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Extended Convergence Of Jarratt Type Methods*

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Abstract

The aim of this article is the extension of the convergence of Jarratt type methods for solving equations with Banach space valued operators. We develop w-continuity conditions and only hypotheses on the first derivative contrasting earlier work where hypotheses of order higher than one are used on the derivatives. We also provide error estimates and uniqueness results on the solution based on Lipschitz type conditions not available before. This is how we extend the applicability of these methods. Numerical experiments complete this study.

1 Introduction

Let X, Y stand for Banach spaces, D an open convex set with $D \subset X$, $L(X, Y)$ denote the space of operaotrs from X into Y that are linear, bounded, and $F : D \longrightarrow Y$ be an operator differentiable according to Fréchet.

One of the most interesting and challenging tasks in mathematics is without a doubt the location of a solution x_* for the equation

$$F(x) = 0. \tag{1.1}$$

It is worth noticing that problems from diverse areas vis mathematical modeling lead to determining x_*. This task usually involves the development of iterative methods, since the closed form solution is obtained only in rare occations. After the introduction of the quadratically convergent method of Newton, the need for faster convergence lead to higher order of convergence methods such as the fourth order Jarratt method [12] defined for $x_0 \in D$ and all $n = 0, 1, 2, \ldots$

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = x_n - J_n F'(x_n)^{-1}F(x_n), \end{cases} \tag{1.2}$$

where $J_n = J(x_n) = (6F'(y_n) - 2F'(x_n))^{-1}(3F'(y_n) + F'(x_n))$. Later the local convergence analysis of three step method

$$\begin{cases} y_n = x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), \\ z_n = x_n - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n), \\ x_{n+1} = z_n - T(\lambda_n)F'(x_n)^{-1}F(z_n), \end{cases} \tag{1.3}$$

was studied in [20], where the sixth order of convergence was shown under some conditions on linear operator T and λ_n but in the special case when $X = Y = \mathbb{R}^k$. The convergence order was established using seventh order derivatives, which significantly limit the applicability of method (1.3). For example: Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define G on Ω by

$$G(x) = x^3 \log x^2 + x^5 - x^4$$

Then, we have $x_* = 1$, and

$$G'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

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$$G''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$
$$G'''(x) = 6 \log x^2 + 60x^2 - 24x + 22.$$

Obviously $G'''(x)$ is not bounded on Ω. So, the convergence of solvers (1.2) and (1.3) are not guaranteed by the analysis in [7, 12]. In this study we use only assumptions on the first derivative to prove our results.

Moreover, the following were not given: estimates on $\|x_n - x_*\|$ useful for determining the number of iterations needed to achieve a predetermined accuracy ε; results on the uniqueness about a ball centered at x_*. Furthermore, a shot in the dark is used to attain the initial point x_0. We handle all these concerns using only the first derivative, w-continuity conditions on F' and utilizing (COC) or (ACOC) to determine the convergence order (to be defined in Remark 1 that need only the first derivative.)

The setting of a Banach space is used and a more general method defined as

$$\begin{cases} y_n = x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), \\ z_n = x_n - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n), \\ x_{n+1} = z_n - H_n F'(x_n)^{-1}F(z_n), \end{cases} \tag{1.4}$$

where $H_n = H(x_n)$, and $H : D \longrightarrow L(X, Y)$. Note that if $H = T(\lambda)$ and $X = Y = \mathbb{R}^k$ method (1.4) reduces to method (1.3) which in turn has generalized many other popular iterative methods both in the scalar and multidimensional case [1–25].

In Section 2 the local convergence of method is given, whereas Section 3 contains numerical experiments.

