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A Uniqueness Theorem For Inverse Sturm–Liouville Problem*

İbrahim Adalar†

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Abstract

In this paper, we discuss the inverse Sturm-Liouville problem. We show a new uniqueness theorem and some results related to it.

1 Introduction

We consider following boundary value problem L on $[0,1]$,

$$-y'' + q(x)y = \lambda y, \tag{1}$$

$$y'(0) - hy(0) = 0, \tag{2}$$

$$y'(1) + Hy(1) = 0, \tag{3}$$

where $h, H \in \mathbb{R}$ and λ is a spectral parameter. $q(x)$ is the real valued function and $q(x) \in L_2(0,1)$. The values of the λ parameter for which (1)–(3) has nonzero solutions are called eigenvalues $\{\lambda_n\}_{n\geq 0}$ and the corresponding nontrivial solutions are called eigenfunctions $\{y_n(x)\}_{n\geq 0}$. Some important results on the properties of eigenvalues and eigenfunctions of Sturm-Liouville problem have been published in various publications (see, [4, Chapter 3], [5], [29]) and the references therein). It is known that the spectrum of such problems consists of countable many real eigenvalues, which have no finite limit point.

Inverse spectral problems consist in recovering the coefficients of an operator from their spectral characteristics. The inverse spectral problem for L is to determine the potential function $q(x)$ from some given data. The first result on this area is given by Ambarzumian [1]. Inverse Sturm-Liouville problems, which appear in mathematical physics and other branches of natural sciences, have now been studied for about 90 years (see, [6], [11], [14]–[22], [24], [28] and the references therein). Borg [2] showed that generally a single spectrum is insufficient to determine the potential. He also proved that if q is symmetric about the midline, $q(1-x) = q(x)$ and if $h = H$, then a single spectrum $\{\lambda_n\}_{n\geq 0}$ uniquely determines the potential. Levinson [14] considerably shortened the proofs using complex analysis techniques. However, if a finite number of eigenvalues in one spectrum is unknown, q is not uniquely determined. Hald [9] showed that the lowest eigenvalue must be taken into account in order to determine the boundary conditions as well as the symmetric potential. He gave a counterexample that there are different symmetric potentials $q(x)$ such that the two Sturm-Liouville problems have the same spectrum except lowest eigenvalues.

The half inverse Sturm-Liouville problem which is one of the important subjects of the inverse spectral theory has been studied firstly by Hochstadt and Lieberman in [12]. They proved that a single spectra and the potential on the interval $[1/2, 1]$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$. Since then, this result has been generalized to various versions. Some uniqueness results as Hochstadt and Lieberman-type theorems have been given in [7], [8], [13], [23]. Castillo discussed the half-inverse problem for (1)–(3). He [3] gave a counterexample that show the necessity of coefficient h. Wei and Xu [25] solved an open problem of missing one eigenvalue presented in [7]. They showed that only one spectrum missing one eigenvalue is sufficient to achieve the Hochstadt and Lieberman's result. Similarly, Wang [27] proved a Borg-type theorem for a missing eigenvalue.

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†Zara Veysel Dursun Colleges of Applied Sciences, Sivas Cumhuriyet University Zara/Sivas, Turkey

The purpose of this note is to give a new uniqueness theorem. Under the light of above studies, our result contains in the cases symmetric potential and half inverse problem. We shall not require a finite number of eigenvalues but rather suppose $\int_0^1 q(t)\cos 2n\pi t dt$ is known for $n = 1, \ldots, k-1$.

2 Main Results

Together with L, we consider a boundary value problem $\widetilde{L} = L(\widetilde{q}(t), h, H)$ of the same form but with different coefficient $\widetilde{q}(t)$. We assume that if a certain symbol s denotes an object related to L , then $\widetilde{s}$ will denote an analogous object related to $\widetilde{L}$. We give following uniqueness theorem.

