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K_a-Convergence For Double Sequences And Korovkin Type Approximation*

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Abstract

In this paper, we introduce the idea of K_a-convergence for double sequences. Then, we use this notion to prove a Korovkin type approximation theorem and present an application that satisfies our new main theorem but does not satisfy classical ones. Finally, we study the rate of convergence of positive linear operators.

1 Introduction and Preliminaries

The following, now a classical result, was proved by P. P. Korovkin [11]: Let I be a compact subset of the real numbers and (L_n) be a sequence of positive linear operators that maps $C(I)$, the space of all continuous real valued functions on I, into itself. Suppose that the sequence $(L_n(f))$ converges to f uniformly on I for the three special functions $f_i : x \to x^i$, where $i = 0, 1, 2$. Then this sequence converges to f uniformly on I for every $f \in C(I)$. Because of its powerful applications, Korovkin's result has been extended in many directions. There is an extensive literature on Korovkin-type theorems (see for example [2, 3, 4, 5, 6, 7, 9, 10, 15]). In this paper, we define the concept of K_a^2-convergence that is a new convergence method and give an example in support of our definition. Then, we use this notion to prove a Korovkin type approximation theorem and present an application that satisfies our new main theorem but does not satisfy classical ones. Finally, we study the rate of convergence of positive linear operators.

Now, we recall well known and important convergence methods; statistical and almost convergence for double sequences.

A double sequence $x = (x_{ij})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $J = J(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $|x_{ij} - L| < \varepsilon$ whenever $i, j > J$, where L is called the P-limit of x and denoted by $P - \lim\limits_{i,j} x_{ij} = L$ (see [16]). We shall call such an x, briefly, "P-convergent". A double sequence is called bounded if there exists a positive number N such that $|x_{ij}| \leq N$ for all $(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Statistical convergence of single sequences was introduced by Fast [8] and Steinhaus [17], independently and studied by many authors. This concept was extended to the double sequences by Moricz [13]. If $E \subset \mathbb{N}^2$ is a two-dimensional subset of positive integers and $|D|$ denotes the cardinality of D, then the double natural density of E is given by

$$\delta_2(E) := P - \lim_{n,k} \frac{|\{i \leq n,\ j \leq k : (i,j) \in E\}|}{nk},$$

if it exists. The number sequence $x = (x_{ij})$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set

$$E := E_{nk}(\varepsilon) := \{i \leq n,\ j \leq k :\ |x_{ij} - L| \geq \varepsilon\}$$

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has natural density zero; in that case we write $st_2 - \lim_{i,j} x_{ij} = L$. Clearly, a P-convergent double sequence is statistically convergent to the same value but its converse is not always true. Also, note that a statistically convergent double sequence may not be bounded.

The definition of almost convergence for double sequences was introduced by Moricz and Rhoades [14] as follows.

A double sequence $x = (x_{ij})$ of real numbers is said to be almost convergent to a limit L if

$$P - \lim_{n,k} \sup_{p,q>0} \left| \frac{1}{nk} \sum_{i=p}^{p+n-1} \sum_{j=q}^{q+k-1} x_{ij} - L \right| = 0.$$

In this case L is called the F_2-limit of x and denoted by $F_2 - \lim_{i,j} x_{ij} = L$. Note that a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent and every almost convergent double sequence is bounded.

Lazic and Jovovic defined the K_a-convergence for single sequences in 1993 [12]. Now, we extend this idea to double sequences. This new convergence method is associated to the four dimensional matrix

$$A = \left(\begin{array}{cccc} \left(\begin{array}{cccc} a_{11} & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{array} \right) & \left(\begin{array}{ccccc} a_{12} & a_{11} & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{array} \right) & \left(\begin{array}{ccccc} a_{13} & a_{12} & a_{11} & 0 & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \end{array} \right) & \cdots \\ \left(\begin{array}{cccc} a_{12} & 0 & 0 & . \\ a_{11} & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{array} \right) & \left(\begin{array}{cccc} a_{22} & a_{21} & 0 & . \\ a_{12} & a_{11} & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{array} \right) & \left(\begin{array}{ccccc} a_{23} & a_{22} & a_{21} & 0 & . \\ a_{13} & a_{12} & a_{11} & 0 & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \end{array} \right) & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{array} \right).$$

Let $a = (a_{nk})$ and $x = (x_{nk})$ are double sequences, set $K_a^2(x) = y$, where $y = (y_{nk})$ and

$$y_{nk} = \sum_{i=1}^{n} \sum_{j=1}^{k} a_{n-i+1 k-j+1} x_{ij} \; (n, k = 1, 2, 3, ...).$$

Then it is said that $y = (y_{nk})$ is the K_a^2-*transformation* of the double sequence $x = (x_{nk})$.