2 Local Convergence Analysis

It is convenient for the analysis to follow the development of some scalar functions and parameters. Consider a function $\omega_0 : [0, \infty) \longrightarrow [0, \infty)$ continuous and increasing satisfying $\omega_0(0) = 0$. Suppose that equation

$$\omega_0(t) = 1 \tag{2.5}$$

has a minimal positive solution r_0. Define functions $\omega : [0, r_0) \longrightarrow [0, \infty)$ and $v : [0, r_0) \longrightarrow [0, \infty)$ continuous and increasing satisfying $\omega(0) = 0$. Define fucntions φ_1 and ψ_1 on the interval $[0, r_0)$ by

$$\psi_1(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta + \frac{1}{3}\int_0^1 v(\theta t)d\theta}{1 - \omega_0(t)}$$

and

$$\psi_1(t) = \varphi_1(t) - 1.$$

Suppose that

$$\frac{v(0)}{3} - 1 < 0. \tag{2.6}$$

Then, in view of these definitions $\psi_1(0) = -1$ and $\psi_1(t) \longrightarrow \infty$ as $t \longrightarrow r_0^-$. Denote by R_0 the minimal solution of equation $\psi_1(t) = 0$ in $(0, r_0)$ assumed to exist by the intermediate value theorem. Suppose that equation

$$p(t) = 0 \tag{2.7}$$

has a minimal positive solution r_p, where

$$p(t) = \frac{1}{2}(3\omega_0(g_1(t)t) + \omega_0(t)).$$

Set

$$r_1 = \min\{r_0, r_p\}.$$

Define functions φ_2 and ψ_2 on the interval $[0, r_1)$ by

$$\varphi_2(t) = \varphi_1(t) + \frac{3(\omega_0(\varphi_1(t)t) + \omega_0(t)) \int_0^1 v(\theta t) d\theta}{4(1 - \omega_0(t))(1 - p(t))}$$

and

$$\psi_2(t) = \varphi_2(t) - 1.$$

Then, we get $\psi_2(0) = -1$ and $\psi_2(t) \longrightarrow \infty$ as $t \longrightarrow r_1^-$. Denote by R_2 the minimal solution of equation $\psi_2(t) = 0$ in $(0, r_1)$. Suppose that equation

$$\omega_0(\varphi_2(t)t) = 1 \tag{2.8}$$

has a minimal positive solution r_2. Set $r = \min\{r_1, r_2\}$. Let $q : [0, r) \longrightarrow [0, \infty)$ be a continuous and increasing function. Define functions φ_3 and ψ_3 on the interval $[0, r)$ by

$$\varphi_3(t) = \left[\frac{\int_0^1 \omega((1-\theta)\varphi_2(t)t) d\theta}{1 - \omega_0(\varphi_2(t)t)} + q(t) \int_0^1 v(\theta \varphi_2(t)t) d\theta \right] \varphi_2(t)$$

and

$$\psi_3(t) = \varphi_3(t) - 1.$$

We obtain again $\psi_3(0) = -1$ and $\psi_3(t) \longrightarrow \infty$ as $t \longrightarrow r^-$. Denote by R_3 the minimal solution of equation $\psi_3(t) = 0$ in the interval $(0, r)$. Define a radius R by

$$R = \min\{R_i\}, \ i = 1, 2, 3. \tag{2.9}$$

We shall show that R is a radius of convergence for method (1.4). By these definitions the following items hold for all $t \in [0, R)$.

$$0 \le \omega_0(t) < 1, \tag{2.10}$$

$$0 \le \omega_0(\varphi_2(t)t) < 1, \tag{2.11}$$

$$0 \le p(t) < 1, \tag{2.12}$$

$$0 \le q(t) < 1, \tag{2.13}$$

and

$$0 \le \varphi_i(t) < 1. \tag{2.14}$$

We use the notation for a ball: $U(h, \rho) = \{x \in X : \|x - h\| < \rho\}$. Moreover, $\bar{U}(h, \rho)$ denotes the closure of $U(h, \rho)$.

Next, we provide the conditions (C) to be used for the local analysis of method (1.4).

(c1) $F : D \longrightarrow Y$ is differentiable. There exists a simple $x_* \in D$ solving equation (1.1).

(c2) There exist function $\omega_0 : [0, \infty) \longrightarrow [0, \infty)$ continuous and increasing satisfying $\omega_0(0) = 0$ and such that for all $x \in D$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \le \omega_0(\|x - x_*\|).$$

Set $D_0 = D \cap U(x_*, r_0)$.

(c3) There exist functions $\omega : [0, r_0) \longrightarrow [0, \infty)$, continuous and increasing satisfying $\omega(0) = 0$ such that for all $x, y \in D_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \le \omega(\|y - x\|)$$

and

$$\|F'(x_*)^{-1}F'(x)\| \le v(\|x - x_*\|).$$

Set $D_1 = D \cap U(x_*, r)$.