Theorem 1 *Let* $a_n = \int_0^1 q(t)\cos 2n\pi t dt$. *We assume that* $h,\ H,\ \int_0^1 |q(t)|\, dt,\ \int_0^1 |\widetilde{q}(t)|\, dt \le \frac{k}{15}$ *for* $k \in \mathbb{N}$ *and* $k \ge 5$. *If* $a_n = \widetilde{a}_n$ *for* $n = 1, \ldots, k-1$ *and* $\lambda_n = \widetilde{\lambda}_n$ *for* $n \ge k$, *then*

$$q(x) + q(1-x) = \widetilde{q}(x) + \widetilde{q}(1-x)$$

almost everywhere on $[0, 1/2]$.

Under the assumptions of the Theorem 1, the following corollaries which are analogies to Levinson and Hochstadt-Lieberman's results can be given.

Corollary 2 *If* $q(x) = q(1-x)$ *and* $\widetilde{q}(x) = \widetilde{q}(1-x)$ *then* $q(x) = \widetilde{q}(x)$ *a.e. on* $[0, 1]$.

Corollary 3 *If* $q(x) = \widetilde{q}(x)$ *on* $[1/2, 1]$ *then* $q(x) = \widetilde{q}(x)$ *a.e. on* $[0, 1]$.

Corollary 4 *If* $q(x) = -q(1-x)$ *and* $\widetilde{q}(x) = \widetilde{q}(1-x)$ *then* $\widetilde{q}(x) = 0$ *a.e. on* $[0, 1]$.

The following lemmas are important for proof of the main result.

Lemma 5 ([10, Lemma 1]) *We assume that* $h,\ H, \int_0^1 |q(t)|\, dt \le \frac{k}{15}$ *for* $k \in \mathbb{N}$ *and* $k \ge 5$. *For* $n \ge k$, *the eigenvalues satisfy*

$$\lambda_n = (n\pi)^2 + 2\left(h + H + \frac{1}{2}\int_0^1 q(t)(1+\cos 2n\pi t)dt\right) \pm \frac{k^2}{10n} \tag{4}$$

and the eigenfunctions satisfy

$$y_n(x) = \cos n\pi x \pm \frac{k}{10n}. \tag{5}$$

Here $r = s \pm \varepsilon$ *means that* $r = s + \theta\varepsilon$ *for some* $|\theta| \le 1$.

The assertion of following Lemma is proven in the proof of [10, Lemma 1].

Lemma 6 *Let* n *be fixed and* $m = n\pi + \frac{a}{n\pi} + \gamma\left(\frac{k}{5n\pi}\right)^2$ *with* $|\gamma| \le 0.69$ *correspond to an eigenvalue. The function* $\cos mx$ *satisfies*

$$\cos mx = (-1)^j \left(\frac{1}{n\pi}\left(a\frac{j-1/2}{n} + b\right) + cd\right) \pm 0.779\left(\frac{k}{5n\pi}\right)^2 \tag{6}$$

for $j = 1, 2, \ldots, n$ *where*

$$a = h + H + \frac{1}{2}\int_0^1 q(t)(1+\cos 2n\pi t)dt,$$

$$b = h + \frac{1}{2} \int_0^{\frac{(j-1/2)}{n}} q(t)(1 + \cos 2n\pi t)dt - a\frac{(j-1/2)}{n},$$

$$|c| = 1, d \leq 0.423\frac{k}{5n\pi}.$$

Proof of Theorem 1. Let us write the equation (1) for y_n and $\widetilde{y}_n$

$$-y_n''(x) + q(x)y_n(x) = \lambda_n y_n(x), \tag{7}$$

$$-\widetilde{y}_n''(x) + \widetilde{q}(x)\widetilde{y}_n(x) = \lambda_n \widetilde{y}_n(x), \tag{8}$$

for $n \geq k$. If we apply the classical procedure:

i) multiply (7) by $\widetilde{y}_n(x)$ and (8) by $y_n(x)$;

ii) subtract from each other, then we get

$$[y_n(x)\widetilde{y}_n'(x) - y_n'(x)\widetilde{y}_n(x)]' = [q(x) - \widetilde{q}(x)]\, y_n(x)\widetilde{y}_n(x).$$