Definition 1 *The double sequence $x = (x_{nk})$ of real numbers is K_a^2-convergent to the number L if, its K_a^2-transformation $y = (y_{nk})$ converges to the number L in Pringsheim's sense, i.e. $P - \lim_{n,k} y_{nk} = L$, and we denote this limit by $K_a^2 - \lim_{n,k} x_{nk} = L$.*

The proof of the following proposition can be easily established from the results concerning the general matrix transformation for double sequences. So, we omit it.

Proposition 1 *Let $a = (a_{nk})$ be a double sequence and assume that*

$$P - \lim_{i,j} \sum_{n=1}^{i} \sum_{k=1}^{j} |a_{nk}| \text{ exists} \tag{1}$$

and

$$\text{there exists a positive integer } M \text{ such that } \sum_{(n,k)\in\mathbb{N}^2} |a_{nk}| < M. \tag{2}$$

(i) *If $x=(x_{nk})$ is P-convergent, $P-\lim_{n,k} x_{nk}=L$ and the conditions (1) and (2) are satisfied, then*

$$K_a^2-\lim_{n,k} x_{nk}=L\sum_{(n,k)\in\mathbb{N}^2} a_{nk},$$

(ii) *A convergence method K_a^2 is regular if and only if the conditions (1), (2) and*

$$\sum_{(n,k)\in\mathbb{N}^2} a_{nk}=1 \tag{3}$$

are valid.

Now, the question arises in the theory of double sequences, which concerns the relationship, if any, between statistical convergence, almost convergence and K_a-convergence. Our answer is "these concepts overlap, but none is implied by the other" and it is important to say that, for these three convergence methods, if a double sequence is bounded convergent then it is statistical convergent, almost convergent and K_a-convergent, too. The following double sequence $x=(x_{nk})$ is K_a^2-convergent, despite this, it is not P-convergent and also, it is not statistical and almost convergent.

Example 1 *Let $a=(a_{nk})$ given by* $(a_{nk})=\begin{pmatrix} -1 & 0 & 0 & . \\ 0 & -1 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}$ *and let $x=(x_{nk})$ given by*

$$(x_{nk})=\begin{pmatrix} \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & . \\ 0 & -\frac{1}{2} & -\frac{2}{2} & -\frac{3}{2} & -\frac{4}{2} & -\frac{5}{2} & . \\ 0 & 0 & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & . \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{2}{2} & -\frac{3}{2} & . \\ . & . & . & . & . & . & . \end{pmatrix}.$$

Then

$$(y_{nk})=\left(\sum_{i=1}^{n}\sum_{j=1}^{k} a_{n-i+1k-j+1}x_{ij}\right)=\begin{pmatrix} -\frac{1}{2} & -\frac{2}{2} & -\frac{3}{2} & -\frac{4}{2} & . \\ 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \end{pmatrix}.$$

Hence, we can write that $K_a^2-\lim_{n,k} x_{nk}=0$. However, it can be easily seen that, $x=(x_{nk})$ is not P-convergent. Also, x is neither statistical convergent nor almost convergent.

2 Korovkin Type Theorem via K_a^2-convergence

In this section we study a Korovkin type approximation theorem via K_a^2-convergence of positive linear operators.

Let $I^2=I\times I$ and I be a compact subset of the real numbers, $C(I^2)$ be the two-dimensional space of all continuous real valued functions on I^2 and $\|f\|_{C(I^2)}$ denote the usual supremum norm of f . Let L be a linear operator from $C(I^2)$ into itself. Then, we say that L is positive linear operator on condition that $f\geq 0$ implies $L(f)\geq 0$. Also, we mean the value of $L(f)$ at a point $(x,y)\in I^2$ by $L(f(s,t);x,y)$ or $L(f;x,y)$. Throughout the paper, we also use the following test functions

$$f_0(x,y)=1,\quad f_1(x,y)=x,\quad f_2(x,y)=y,\ \ f_3(x,y)=x^2+y^2.$$

Now, we begin with the following well-known Korovkin type theorems.