(c4) There exists function $q : [0, r) \longrightarrow [0, \infty)$ continuous and increasing, $H : D \longrightarrow L(X, Y)$ such that for all $x, z \in D_1$

$$\|(F'^{-1} - H(x)F'^{-1})F'(x_*)\| \leq q(\|x - x_*\|).$$

(c5) $\bar{U}(x_*, R) \subseteq D$, where R is defined in (2.9), (2.6) holds, r_0, r_p, r_2 exist and are given by (2.5), (2.7) and (2.8), respectively.

(c6) There exists $\bar{R} \geq R$ such that

$$\int_0^1 \omega_0(\theta\bar{R})d\theta < 1.$$

Set $D_2 = D \cap \bar{U}(x_*, \bar{R})$.

Next, the local convergence analysis is given based on condition (C).

Theorem 1 *Under the conditions (C) further assume $x_0 \in U(x^*, R) - \{x^*\}$. Then the following assertions hold*

$$\{x_n\} \subset U(x_*, R), \tag{2.15}$$

$$\lim_{n \longrightarrow \infty} x_n = x_*, \tag{2.16}$$

$$\|y_n - x^*\| \leq \varphi_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < R, \tag{2.17}$$

$$\|z_n - x^*\| \leq \varphi_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.18}$$

$$\|x_{n+1} - x^*\| \leq \varphi_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.19}$$

and x_* solves equation (1.1) uniquely in D_2 a set given in (c6).

Proof. Let us choose $x \in U(x^*, R)$. Using (2.9), (2.10) and (c2), we get that

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \omega_0(\|x_0 - x^*\|) \leq \omega_0(R) < 1, \tag{2.20}$$

leading together with a Lemma on invertible operators due to Banach [16] that $F'(x)$ is invertible and

$$\|F'^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\|x_0 - x^*\|)}. \tag{2.21}$$

In view of (2.21), y_0 exists by the first substep of method (1.4) if $n = 0$. We also have

$$F(x) = F(x) - F(x_*) = \int_0^1 F'(x_* + \theta(x - x_*))(x - x_*)d\theta,$$

so by (c3)

$$\|F'(x_*)^{-1}F(x)\| \leq \int_0^1 v(\theta\|x - x_*\|)d\theta\|x - x_*\|. \tag{2.22}$$

Then, by (2.9), (2.14) (for $i = 1$), (2.21), (2.22) and method (1.4) for $n = 0$, we obtain in turn that

$$\begin{aligned}
\|y_0 - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0) + \frac{1}{3}F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x_*)\| \left[\int_0^1 \|F'(x_*)^{-1}[F'(x_0 + \theta(x_0 - x_*)) \right. \\
&\quad \left. -F'(x_0))(x_0 - x_*)d\theta\| + \frac{1}{3}\|F'(x_*)^{-1}F(x_0)\|\right] \\
&\leq \frac{\int_0^1 \omega((1-\theta)\|x_0 - x_*\|)d\theta + \frac{1}{3}\int_0^1 v(\theta\|x_0 - x_*\|)d\theta}{1 - \omega_0(\|x_0 - x_*\|)}\|x_0 - x_*\| \\
&\leq \varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < R, \qquad (2.23)
\end{aligned}$$

showing $y_0 \in U(x_*, R)$ and the validity of (2.17) for $n=0$. We must show that $3F'(y_0)-F'(x_0)$ is invertible. Indeed, by (2.9), (2.12), (2.23) and (c2), we get that

$$\begin{aligned}
& \|(2F'(x_*))^{-1}(3(F'(y_0)-F'(x_*))+(F'(x_*)-F'(x_0))\| \\
\leq\ & \frac{1}{2}[3\|F'(x_*)^{-1}(F'(y_0)-F'(x_*))\|+\|F'(x_*)^{-1}(F'(x_*)-F'(x_0))\|] \\
\leq\ & \frac{1}{3}(3\omega_0(\|y_0-x_*\|)+\omega_0(\|x_0-x_*\|)) \\
\leq\ & \frac{1}{3}(3(\omega_0(\varphi_1(\|x_0-x_*\|)\|x_0-x_*\|)+\omega_0(\|x_0-x_*\|)) \\
\leq\ & p(\|x_0-x_*\|) \leq p(R) < 1,
\end{aligned} \tag{2.24}$$

