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Dependence Of Eigenvalues Of Some Boundary Value Problems*

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Abstract

In this work we deal with a system of two first-order differential equations containing the same eigenvalue parameter. We consider some suitable separated real and complex coupled boundary conditions, and show that the eigenvalues generated by this system are continuous in an eigenvalue branch. Also we introduce the ordinary and Frechet derivatives of these eigenvalues with respect to some elements of the data.

1 Introduction

In this paper we will concern with a system of two first-order differential equations that contains the continuous analogous of the orthogonal polynomials on the unit circle. In the literature, this construction has been introduced by Atkinson in his book [1], Chap. 7. Indeed, the recurrence equations

$$y_{n+1} = \lambda y_n + b_n z_n,$$

$$z_{n+1} = \lambda \bar{b}_n y_n + z_n,$$

where $n = 0, 1, 2, ...$, imply the equations

$$\begin{cases} y_{n+1} - y_n = 2i\dfrac{\widetilde{\lambda}}{\left(1 - i\widetilde{\lambda}\right)} y_n + b_n z_n, \\ z_{n+1} - z_n = \dfrac{\left(1 + i\widetilde{\lambda}\right)}{\left(1 - i\widetilde{\lambda}\right)} \bar{b}_n y_n, \end{cases} \tag{1}$$

where $\lambda = \left(1 + i\widetilde{\lambda}\right) / \left(1 - i\widetilde{\lambda}\right)$. Considering $y_{n+1} - y_n$ as the discrete version of $y(x)dx$, b_n as $b(x)dx$ and $2\widetilde{\lambda}$ as μdx, the system of equations (1) leads to the continuous version of the recurrence relations as

$$\begin{cases} y' = i\mu y + b(x)z, \\ z' = \overline{b(x)}y, \end{cases} \tag{2}$$

where $b(x)$ is a continuous function on a given interval. However, there does not exist a detailed analysis of the system of equations (2) both in the literature and in [1]. Beside this, the system of equations that we will study is much more general than (2) as can be seen in (3). In this paper, we focus on the differentiability property of the eigenvalues of some boundary value problems generated by (3) and separated, real and complex-coupled boundary conditions. Recently, such an investigation has been introduced in [2]–[4] for odd-order boundary-value problems and some background information on differentiable properties of eigenvalues of even-order boundary-value problems can be found in, for example, [2].

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2 System of Equations

In this paper we consider the following system of equations

$$\begin{cases} y_1' = i\lambda w(x)y_1 + ia(x)y_1 + b(x)y_2, \\ y_2' = -i\lambda r(x)y_2 + \overline{b(x)}y_1 + ic(x)y_2, \end{cases} \tag{3}$$

where $x \in [t_1, t_2]$, a, c, r, w are real-valued, integrable functions on $[t_1, t_2]$, b is a complex-valued function, $w > 0$ and $r > 0$ for almost all x on $[t_1, t_2]$.

The system of equations (3) can also be handled as the following first-order equation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \left(-i\lambda \begin{bmatrix} -w & 0 \\ 0 & r \end{bmatrix} + \begin{bmatrix} ia & b \\ \overline{b} & ic \end{bmatrix}\right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{4}$$

The assumptions on the coefficients of (3) and (4) allow us to introduce the following lemma.

Lemma 1 *The system of equations (3) has one and only one solution*

$$Y(x,\lambda) = \begin{bmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{bmatrix} \tag{5}$$

satisfying the initial conditions

$$y_1(l,\lambda) = l_1, \ \ y_2(l,\lambda) = l_2,$$

where $l \in [t_1, t_2]$ and l_1, l_2 are arbitrary complex numbers. Moreover, $y_1(.,\lambda)$ and $y_2(.,\lambda)$ are entire functions of λ.

