Surfaces in $\mathbb{E}^4$

Let M be a general rotational surface defined by the following parametrization

$$X(u,v) = (x_1(u), x_2(u), x_3(u)\cos v - x_4(u)\sin v, x_3(u)\sin v + x_4(u)\cos v) \tag{10}$$

where

$$\gamma(u) = (x_1(u), x_2(u), x_3(u), x_4(u)),$$

is the meridian curve of the surface M.

In the sequel, two situations have been dealt with in relation to the case of the meridian curve of the general rotation surface. Some results related to the rotation surface in each case have been obtained.

Case I. Suppose

$$x_3(u) = r(u)\cos\varphi(u) \quad \text{and} \quad x_4(u) = r(u)\sin\varphi(u). \tag{11}$$

Then the resultant rotational surface should have the following parametrization

$$M : X(u,v) = (x_1(u), x_2(u), r(u)\cos\alpha, r(u)\sin\alpha) \tag{12}$$

where $\alpha(u,v) = \varphi(u) + v$ and $r(u), \varphi(u)$ are smooth functions.

The tangent space T_pM of M is spanned by

$$\begin{cases} X_u = (x_1', x_2', r'cos\alpha - \varphi' r\sin\alpha, r'\sin\alpha + \varphi' r\cos\alpha), \\ X_v = (0, 0, -r\sin\alpha, r\cos\alpha). \end{cases} \tag{13}$$

We may consider the profile curve $\gamma(u)$ has arclength parameter, i.e.,

$$\|\gamma'(u)\|^2 = (x_1')^2 + (x_2')^2 + (r')^2 + r^2(\varphi')^2 = 1. \tag{14}$$

Consequently, the coefficients of first fundamental form become

$$\begin{cases} E = \langle X_u, X_u \rangle = 1, \\ F = \langle X_u, X_v \rangle = r^2(u)\varphi'(u), \\ G = \langle X_v, X_v \rangle = r^2(u), \end{cases} \tag{15}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^4$.

The second partial derivatives of $X(u,v)$ are expressed as follows

$$\begin{cases} X_{uu}(u,v) = (x_1'', x_2'', A(u,v), B(u,v)), \\ X_{uv}(u,v) = (0, 0, -r'\sin\alpha - r\varphi'\cos\alpha\varphi', r'\cos\alpha - r\varphi'\sin\alpha), \\ X_{vv}(u,v) = (0, 0, -r\cos\alpha, -r\sin\alpha), \end{cases} \tag{16}$$

where $A(u,v)$ and $B(u,v)$ are differentiable functions defined by

$$\begin{cases} A(u,v) = r''\cos\alpha - 2r'\varphi'\sin\alpha - r(\varphi')^2\cos\alpha - r\varphi''\sin\alpha, \\ B(u,v) = r''\sin\alpha + 2r'\varphi'\cos\alpha - r(\varphi')^2\sin\alpha + r\varphi''\cos\alpha. \end{cases}$$

The normal space is spanned by the vector fields

$$N_1 = \frac{1}{\lambda}(-x_2', x_1', 0, 0) \quad \text{and} \quad N_2 = \frac{1}{\lambda\sqrt{1 - r^2(\varphi')^2}}\left(r'x_1', r'x_2', -\lambda^2\cos\alpha, -\lambda^2\sin\alpha\right). \tag{17}$$

where

$$\lambda^2 = (x_1')^2 + (x_2')^2, \tag{18}$$

is the differentiable function. Hence, the coefficients of the second fundamental form of the surface are

$$\begin{cases} L_{11}^1 = \langle X_{uu}(u,v), N_1(u,v) \rangle = \frac{\kappa_1(u)}{\lambda}, \\ L_{12}^1 = \langle X_{uv}(u,v), N_1(u,v) \rangle = 0, \\ L_{22}^1 = \langle X_{vv}(u,v), N_1(u,v) \rangle = 0, \\ L_{11}^2 = \langle X_{uu}(u,v), N_2(u,v) \rangle = \frac{r'\lambda\lambda' - \lambda^2\left(r'' - r(\varphi')^2\right)}{\lambda\sqrt{1-r^2(\varphi')^2}}, \\ L_{12}^2 = \langle X_{uv}(u,v), N_2(u,v) \rangle = \frac{\lambda r\varphi'}{\sqrt{1-r^2(\varphi')^2}}, \\ L_{22}^2 = \langle X_{vv}(u,v), N_2(u,v) \rangle = \frac{\lambda r}{\sqrt{1-r^2(\varphi')^2}}, \end{cases} \tag{19}$$

where

$$\kappa_1(u) = x_1'(u)x_2''(u) - x_1''(u)x_2'(u),$$

is the smooth function.

Consequently, by the use the equations (15) and (19) with (8) we obtain the following result.