Theorem 2 **([18])** *Suppose that (L_{nk}) is a double sequence of positive linear operators from $C\left(I^2\right)$ into itself, satisfying the following conditions:*

$$P-\lim_{n,k}\|L_{nk}(f_r)-f_r\|_{C(I^2)}=0, \quad (r=0,1,2,3).$$

Then, for all $f\in C\left(I^2\right)$,

$$P-\lim_{n,k}\|L_{nk}(f)-f\|_{C(I^2)}=0.$$

Theorem 3 **([6])** *Assume that (L_{nk}) is a double sequence of positive linear operators acting from $C\left(I^2\right)$ into itself, satisfying the following conditions:*

$$st_2-\lim_{n,k}\|L_{nk}(f_r)-f_r\|_{C(I^2)}=0, \quad (r=0,1,2,3).$$

Then, for all $f\in C\left(I^2\right)$,

$$st_2-\lim_{n,k}\|L_{nk}(f)-f\|_{C(I^2)}=0.$$

Theorem 4 **([1])** *Suppose that (L_{nk}) is a double sequence of positive linear operators from $C\left(I^2\right)$ into itself, satisfying the following conditions:*

$$F_2-\lim_{n,k}\|L_{nk}(f_r)-f_r\|_{C(I^2)}=0, \ (r=0,1,2,3).$$

Then, for all $f\in C\left(I^2\right)$,

$$F_2-\lim_{n,k}\|L_{nk}(f)-f\|_{C(I^2)}=0.$$

Now we give the following Korovkin type approximation theorem for method K_a^2 that is our main result.

Theorem 5 *Let $a=(a_{nk})$ be a double sequence and the conditions (1) and (2) are satisfied. Suppose that (L_{nk}) is a double sequence of positive linear operators acting from $C\left(I^2\right)$ into itself, satisfying the following conditions:*

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}|a_{n-i+1k-j+1}|\,L_{ij}(f_r)-f_r\right\|_{C(I^2)}=0, \quad (r=0,1,2,3). \tag{4}$$

Then, for all $f\in C\left(I^2\right)$, we have

$$K_a^2-\lim_{n,k}\|L_{nk}(f)-f\|_{C(I^2)}=0, \quad i.e.,$$

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}a_{n-i+1k-j+1}L_{ij}(f)-f\right\|_{C(I^2)}=0.$$

Proof. Let $f\in C\left(I^2\right)$ and $(x,y)\in I^2$ be fixed. By the continuity of f on I^2, we can write

$$|f(x,y)|\leq M_f. \tag{5}$$

Therefore

$$|f(s,t)-f(x,y)|\leq 2M_f.$$

Also, since f is continuous on I^2, we write that for every $\varepsilon>0$, there exists a number $\delta>0$ such that $|f(s,t)-f(x,y)|<\varepsilon$ holds for all $(s,t)\in I^2$ satisfying $|s-x|<\delta$ and $|t-y|<\delta$. Hence, we get

$$|f(s,t)-f(x,y)|<\varepsilon+\frac{2M_f}{\delta^2}\left\{(s-x)^2+(t-y)^2\right\}. \tag{6}$$

This means

$$-\varepsilon-\frac{2M_f}{\delta^2}\left\{(s-x)^2+(t-y)^2\right\}<f(s,t)-f(x,y)<\varepsilon+\frac{2M_f}{\delta^2}\left\{(s-x)^2+(t-y)^2\right\}.$$

Using the linearity and the positivity of the operators L_{nk} and the inequality (6), we get