Lemma 2 *Let $y_1(x,\lambda)$, $y_2(x,\lambda)$ and $z_1(x,\mu)$, $z_2(x,\mu)$ be the solutions of (3) corresponding to the parameters λ and μ, respectively. Then we have*

$$\begin{aligned} & -i\left(\lambda - \overline{\mu}\right) \int_{t_1}^{t_2} \begin{bmatrix} \overline{z_1(x,\mu)} & \overline{z_2(x,\mu)} \end{bmatrix} \begin{bmatrix} w(x) & 0 \\ 0 & r(x) \end{bmatrix} \begin{bmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{bmatrix} dx \\ = & \left[Y(x,\lambda), \overline{Z(x,\mu)}\right] |_{t_1}^{t_2}, \end{aligned} \tag{6}$$

where $Y(x,\lambda)$ and $Z(x,\mu)$ are the vectors given by the rule (5) corresponding with $y_1(x,\lambda)$, $y_2(x,\lambda)$ and $z_1(x,\mu)$, $z_2(x,\mu)$, respectively and

$$[Y(x,\lambda), Z(x,\mu)] := -y_1(x,\lambda)z_1(x,\mu) + y_2(x,\lambda)z_2(x,\mu). \tag{7}$$

Proof. Using the equations in (3) we get

$$\begin{aligned} \frac{d}{dx}\left(-y_1\overline{z_1} + y_2\overline{z_2}\right) &= -\left(i\lambda w y_1 + iay_1 + by_2\right)\overline{z_1} - y_1\left(-i\overline{\mu}w\overline{z_1} - ia\overline{z_1} + \overline{b}\overline{z_2}\right) \\ &\quad + \left(-i\lambda r y_2 + \overline{b}y_1 + icy_2\right)\overline{z_2} + y_2\left(i\overline{\mu}r\overline{z_2} + b\overline{z_1} - ic\overline{z_2}\right) \\ &= -i\left(\lambda - \overline{\mu}\right)y_1\overline{z_1}w - i\left(\lambda - \overline{\mu}\right)y_2\overline{z_2}r. \end{aligned} \tag{8}$$

Integrating both sides of (8) from t_1 to t_2 we obtain the result. ■

We shall note that the representation (7) can also be introduced as follows

$$[Y(x,\lambda), Z(x,\mu)] = Z^t(x,\mu)JY(x,\lambda) \tag{9}$$

where the superscript denotes the transpose of the vector and

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Representation (9) will allow us to impose real and complex-coupled boundary conditions.

Corollary 3 *If $y_1(x,\lambda)$, $y_2(x,\lambda)$ and $z_1(x,\lambda)$, $z_2(x,\lambda)$ are the solutions of (3) corresponding with the same parameter λ, then $[Y(.,\lambda), Z(.,\lambda)]$ is independent from x and depends only on λ.*

3 Boundary Conditions

In this section we will share some suitable separated, real-coupled and complex-coupled boundary conditions for the solutions of (3).

Firstly, we can impose the separated boundary conditions as

$$\begin{cases} (i+\tan\alpha)y_1(t_1)-(1+i\tan\alpha)y_2(t_1)=0, \\ (i+\tan\beta)y_1(t_2)-(1+i\tan\beta)y_2(t_2)=0, \end{cases} \tag{10}$$

where α and β are some real numbers.

Secondly, the real-coupled boundary conditions can be given as

$$\begin{bmatrix} y_1(t_2) \\ y_2(t_2) \end{bmatrix} = M \begin{bmatrix} y_1(t_1) \\ y_2(t_1) \end{bmatrix}, \tag{11}$$

where $M \in E_2(\mathbb{C})$, i.e.

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \; m_{ij} \in \mathbb{C}, \tag{12}$$

such that

$$M^* J M = J, \tag{13}$$

where $M^* = \overline{M^t}$.

Finally, the complex-coupled boundary conditions can be introduced as

$$\begin{bmatrix} y_1(t_2) \\ y_2(t_2) \end{bmatrix} = e^{i\theta} M \begin{bmatrix} y_1(t_1) \\ y_2(t_1) \end{bmatrix}, \tag{14}$$

where θ is a real number and $M \in E_2(\mathbb{C})$.