Theorem 1 *Let M be a general rotational surface in $\mathbb{E}^4$ given with the parametrization (12). Then the Gaussian curvature of M at point p is*

$$K = -\frac{r''\left(1 - r^2(\varphi')^2\right) + rr'\varphi'(r\varphi'' + r'\varphi')}{r\left(1 - r^2(\varphi')^2\right)^2}. \tag{20}$$

For the vanishing Gaussian curvature we give the following examples.

Example 1 *The general rotational surfaces given with the following parametrization have vanishing Gaussian curvatures*

(i) $X(u,v) = (x_1(u), x_2(u), c\cos\alpha, c\sin\alpha)$,

(ii) $X(u,v) = (x_1(u), x_2(u), (au+b)\cos(c+v), (au+b)\sin(c+v))$,

(iii) $X(u,v) = (x_1(u), x_2(u), (au+b)\cos(c + d\ln(u + \frac{b}{a}) + v), (au+b)\sin(c + d\ln(u + \frac{b}{a}) + v))$,

where a, b, c and d are real constants.

Let us denote be $H = \left\|\overrightarrow{H}\right\|$ the mean curvature of the general rotational surface M in $\mathbb{E}^4$. Consequently, by (15) and (19) with (9) we obtain the following result.

Theorem 2 *Let M be a general rotational surface in $\mathbb{E}^4$ given with the parametrization (12). Then the mean curvature of M at point p is*

$$\begin{aligned}\left\|\overrightarrow{H}\right\|^2 &= \frac{1}{4r^2\left(1 - r^2(\varphi')^2\right)^3}\{1 - (r')^2 - 3r^2(\varphi')^2 + 2r^4(\varphi')^4 + 4r^4(r')^2(\varphi')^4 \\ &\quad -4r^2(r')^2(\varphi')^2 - 6r^3r'\varphi'\varphi'' + 4r^5r'(\varphi')^3\varphi'' - r^4(\varphi'')^2 \\ &\quad +r^2\kappa_\gamma^2 - 2rr'' - r^4\kappa_\gamma^2(\varphi')^2 + 6r''r^3(\varphi')^2 - 4r^5r''(\varphi')^4\}.\end{aligned} \tag{21}$$

However, by taking $r = 1$ in the equation (21), the following result is obtained.

Corollary 1 *Let M be a general rotational surface in $\mathbb{E}^4$ given with the radius function $r(u) = 1$. Then the mean curvature of M at point p becomes*

$$\left\|\overrightarrow{H}\right\|^2 = \frac{1}{4\left(1 - (\varphi'(u))^2\right)^2}\left(\kappa_\gamma^2 + 1 - 2(\varphi'(u))^2 - \frac{(\varphi''(u))^2}{1 - (\varphi'(u))^2}\right) \tag{22}$$

where κ_γ is the curvature of the meridian curve γ.

As a consequence of (22) we obtain the following result.

Corollary 2 *Let M be a general rotational surface in $\mathbb{E}^4$ given with the radius function $r(u) = 1$. Then M has positive mean curvature $\left\|\overrightarrow{H}\right\| > \frac{1}{2}$.*

Proof. Now, one can obtain this expression in terms of κ_γ and the derivatives of ϕ. Because the parameter u is the arc length of γ, then

$$(x_1')^2 + (x_2')^2 + (x_3')^2 + (x_4')^2 = 1,$$

and $\lambda^2 = 1 - (\varphi'(u))^2$. Remark that, $\|\gamma''(u)\| = \kappa_\gamma$. Hence, we have

$$\kappa_\gamma^2 = (x_1'')^2 + (x_2'')^2 + (\varphi')^4 + (\varphi'')^2 . \tag{23}$$

Furthermore, we can write

$$x_1' = \sqrt{1-(\varphi')^2}\cos\theta \quad \text{and} \quad x_2' = \sqrt{1-(\varphi')^2}\sin\theta. \tag{24}$$

Then, we have

$$x_1'' = -\frac{\varphi'\varphi''}{\lambda}\cos\theta - \lambda\theta'\sin\theta \quad \text{and} \quad x_2'' = -\frac{\varphi'\varphi''}{\lambda}\sin\theta + \lambda\theta'\cos\theta. \tag{25}$$

We have two expressions

$$\kappa_\gamma^2 = \frac{(\varphi')^2(\varphi'')^2}{(1-(\varphi')^2)} + (1-(\varphi')^2)(\theta')^2 + (\varphi')^4 + (\varphi'')^2 . \tag{26}$$

Consequently, summing up (22)–(26) we obtain

$$\left\|\vec{H}\right\|^2 = \frac{(\theta')^2}{4(1-(\varphi')^2)} + \frac{1}{4}. \tag{27}$$

This completes the proof of the corollary. ■

Case II. Suppose $x_4 = \lambda x_3$, $\lambda \in \mathbb{R}$, then the position vector of the rotational surface M is represented by

$$X(u,v) = (x_1(u), x_2(u), x_3(u)(\cos v - \lambda\sin v), x_3(u)(\sin v + \lambda\cos v)) . \tag{28}$$