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\sum_{j=1}^{k}a_{n-i+1k-j+1}L_{ij}\left(f;x,y\right)-f\left(x,y\right)\right|=\left|\sum_{i=1}^{n}\sum_{j=1}^{k}a_{n-i+1k-j+1}\left(L_{ij}\left(f\left(s,t\right);x,y\right)\right.\right.\\
&\quad\left.-L_{ij}\left(f\left(x,y\right);x,y\right)+L_{ij}\left(f\left(x,y\right);x,y\right)\right)-f\left(x,y\right)\big|\\
\leq\ &\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|L_{ij}\left(\left|f\left(s,t\right)-f\left(x,y\right)\right|;x,y\right)\\
&+\left|f\left(x,y\right)\right|\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|L_{ij}\left(f_0;x,y\right)-f_0\left(x,y\right)\right|\\
\leq\ &\varepsilon+\left(\varepsilon+M_f+\frac{2M_f\left\|f_3\right\|_{C(I^2)}}{\delta^2}\right)\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|L_{ij}\left(f_0;x,y\right)-f_0\left(x,y\right)\right|\\
&+\frac{4M_f\left\|f_1\right\|_{C(I^2)}}{\delta^2}\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|L_{ij}\left(f_1;x,y\right)-f_1\left(x,y\right)\right|\\
&+\frac{4M_f\left\|f_2\right\|_{C(I^2)}}{\delta^2}\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|L_{ij}\left(f_2;x,y\right)-f_2\left(x,y\right)\right|\\
&+\frac{2M_f}{\delta^2}\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|L_{ij}\left(f_3;x,y\right)-f_3\left(x,y\right)\right|.
\end{aligned}$$

Then taking supremum over $(x,y)\in I^2$, we have

$$\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}a_{n-i+1k-j+1}L_{ij}\left(f\right)-f\right\|_{C(I^2)}\leq\varepsilon+K\left\{\sum_{r=0}^{3}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|L_{ij}\left(f_r\right)-f_r\right\|_{C(I^2)}\right\}$$

where

$$K:=\max\left\{\varepsilon+M_f+\frac{2M_f\left\|f_3\right\|_{C(I^2)}}{\delta^2},\frac{4M_f\left\|f_1\right\|_{C(I^2)}}{\delta^2},\frac{4M_f\left\|f_1\right\|_{C(I^2)}}{\delta^2},\frac{2M_f}{\delta^2}\right\}.$$

Then using the hypothesis (4), we get

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}a_{n-i+1k-j+1}L_{ij}\left(f\right)-f\right\|_{C(I^2)}=0.$$

The proof is complete. ■

We now present an example of a sequence of positive linear operators that satisfies the conditions of Theorem 5 but does not satisfy the conditions of Theorem 2, Theorem 3 and Theorem 4.

Example 2 *Let* $a=(a_{nk})$ *given by* $(a_{nk})=\begin{pmatrix} -1 & 0 & 0 & . \\ 0 & -1 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}$ *and* $x=(x_{nk})$ *given by*

$$(x_{nk})=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & . \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & . \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & . \\ 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & . \\ 1 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & . \\ . & . & . & . & . & . & . & . \end{pmatrix}.$$

Observe now that, $\sum_{(n,k)\in\mathbb{N}^2}|a_{nk}|=2$ *and* $\sum_{(n,k)\in\mathbb{N}^2}a_{nk}=-2$. *Then, we consider the double Bernstein operators:*

$$B_{nk}(f;x,y)=\sum_{i=0}^{n}\sum_{j=0}^{k}f\left(\frac{i}{n},\frac{j}{k}\right)\binom{n}{i}\binom{k}{j}x^iy^j(1-x)^{n-i}(1-y)^{k-j}$$

where $(x,y)\in I^2=[0,1]^2=[0,1]\times[0,1]$, $f\in C\left(I^2\right)$ *and* $n,k\in\mathbb{N}$. *Using these polynomials, we introduce the following positive linear operators on* $C\left(I^2\right)$:

$$T_{nk}(f;x,y)=x_{nk}B_{nk}(f;x,y),\qquad (x,y)\in I^2,\quad f\in C\left(I^2\right). \tag{7}$$

We now claim that

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}|a_{n-i+1k-j+1}|\,T_{ij}(f_r)-f_r\right\|_{C(I^2)}=0\ , \tag{8}$$

for each $r=0,1,2,3$. *Indeed, we first observe that*

$$\begin{aligned} T_{nk}(f_0;x,y) &= x_{nk}f_0(x,y), \\ T_{nk}(f_1;x,y) &= x_{nk}f_1(x,y), \\ T_{nk}(f_2;x,y) &= x_{nk}f_2(x,y), \\ T_{nk}(f_3;x,y) &= x_{nk}\left[f_3(x,y)+\frac{x-x^2}{n}+\frac{y-y^2}{k}\right]. \end{aligned}$$

Hence,

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{k}|a_{n-i+1k-j+1}|\,T_{ij}(f_0;x,y)-f_0(x,y)\right|=\left|\sum_{i=1}^{n}\sum_{j=1}^{k}|a_{n-i+1k-j+1}|\,x_{nk}-1\right|.$$