Before passing to the details on the boundary-value problems we shall pay a special attention on the equation (13) together with the matrix (12). The equation (13) can also be written as

$$\begin{bmatrix} -|m_{11}|^2+|m_{21}|^2 & -\overline{m_{11}}m_{12}+\overline{m_{21}}m_{22} \\ -m_{11}\overline{m_{12}}+m_{21}\overline{m_{22}} & -|m_{12}|^2+|m_{22}|^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

which implies that

$$\begin{cases} -|m_{11}|^2+|m_{21}|^2=-1, \\ m_{11}\overline{m_{12}}=m_{21}\overline{m_{22}}, \\ -|m_{12}|^2+|m_{22}|^2=1. \end{cases} \tag{15}$$

It is possible to consider a matrix M with entries m_{ij} satisfying (15). Indeed, for the matrix

$$M = \begin{bmatrix} \sqrt{2}\dfrac{i+\tan\gamma}{1+i\tan\gamma} & \dfrac{1-i\tan\gamma}{-i+\tan\gamma} \\ \dfrac{i+\tan\delta}{1+i\tan\delta} & \sqrt{2}\dfrac{1-i\tan\delta}{-i+\tan\delta} \end{bmatrix}$$

equations in (15) are satisfied, where γ and δ are some real numbers.

Finally, we should note that all the conditions (10), (11) and (14) can be embedded into the following boundary conditions

$$\begin{bmatrix} y_1(t_1) \\ y_2(t_1) \end{bmatrix} = A.v, \; \begin{bmatrix} y_1(t_2) \\ y_2(t_2) \end{bmatrix} = B.v \tag{16}$$

where A and B are some 2×2 complex matrices and v is a 2×1 vector.

Now we can introduce the following theorem.

Theorem 4 *Each boundary-value problem generated by (3) and (10), (11), (14) has only real and discrete eigenvalues with possible accumulation points at positive and negative infinity.*

Proof. Let $Y(x,\lambda)$ be the corresponding vector given by (5) and generated by $y_1(x,\lambda)$ and $y_2(x,\lambda)$ of the solutions of (3). For each problem we will use the following equation

$$2\operatorname{Im}\lambda\left(\int_{t_1}^{t_2}|y_1(x,\lambda)|^2\,w(x)dx+\int_{t_1}^{t_2}|y_2(x,\lambda)|^2\,r(x)dx\right)=\left[Y(x,\lambda),\overline{Y(x,\lambda)}\right]\big|_{t_1}^{t_2}. \tag{17}$$

Separated boundary conditions give that

$$\begin{aligned}&-y_1(t_2,\lambda)\overline{y_1(t_2,\lambda)}+y_2(t_2,\lambda)\overline{y_2(t_2,\lambda)}+y_1(t_1,\lambda)\overline{y_1(t_1,\lambda)}-y_2(t_1,\lambda)\overline{y_2(t_1,\lambda)}\\=&\left[-\frac{1+i\tan\beta}{i+\tan\beta}\frac{1-i\tan\beta}{-i+\tan\beta}+1\right]|y_2(t_2,\lambda)|^2+\left[\frac{1+i\tan\alpha}{i+\tan\alpha}\frac{1-i\tan\alpha}{-i+\tan\alpha}-1\right]|y_2(t_1,\lambda)|^2\\=&\ 0.\end{aligned} \tag{18}$$

Therefore (17) and (18) show that $\operatorname{Im}\lambda=0$. This completes the proof for the problem (3), (10).

Real-coupled boundary conditions implies that

$$\begin{aligned}&Y^*(t_2,\lambda)JY(t_2,\lambda)-Y^*(t_1,\lambda)JY(t_1,\lambda)\\=&\ Y^*(t_2,\lambda)M^*JMY(t_2,\lambda)-Y^*(t_1,\lambda)JY(t_1,\lambda)=0.\end{aligned} \tag{19}$$

Therefore (17) and (19) prove the result for the problem (3), (11).

The proof for the problem (3), (14) can now be given similarly. Therefore the reality of the eigenvalues of each problem has been proved.

For the other assertion we shall use the conditions in (16).

Let $\mathcal{Y}(x,\lambda)$ be a fundamental matrix solution of (4) satisfying $\mathcal{Y}(t_1,\lambda)=I$, where I is the 2×2 identity matrix. For any other solution $Y(x,\lambda)$ of (4) one may write the following

$$Y(x,\lambda)=\mathcal{Y}(x,\lambda)Y(t_1,\lambda).$$

Considering the conditions (16) we get

$$\{B-\mathcal{Y}(t_2,\lambda)A\}.v=0$$

and for $v\neq 0$ we obtain that

$$\Delta(\lambda):=\det\{B-\mathcal{Y}(t_2,\lambda)A\}=0.$$

Clearly, the zeros of $\Delta(\lambda)$ coincide with the eigenvalues of the problem (4), (16) and vice versa. Finally using Lemma 2.1 we complete the proof. ■

Moreover, for the eigenvalues of the corresponding problems we can introduce the following theorem.