so

$$\|(3F'(y_0)-F'(x_0))^{-1}F'(x_*)\| \leq \frac{1}{2(1-p(\|x_0-x_*\|))}, \tag{2.25}$$

and z_0 exists by the second substep of method (1.4) for $n=0$. Then, in view of (2.9), (2.14) (for $i=2$), (2.23), (2.25) and the second substep of method (1.4) for $n=0$, we have in turn that

$$\begin{aligned}
& \|z_0-x_*\| \\
=\ & \|(x_0-x_*-F'(x_0)^{-1}F(x_0)) \\
& +[I-\frac{1}{2}(3F'(y_0)-F'(x_0))^{-1}(3F'(y_0)+F'(x_0))]F'(x_0)^{-1}F(x_0) \\
=\ & (x_0-x_*-F'(x_0)^{-1}F(x_0))+\frac{3}{2}(3F'(y_0)-F'(x_0))^{-1} \\
& \times[(F'(y_0)-F'(x_*))+(F'(x_*)-F'(x_0))]F'(x_0)^{-1}F(x_0)\| \\
\leq\ & \left[\varphi_1(\|x_0-x_*\|)+\frac{3}{4}\frac{(\omega_0(\|y_0-x_*\|)+\omega_0(\|x_0-x_*\|))\int_0^1 v(\theta\|x_0-x_*\|)d\theta}{(1-\omega_0(\|x_0-x_*\|))(1-p(\|x_0-x_*\|))}\right] \\
& \times\|x_0-x_*\| \\
\leq\ & \varphi_2(\|x_0-x_*\|)\|x_0-x_*\| \leq \|x_0-x_*\| < R,
\end{aligned} \tag{2.26}$$

so $z_0 \in U(x_*, R)$ and (2.18) hold for $n=0$. As in (2.21) for $x=z_0$, we have

$$\begin{aligned}
\|F'(z_0)^{-1}F'(x_*)\| &\leq \frac{1}{1-w_0(\|z_0-x_*\|)} \\
&\leq \frac{1}{1-\omega_0(\varphi_2(\|x_0-x_*\|)\|x_0-x_*\|)},
\end{aligned} \tag{2.27}$$

and x_1 exists by the third substep of method (1.4) for $n=0$. Moreover, using (2.9), (c4), (2.14) (for $i=3$), (2.22) (for $x=z_0$) and (2.27), we have in turn that

$$\begin{aligned}
\|x_1-x_*\| &\leq \|(z_0-x_*-F'(z_0)^{-1}F(z_0))\| \\
&\quad +\|(F'(z_0)^{-1}-H(x_0)F'(x_0)^{-1})F'(x_*)\|\|F'(x_*)^{-1}F(z_0)\| \\
&\leq \left[\frac{\int_0^1 \omega((1-\theta)\|z_0-x_*\|)d\theta}{1-\omega_0(\|z_0-x_*\|)}\right. \\
&\quad \left. q(\|x_0-x_*\|)\int_0^1 v(\theta\|z_0-x_*\|)d\theta\right]\|z_0-x_*\| \\
&\leq \varphi_3(\|x_0-x_*\|)\|x_0-x_*\| \leq \|x_0-x_*\|,
\end{aligned} \tag{2.28}$$

so $x_1 \in U(x_*, R)$ and (2.19) holds for $n=0$. The induction for (2.17)–(2.19) is terminated if x_0, y_0, z_0, x_1 are switched with x_k, y_k, z_k, x_{k+1}, respectively in the above estimates. Then, by the estimate

$$\|x_{k+1}-x^*\| \leq \rho\|x_n-x^*\| \leq R, \tag{2.29}$$

where $\rho = \varphi_3(\|x_0 - x^*\|) \in [0,1)$, we conclude that $\lim_{k\longrightarrow\infty} x_k = x_*$ and $x_{k+1} \in U(x^*, R)$. The uniqueness of the solution x_* is shown by considering $y_* \in D_2$ so that $F(y_*) = 0$ and setting $Q = \int_0^1 F'(x_* + \theta(y_* - x_*))d\theta$. Then, by (c6), we have