Then the coefficients of the first fundamental form of M become

$$\begin{cases} E = \langle X_u, X_u\rangle = 1, \\ F = \langle X_u, X_v\rangle = 0, \\ G = \langle X_v, X_v\rangle = (1+\lambda^2)\, x_3^2(u). \end{cases} \tag{29}$$

The second partial derivatives of $X(u,v)$ are expressed as follows

$$\begin{cases} X_{uu}(u,v) = ({x_1}'', {x_2}'', x_3''(\cos v - \lambda\sin v), x_3''(\sin v + \lambda\cos v)), \\ X_{uv}(u,v) = (0,0,x_3'(-\sin v - \lambda\cos v), x_3'(\cos v - \lambda\sin v)), \\ X_{vv}(u,v) = (0,0,x_3(-\cos v + \lambda\sin v), x_3(-\sin v - \lambda\cos v)). \end{cases} \tag{30}$$

The normal space of M is spanned by the vector fields

$$\begin{cases} N_1 = \frac{1}{\kappa_\gamma}(x_1'', x_2'', x_3''(\cos v - \lambda\sin v), x_3''(\sin v + \lambda\cos v)), \\ N_2 = \frac{\sqrt{1+\lambda^2}}{\kappa_\gamma}(x_2'x_3'' - x_2''x_3', x_3'x_1'' - x_1'x_3'', \frac{\kappa_1(\cos v - \lambda\sin v)}{1+\lambda^2}, \frac{\kappa_1(\sin v + \lambda\cos v)}{1+\lambda^2}), \end{cases} \tag{31}$$

where

$$\kappa_\gamma = \sqrt{(x_1'')^2 + (x_2'')^2 + (1+\lambda^2)(x_3'')^2},$$

is the curvature of the curve γ and

$$\kappa_1 = x_1'x_2'' - x_2'x_1'',$$

is a smooth function.

As in the previous case, the coefficients of the second fundamental form of the surface are given by

$$\begin{cases} L_{11}^1 = \langle X_{uu}(u,v), N_1(u,v)\rangle = \kappa_\gamma, \\ L_{12}^1 = \langle X_{uv}(u,v), N_1(u,v)\rangle = 0, \\ L_{22}^1 = \langle X_{vv}(u,v), N_1(u,v)\rangle = \frac{-x_3 x_3''(1+\lambda^2)}{\kappa_\gamma}, \\ L_{11}^2 = \langle X_{uu}(u,v), N_2(u,v)\rangle = 0, \\ L_{12}^2 = \langle X_{uv}(u,v), N_2(u,v)\rangle = 0, \\ L_{22}^2 = \langle X_{vv}(u,v), N_2(u,v)\rangle = \frac{-x_3\kappa_1\sqrt{1+\lambda^2}}{\kappa_\gamma}. \end{cases} \tag{32}$$

Substituting (32) with (29) into (8) we get the following result.

Theorem 3 *Let M be a surface given with the parametrization (28). Then, the Gaussian curvature of M is given by*

$$K = -\frac{x_3''(u)}{x_3(u)}.$$

As a consequence of Theorem 3 we obtain the following result.

Corollary 3 ([1]) *Let M be a surface given with the position vector (28). Then the following statements are valid;*

i) If $x_3(u) = ae^{cu} + be^{-cu}$ then the corresponding surface is pseudo-spherical, i.e., it has negative Gaussian curvature $K = -\frac{1}{c^2}$,

ii) If $x_3(u) = a\cos cu + b\sin cu$ then the corresponding surface is spherical, i.e., it has positive Gaussian curvature $K = \frac{1}{c^2}$,

iii) If $x_3(u) = au + b$ then the corresponding surface is flat,

where a, b and c are real constants.

The following examples are due to T. Otsuki given in [18].

Example 2 *For the case $\lambda = 0$, $x_3(u) = \sin u$ the surface patch (28) becomes*

$$X(u,v) = (x_1(u), x_2(u), \sin u\cos v, \sin u\sin v)$$

where, $u \in I$, $0 \le v < 2\pi$. In [18] T. Otsuki considers the following surface patches

a) $x_1(u) = \frac{4}{3}\cos^3(\frac{u}{2})$, $x_2(u) = \frac{4}{3}\sin^3(\frac{u}{2})$, $x_3(u) = \sin u$,

b) $x_1(u) = \frac{1}{2}\sin^2 u\cos(2u)$, $x_2(u) = \frac{1}{2}\sin^2 u\sin(2u)$, $x_3(u) = \sin u$.

One can show that these surfaces both have constant Gaussian curvature $K = 1$. The surface given in (a) is called Otsuki (non-round) sphere in $\mathbb{E}^4$ which does not lie in a 3-dimensional subspace of $\mathbb{E}^4$.

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