Theorem 5 *If we denote the eigenvalues of the problem (3), (16) by $\lambda_0,\ \lambda_1,\ \lambda_2,\ldots$ then the series*

$$\sum_{\lambda_r\neq 0}|\lambda_r|^{-1-\epsilon}$$

is convergent for each $\epsilon>0$.

Proof. The proof follows from Theorem 4, representation (4) and Gronwall's inequality. ■

4 Banach Space

In this section we will share some suitable Banach spaces to introduce the ordinary and Frechet derivatives of the corresponding boundary-value problems.

We shall denote by $\mathbb{B}$ the Banach space consisting of all vectors ν such that

$$\nu = (t_1, t_2, A_1, A_2, f_1, g_1, h_1, m_1, n_1)$$

with the norm

$$\|\nu\| = |r_1| + |r_2| + \|A_1\| + \|B_1\| + \int_{\widetilde{t}_1}^{\widetilde{t}_2} (|f_1| + |g_1| + |h_1| + |m_1| + |n_1|)\,,$$

where

$$\begin{aligned} B \quad &= \quad \mathbb{R} \times \mathbb{R} \times M_{2,2}(\mathbb{C}) \times M_{2,2}(\mathbb{C}) \times L^1(\widetilde{t}_1, \widetilde{t}_2) \times L^1(\widetilde{t}_1, \widetilde{t}_2) \\ & \qquad \times L^1(\widetilde{t}_1, \widetilde{t}_2) \times L^1(\widetilde{t}_1, \widetilde{t}_2) \times L^1(\widetilde{t}_1, \widetilde{t}_2), \end{aligned}$$

$M_{2,2}(\mathbb{C})$ is the set of all complex 2×2 matrices and $(\widetilde{t}_1, \widetilde{t}_2) \supset [t_1, t_2]$. Here the functions f_1, g_1, h_1, m_1, n_1 are defined as f, g, h, m, $n \in L^1(t_1, t_2)$ on $[t_1, t_2]$ and zero otherwise.

Let X_1 be a subset of $\mathbb{B}$ consisting of all vectors

$$\omega_1 = (t_1, t_2, A, B, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{w}, \widetilde{r}).$$

There is no confusion with identifying the subset X_1 with X that is a subset of $\mathbb{B}$ with the vectors

$$\omega_1 = (t_1, t_2, A, B, a, b, c, w, r).$$

Therefore we will identify X with X_1 to inherit the norm from $\mathbb{B}$ and convergence in X.

Lemma 6 *The solutions $y_1(x, \lambda)$, $y_2(x, \lambda)$ of (4), (16) satisfying the initial conditions*

$$y_1(\xi, \lambda) = \zeta_1, \;\; y_2(\xi, \lambda) = \zeta_2,$$

where $\xi \in (\widetilde{t}_1, \widetilde{t}_2)$ and ζ_1, ζ_2 are complex numbers, are continuous functions of the variables ξ, ζ_1, ζ_2, a, b, c, w, r.

Proof. The proof follows from (4) and Theorem 2.7 in [5]. ■

Lemma 7 *The eigenvalue $\lambda = \lambda(\omega_0)$ of (4), (16) is continuous at $\omega_0 \in X$.*

Proof. Consider the function

$$\Phi(\omega, \lambda) = \det \{B - \mathcal{Y}(t_1, t_2, a, b, c, w, r, \lambda)A\}\,.$$

The zeros of Φ coincide with the eigenvalues of (4), (16). As we have discussed earlier that Φ is an entire function and therefore it is not constant in λ.

Now let $\omega_0 \in X$ and $\Phi(\omega_0, \mu) = 0$. Then for the values of λ satisfying $|\lambda - \mu| > 0$ we have $\Phi(\omega_0, \lambda) \neq 0$ and applying the well-known result on continuity of roots of an equation of [6] we complete the proof. ■

Remark 1 *In Lemma 7, we, in fact, prove that there exists an eigenvalue branch such that it is continuous at ω_0. Therefore from now on we will only consider such continuous eigenvalue branches.*

Finally, using Lemma 6, Lemma 7 and the results in [5] we can introduce the following results.