$$\|F'(x_*)^{-1}(Q - F'(x_*))\| \leq \int_0^1 \omega_0(\theta\|y_* - x_*\|)d\theta \leq \int_0^1 \omega_0(\theta\bar{R})d\theta < 1,$$

so Q is invertible. Finally, $x_* = y_*$ by the identity

$$0 = F(y_*) - F(x_*) = Q(y_* - x_*).$$

■

Remark 1 1. *We can compute the computational order of convergence (COC) [24] defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence without resorting to the computation of higher order derivatives appearing in the method or in the sufficient convergence criteria usually appearing in the Taylor expansions for the proofs of those results.

2. *Let us consider a choice for H that will also be used on all examples:*

$$H(x_n) = I.$$

Then, we get in turn

$$\begin{aligned}
& \|(F'(z_n)^{-1} - H(x_n)F'(x_n)^{-1})F'(x_*)\| \\
= \; & \|(F'(z_n)^{-1} - F'(x_n)^{-1})F'(x_*)\| \\
= \; & \|F'(z_n)^{-1}(F'(x_n) - F'(z_n))F'(x_n)^{-1}F'(x_*)\| \\
\leq \; & \|F'(z_n)^{-1}F'(x_*)\|[\|F'(x_*)^{-1}(F'(x_n) - F'(x_*))\| \\
& + \|F'(x_*)^{-1}(F'(z_0) - F'(x_*))\|]\|F'(x_*)^{-1}F'(x_n)\| \\
\leq \; & \frac{\omega_0(\|x_n - x_*\|) + \omega_0(\|z_0 - x_*\|)}{(1 - \omega_0(\|z_0 - x_*\|))(1 - \omega_0(\|x_n - x_*\|))} \\
\leq \; & \frac{\omega_0(\|x_n - x_*\|) + \omega_0(\varphi_1(\|x_0 - x_*\|)\|x_n - x_*\|)}{(1 - \omega_0(\varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\|)(1 - \omega_0(\|x_n - x_*\|))}.
\end{aligned}$$

Therefore, we can choose

$$q(t) = \frac{\omega_0(t) + \omega_0(\varphi_1(t)t)}{(1 - \omega_0(\varphi_1(t)t))(1 - \omega_0(t))}.$$

3 Numerical Examples

Example 1 *Let us consider a system of differential equations governing the motion of an object and given by*

$$F_1'(x) = e^x, \;\; F_2'(y) = (e-1)y + 1, \;\; F_3'(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. *Let* $F = (F_1, F, F_3)$. *Let* $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1), p = (0,0,0)^T$. *Define function* F *on* D *for* $w = (x,y,z)^T$ *by*

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the $(\mathcal{A})$ *conditions, we get for* $\alpha = 1$, $w_0(t) = (e-1)t, w(t) = e^{\frac{1}{e-1}}t, v(t) = e^{\frac{1}{e-1}}$. *The radii are*

$$R_1 = 0.154407, \quad R_2 = 0.0555405, \quad R_3 = 0.0853234 \text{ and } R = R_2.$$

Example 2 *Let* $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, *the space of continuous functions defined on* $[0,1]$ *be equipped with the max norm. Let* $D = \overline{U}(0,1)$. *Define function* F *on* D *by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.30}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \ \textit{for each } \xi \in D.$$

Then, we get that $x^* = 0$, *so* $w_0(t) = 7.5t, w(t) = 15t$ *and* $v(t) = 2$. *Then the radii are*

$$R_1 = 0.02222, R_2 = 0.00886359, R_3 = 0.0154587 \text{ and } R = R_2.$$

Example 3 *Returning back to the motivational example at the introduction of this study, we have* $w_0(t) = w(t) = 96.6629073t$ *and* $v_1(t) = 2$. *The parameters for method (1.2) are*

$$R_1 = 0.002229894, R_2 = 0.000765558, R_3 = 0.00294163 \textit{ and } R = R_2.$$

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