Then we get the following double sequence

$$\left(\left|\sum_{i=1}^{n}\sum_{j=1}^{k}|a_{n-i+1k-j+1}|\,x_{nk}-1\right|\right)=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & . \\ 0 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{pmatrix}. \tag{9}$$

We get

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}|a_{n-i+1k-j+1}|\,T_{ij}(f_0)-f_0\right\|_{C(I^2)}=0,$$

which guarantees that (8) holds true for $r=0$. Also, since

$$\begin{aligned}\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|T_{ij}\left(f_1;x,y\right)-f_1\left(x,y\right)\right| &= \left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|xx_{nk}-x\right| \\ &= |x|\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|x_{nk}-1\right|,\end{aligned}$$

then

$$\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|T_{ij}\left(f_1\right)-f_1\right\|_{C(I^2)}=\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|x_{nk}-1\right|.$$

By (9), we have

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|T_{ij}\left(f_1\right)-f_1\right\|_{C(I^2)}=0.$$

Similary, we have

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|T_{ij}\left(f_2\right)-f_2\right\|_{C(I^2)}=0.$$

Hence (8) is valid for $r=1,2$. Finally, we get

$$\begin{aligned}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|T_{ij}\left(f_3\right)-f_3\right\|_{C(I^2)} &\leq 2\left|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|x_{nk}-1\right| \\ +\frac{1}{4}\left\{\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|\frac{x_{nk}}{n}+\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|\frac{x_{nk}}{k}\right\}&, \qquad (10)\end{aligned}$$

$$\left(\frac{x_{nk}}{n}\right)=\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & . \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & . \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & . \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & . \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 & \frac{1}{10} & \frac{1}{10} & \frac{1}{5} & . \\ . & . & . & . & . & . & . & .\end{pmatrix},$$

$P-\lim_{n,k}\frac{x_{nk}}{n}=0$ *and similary* $P-\lim_{n,k}\frac{x_{nk}}{k}=0$. *Then, from Proposition 1, we obtain*

$$K_a^2-\lim_{n,k}\frac{x_{nk}}{n}=0 \text{ and } K_a^2-\lim_{n,k}\frac{x_{nk}}{k}=0. \qquad (11)$$

From the inequality (10) and using (9), (11), we have

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{n-i+1k-j+1}\right|T_{ij}\left(f_3\right)-f_3\right\|_{C(I^2)}=0.$$

So, our claim (8) holds true for each $r=0,1,2,3$. Now, from (8), we can say that our sequence (T_{nk}) defined by (7) satisfies all assumptions of Theorem 5. Using these facts, we conclude that

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k}a_{n-i+1k-j+1}T_{ij}\left(f\right)-f\right\|_{C(I^2)}=0$$

holds for any $f \in C\left(I^2\right)$ *. However, since* $\|T_{nk}(f_0) - f_0\|_{C(I^2)} = |x_{nk} - 1|$ *and a double sequence* $\left(\|T_{nk}(f_0) - f_0\|_{C(I^2)}\right)$ *does not converge in Pringsheim's sense, Theorem 2 (the classical Korovkin theorem for double sequences) does not work for the sequence* (T_{nk}) *. The double sequence* $x = (x_{nk})$ *can be also given as follows:*

$$x_{nk} = \begin{cases} \frac{1}{2}, & \text{if } k = n \text{ or } k = n+1, \\ 1, & \text{if } k > n+1 \text{ and } n \text{ is odd}, \\ 0, & \text{if } k > n+1 \text{ and } n \text{ is even}, \\ 1, & \text{if } k < n \text{ and } k \text{ is odd}, \\ 0, & \text{if } k < n \text{ and } k \text{ is even}, \end{cases}$$

$n, k = 1, 2, 3, \ldots$ *. Hence, we get that the double sequence* $\left(\|T_{nk}(f_0) - f_0\|_{C(I^2)}\right)$ *is not statistically convergent and Theorem 3 (the statistical Korovkin theorem) does not work for the sequence* (T_{nk}) *. Also, since*

$$P - \lim_{n,k} \left\| \frac{1}{nk} \sum_{i=p}^{p+n-1} \sum_{j=q}^{q+k-1} T_{ij}(f_0) - f_0 \right\|_{C(I^2)} \neq 0, (p, q \in \mathbb{N}),$$

Theorem 4 does not work for the sequence (T_{nk}) *, too.*

3 Rate of Convergence

The main aim of this section is to study the rate of K_a^2-convergence with the aid of the modulus of continuity that is defined by

$$\omega(f, \delta) = \sup_{\sqrt{(s-x)^2 + (t-y)^2} \leq \delta} |f(s,t) - f(x,y)| \quad (\delta > 0), \ f \in C(I^2).$$

It is readily seen that, for any $\lambda > 0$ and for all $f \in C(I^2)$

$$\omega(f, \lambda\delta) \leq (1 + [\lambda])\, \omega(f, \delta)$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ. Then the result is stated as follows.