Lemma 8 *Suppose that $\lambda = \lambda(\omega_0)$, $\omega_0 \in X$, is an eigenvalue of (4), (16) of order one or two. There exist normalized eigenfunctions $u_1(x,\omega)$, $u_2(x,\omega)$ of λ of order one or $u_1^j(x,\omega)$, $u_2^j(x,\omega)$, $j = 1, 2$, of λ of order two such that $u_k(x,\omega)$ converges uniformly to $u_k(x,\omega_0)$ and $u_k^j(x,\omega)$ converges uniformly to $u_k^j(x,\omega_0)$ on any compact subintervals of $(\widetilde{t}_1, \widetilde{t}_2)$, where $k = 1, 2$.*

We recall that an operator T acting on Banach spaces K_1, K_2 as $T : K_1 \to K_2$ is said to be differentiable at $f \in K_1$ if

$$\|T(f+h) - Tf - T'(f)\| = o(h) \text{ as } h \to 0,$$

where $T' : K_1 \to K_2$ is a bounded operator and it is said to be the Frechet derivative of T.

Finally in the next theorem we will fix all the other variables except the variable that the derivative of the eigenvalue is being taken with respect to it.

Theorem 9 *Let $\lambda = \lambda(\omega)$, $\omega \in X$, be the eigenvalue of (3), (16). Then with the normalized eigenfunctions $u_1(x,\lambda)$, $u_2(x,\lambda)$ we have the following equations:*

(i) $\lambda'(\alpha) = -2\,|u_1(t_1)|^2 = -2\,|u_2(t_1)|^2$,

(ii) $\lambda'(\beta) = 2\,|u_1(t_2)|^2 = 2\,|u_2(t_2)|^2$,

(iii) $\lambda'(M) = -i \begin{bmatrix} \overline{u_1(t_1)} & \overline{u_2(t_1)} \end{bmatrix} H^* J M \begin{bmatrix} u_1(t_1) \\ u_2(t_1) \end{bmatrix}$, $H \in E_2(\mathbb{C})$,

(iv) $\lambda'(\theta) = |u_1(t_1)|^2 - |u_2(t_1)|^2$,

(v) $\lambda'(a) = \int\limits_{t_1}^{t_2} |u_1|^2\, h$, $h \in L^1(t_1,t_2)$,

(vi) $\lambda'(b) = \int\limits_{t_1}^{t_2} 2\,\mathrm{Im}(u_1 \overline{u_2}) h$, $h \in L^1(t_1,t_2)$,

(vii) $\lambda'(c) = -\int\limits_{t_1}^{t_2} |u_2|^2\, h$, $h \in L^1(t_1,t_2)$,

(viii) $\lambda'(w) = \lambda \int\limits_{t_1}^{t_2} |u_1|^2\, h$, $h \in L^1(t_1,t_2)$,

(ix) $\lambda'(r) = \lambda \int\limits_{t_1}^{t_2} |u_2|^2\, h$, $h \in L^1(t_1,t_2)$.

Proof. To prove the assertions we will use the equation (6).

Let $u_1 = u_1(., \lambda(\alpha))$, $u_2 = u_2(., \lambda(\alpha))$ and $v_1 = u_1(., \lambda(\alpha+h))$, $v_2 = u_2(., \lambda(\alpha+h))$ for $h > 0$. Then we obtain that

$$\begin{aligned} & -i\,(\lambda(\alpha) - \lambda(\alpha+h)) \left(\int\limits_{t_1}^{t_2} u_1 \overline{v_1} w dx + \int\limits_{t_1}^{t_2} u_2 \overline{v_2} r dx \right) \\ = \;& u_1(t_1)\overline{v_1(t_1)} - u_2(t_1)\overline{v_2(t_1)} \\ = \;& \left[\frac{1 + i\tan\alpha}{i + \tan\alpha} \frac{1 - i\tan(\alpha+h)}{-i + \tan(\alpha+h)} - 1 \right] u_2(t_1)\overline{v_2(t_1)}. \end{aligned} \tag{20}$$

Dividing the equation (20) by h letting $h \to 0$ we obtain the result and this proves (i).

(ii) can be proved similarly.