By integrating both sides of this equality on $[0,1]$, we obtain

$$[y_n(x)\widetilde{y}_n'(x) - y_n'(x)\widetilde{y}_n(x)]_0^1 = \int_0^1 [q(x) - \widetilde{q}(x)]\, y_n(x)\widetilde{y}_n(x)dx.$$

Now let us see which boundary conditions satisfy $\widetilde{y}_n(x)$ and by $y_n(x)$ at 0 with $x = 1$. Since $h = \widetilde{h}$, $H = \widetilde{H}$, we have that

$$\int_0^1 [q(x) - \widetilde{q}(x)]\, y_n(x)\widetilde{y}_n(x)dx = 0,\ n \geq k.$$

Let k and $\widetilde{k}$ be defined as in Lemma 5 and assume that $k \geq \widetilde{k}$. Also, we have that

$$y_n(x)\widetilde{y}_n(x) = \frac{1 + \cos 2n\pi x}{2} \pm \frac{k}{(5n)^2} \pm \left(\frac{k}{10n}\right)^2 \pm \left(\frac{k}{5n}\right)^3$$

from (5) and (6). It is obvious that

$$\int_0^1 [q(x) - \widetilde{q}(x)] \left(\frac{1 + \cos 2n\pi x}{2} \pm \frac{k}{(5n)^2} \pm \left(\frac{k}{10n}\right)^2 \pm \left(\frac{k}{5n}\right)^3\right) dx = 0,\ n \geq k. \tag{9}$$

On the other hand, one can show by using (4) that

$$\lambda_n - \widetilde{\lambda}_n = \int_0^1 (q(x) - \widetilde{q}(x))\, dx = 0,\ n \geq k.$$

It is easy to check that

$$\lim_{n\to\infty} \left(\lambda_n - \widetilde{\lambda}_n\right) = \int_0^1 (q(x) - \widetilde{q}(x))\, dx = 0. \tag{10}$$

We obtain that

$$\left(\frac{1}{2} \pm \frac{k}{(5n)^2} \pm \left(\frac{k}{10n}\right)^2 \pm \left(\frac{k}{5n}\right)^3\right) \int_0^1 [q(x) - \widetilde{q}(x)]\, dx = 0, \;\; n \geq k. \tag{11}$$

Using (9), (10) and (11) we have that

$$\int_0^1 [q(x) - \widetilde{q}(x)] \cos 2n\pi x dx = 0, \;\; n = 0 \;\; \text{and } n \geq k. \tag{12}$$

This result and the assumptions of the theorem show that

$$\int_0^1 [q(x) - \widetilde{q}(x)] \cos 2n\pi x dx = 0, \;\; n \geq 0$$

and so

$$\int_0^{1/2} [q(x) - \widetilde{q}(x)] \cos 2n\pi x dx + \int_0^{1/2} [q(1-x) - \widetilde{q}(1-x)] \cos 2n\pi x dx = 0, \;\; n \geq 0.$$

Thus can be rewritten as

$$\int_0^{1/2} \left[\phi(x) - \widetilde{\phi}(x)\right] \cos 2n\pi x dx = 0, \;\; n \geq 0$$

where $\phi(x) = q(x) + q(1-x)$. By the completeness of the functions $\{\cos 2n\pi x\}_{n=0}^{\infty}$ on $[0, 1/2]$ (see [26]), we have that

$$\phi(x) = \widetilde{\phi}(x)$$

and so $q(x) + q(1-x) = \widetilde{q}(x) + \widetilde{q}(1-x)$ on $[0, 1/2]$ almost everywhere. The proof is complete. ■

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