Theorem 6 *Let* $a = (a_{nk})$ *be a double sequence and the conditions (1) and (2) are satisfied. Assume that* (L_{nk}) *be a double sequence of positive linear operators acting from* $C\left(I^2\right)$ *into itself, satisfying the following conditions:*

$$(i) \;\; P - \lim_{n,k} \left\| \sum_{i=1}^{n} \sum_{j=1}^{k} |a_{n-i+1k-j+1}| L_{ij}(f_0) - f_0 \right\|_{C(I^2)} = 0,$$

$$(ii) \;\; P - \lim_{n,k} \omega(f; \alpha_{n,k}) = 0,$$

where

$$\alpha_{n,k} := \sqrt{\left\| \sum_{i=1}^{n} \sum_{j=1}^{k} |a_{n-i+1k-j+1}| L_{ij}\left((s - .)^2 + (t - .)^2\right) \right\|_{C(I^2)}}.$$

Then for all $f \in C\left(I^2\right)$,

$$K_a^2 - \lim_{n,k} \|L_{nk}(f) - f\|_{C(I^2)} = 0.$$

Proof. Let $f \in C\left(I^2\right)$ and $(x,y) \in I^2$ be fixed. Using (5), the properties of ω, and the positivity and monotonicity of L_{nk}, we get that

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\sum_{j=1}^{k} a_{n-i+1k-j+1} L_{ij}\left(f;x,y\right) - f\left(x,y\right)\right| \leq \sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(\left|f\left(s,t\right) - f\left(x,y\right)\right|;x,y\right) \\
&+M_f \left|\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(f_0;x,y\right) - f_0\left(x,y\right)\right| \\
\leq\; & \sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(\omega\left(f;\delta\frac{\sqrt{\left(s-x\right)^2+\left(t-y\right)^2}}{\delta}\right);x,y\right) \\
&+M_f \left|\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(f_0;x,y\right) - f_0\left(x,y\right)\right| \\
\leq\; & \omega\left(f;\delta\right)\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(1+\frac{\left(s-x\right)^2+\left(t-y\right)^2}{\delta^2};x,y\right) \\
&+M_f \left|\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(f_0;x,y\right) - f_0\left(x,y\right)\right| \\
\leq\; & \omega\left(f;\delta\right)\left|\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(f_0;x,y\right) - f_0\left(x,y\right)\right| + \omega\left(f;\delta\right) \\
&+\frac{\omega\left(f;\delta\right)}{\delta^2}\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(\left(s-x\right)^2+\left(t-y\right)^2;x\right) \\
&+M_f \left|\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(f_0;x,y\right) - f_0\left(x,y\right)\right|.
\end{aligned}$$

Then taking supremum over $(x,y) \in I^2$, we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{n}\sum_{j=1}^{k} a_{n-i+1k-j+1} L_{ij}\left(f\right) - f\right\|_{C(I^2)} \\
\leq\; & \omega\left(f;\delta\right)\left\|\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(f_0\right) - f_0\right\| + 2\omega\left(f;\delta\right) + M_f\left\|\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(f_0\right) - f_0\right\|
\end{aligned}$$

where

$$\delta := \alpha_{n,k} := \sqrt{\left\|\sum_{i=1}^{n}\sum_{j=1}^{k} \left|a_{n-i+1k-j+1}\right| L_{ij}\left(\left(s-.\right)^2+\left(t-.\right)^2\right)\right\|_{C(I^2)}}.$$

Then, from the hypotheses, we conclude that

$$P-\lim_{n,k}\left\|\sum_{i=1}^{n}\sum_{j=1}^{k} a_{n-i+1k-j+1} L_{ij}\left(f\right) - f\right\|_{C(I^2)} = 0,$$

we obtain the assertion. ■

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