For the proof of (iii) we let $u_1 = u_1(.,\lambda(M))$, $u_2 = u_2(.,\lambda(M))$ and $v_1 = u_1(.,\lambda(M+H))$, $v_2 = u_2(.,\lambda(M+H))$ for $H \in E_2(\mathbb{C})$. Then we have

$$\begin{aligned}
& -i\left(\lambda(M)-\lambda(M+H)\right)\left(\int_{t_1}^{t_2} u_1\overline{v_1}w dx + \int_{t_1}^{t_2} u_2\overline{v_2}r dx\right) \\
= \;& (V^*JU)(t_2)-(V^*JU)(t_1) \\
= \;& V^*(t_1)(M+H)^*JMU(t_1)-V^*(t_1)JU(t_1) \\
= \;& V^*(t_1)H^*JMU(t_1). \qquad (21)
\end{aligned}$$

Therefore (21) completes the proof for (iii).

Now we let $u_1 = u_1(.,\lambda(\theta))$, $u_2 = u_2(.,\lambda(\theta))$ and $v_1 = u_1(.,\lambda(\theta+h))$, $v_2 = u_2(.,\lambda(\theta+h))$ for $h > 0$. Then we obtain that

$$\begin{aligned}
& -i\left(\lambda(\theta)-\lambda(\theta+h)\right)\left(\int_{t_1}^{t_2} u_1\overline{v_1}w dx + \int_{t_1}^{t_2} u_2\overline{v_2}r dx\right) \\
= \;& (V^*JU)(t_2)-(V^*JU)(t_1) \\
= \;& (e^{-ih}-1)V^*(t_1)JU(t_1). \qquad (22)
\end{aligned}$$

Then we divide the equation (22) by h we let $h \to 0$ to obtain the result (iv).

For $u_1 = u_1(.,\lambda(a))$, $u_2 = u_2(.,\lambda(a))$ and $v_1 = u_1(.,\lambda(a+h))$, $v_2 = u_2(.,\lambda(a+h))$, $h \in L^1(t_1,t_2)$, we have

$$-i\left(\lambda(a)-\lambda(a+h)\right)\left(\int_{t_1}^{t_2} u_1\overline{v_1}w dx + \int_{t_1}^{t_2} u_2\overline{v_2}r dx\right) = -i\int_{t_1}^{t_2} u_1\overline{v_1}h dx \qquad (23)$$

and (23) completes the proof of (v).

For the rest we can apply a similar way and this completes the proof. ∎

5 Conclusion and Remarks

In this paper we have considered a system of first-order differential equations containing the same eigenvalue parameter together with some suitable separated and coupled boundary conditions. As we have stressed in the Introduction the motivation of this system has been introduced by Atkinson [1]. The special form of (3) coincides with (2). In fact, $w \equiv 1$, $r = a = c = 0$ on the given interval $[t_1, t_2]$ gives the system of equations (2). However, in this case some modifications should be done. For example (6) should be given as

$$-(\lambda-\overline{\mu})\int_{t_1}^{t_2} y_1(x,\lambda)\overline{z_1(x,\mu)}dx = \left[Y(x,\lambda), \overline{Z(x,\mu)}\right]\Big|_{t_1}^{t_2}$$

and this form has been given in [1].

We need to pay attention to some issues here. In the equation (3), we considered a first-order differential system corresponding to some boundary conditions and such first-order systems have been investigated in the literature, e.g. orthogonality of eigenvectors, self adjointness property (but not continuity property). However, all these resources contain abstract boundary conditions such as

$$A^*EA = B^*EA, \qquad (24)$$

where A and B are some suitable abstract matrices corresponding to some boundary conditions and E is some special matrix.

It is an ordinary way to obtain that the boundary value problem constructed by some matrices satisfying (24) is selfadjoint (symmetric) and say that all eigenvalues are real. However, it is not possible to investigate the continuity dependence of eigenvalues on data A, B. In other words, it is not clear how the continuity depends if one replaces A by some matrices $A+H$. For this reason, the papers cited as [2]–[5], all consider separated and coupled boundary conditions to investigate the dependence of eigenvalues.

On the other side, Eq. (3) is not a standart Hamiltonian system. This can be seen easily from the system (3) and from the boundary conditions.

In this work, we have achieved to impose suitable, well-defined boundary conditions in such an unusual way for the solutions of the equation. Paper therefore includes new types of boundary conditions that can not be embedded into the known ones.

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