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Concavity And Generalized Entropy*

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Abstract

We introduce a two-parameter class of entropic functions involving Euler's gamma function and solve a concavity problem stated by Ferreira & Tenreiro Machado in 2019.

1 Introduction

The concept of entropy plays a central role in many-particle physics; see [6]. The archetype of entropy is the nowadays known as Boltzmann-Gibbs (BG) entropy, namely,

$$S_{BG}(p_1, ..., p_n) = -k \sum_{i=1}^{n} p_i \log(p_i), \tag{1}$$

where $k = 1.3807 \times 10^{-23} J/K$ is the Boltzmann constant, n is the number of microstates consistent with the macroscopic constraints of a given thermodynamical system, and p_i is the probability that the system is in the microstate i. Apart from the constant factor, the mathematical properties of S_{BG} follow from the study of the function $f(x) = -x \log(x)$. In the past 30 years, many formulations appeared in the literature extending the formula (1) (see, for example, [1, 2, 6, 7, 8, 12]). Here, we consider entropic functions of the form $S(p_1, ..., p_n) = \sum_{i=1}^{n} f(p_i)$, where f is defined on $[0, 1]$. Now, the question arises: can f be any function defined on $[0, 1]$? From the mathematical perspective the answer is indeed "yes", but from the physical perspective the answer is "no". For instance, an entropy is a function that measures some kind of feature in a physical system (such as energy that cannot produce work, disorder, uncertainty, randomness, complexity, etc.), therefore, f should be a nonnegative function. However, there does not exist a complete list of properties that an entropic function must satisfy. This is the reason why different entropic functions come on the scene and attract the attention of numerous researchers. It is nevertheless usual for an entropic function S_n to satisfy the following three Shannon–Khinchin axioms (see [6, section 2]):

Let $\Delta_n = \{(p_1, ..., p_n) \in \mathbb{R}^n \,|\, p_i \geq 0,\, p_1 + ... + p_n = 1\}$.

(i) S_n is nonnegative and continuous on Δ_n.

(ii) For all $(p_1, ..., p_n) \in \Delta_n$: $\quad S_n(p_1, ..., p_n) \leq S_n(1/n, ..., 1/n)$.

(iii) For all $(p_1, ..., p_n) \in \Delta_n$: $\quad S_{n+1}(p_1, ..., p_n, 0) = S_n(p_1, ..., p_n)$.

This paper is motivated by a recently published article of Ferreira & Tenreiro Machado [7]. Inspired by Abe [1] and by fractional calculus theory [9] they studied the entropic function

$$S_\alpha(p_1, ..., p_n) = \sum_{i=1}^{n} p_i \frac{\Gamma(1 - \log(p_i))}{\Gamma(1 - \alpha - \log(p_i))} \quad (0 < \alpha \leq 1), \tag{2}$$

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and applied it to study the Dow Jones Industrial Average taking into account the variation of the parameter α (see [7, section 3] for details). Here, as usual, Γ denotes Euler's gamma function. Numerous computer calculations led Ferreira & Tenreiro Machado to the conjecture that the function

$$R_\alpha(x) = x\frac{\Gamma(1-\log(x))}{\Gamma(1-\alpha-\log(x))} \quad (0<\alpha\leq 1) \tag{3}$$

is concave on $(0,1]$, but "a rigorous proof of that fact was not yet obtained" [7, p. 3]. It is our aim to prove that for each parameter $\alpha \in (0,1]$ the function R_α is strictly concave on $[0,1]$. An application of Jensen's inequality for concave functions gives that S_α satisfies the second Shannon–Khinchin axiom.

Actually, we do not only prove that R_α is concave, but we show that for two real parameters a and b and a function f satisfying certain assumptions, the function

$$F_{a,b,f}(x) = x\frac{\Gamma(a+f(x))}{\Gamma(b+f(x))}$$

is concave on $(0,1]$. Inspired by this result we introduce a two-parameter class of entropy functions which includes (2),

$$S_{a,b,f}(p_1,...,p_n) = \sum_{i=1}^n p_i \frac{\Gamma(a+f(p_i))}{\Gamma(b+f(p_i))}. \tag{4}$$

Since

$$S_{1,0,-\log}(p_1,...,p_n) = -\sum_{i=1}^n p_i \log(p_i),$$

we conclude that the Boltzmann–Gibbs entropy is a special case of (4).

In the next section, we present two helpful lemmas. Our main results are given in Section 3.

2 Two Lemmas

The digamma function ψ is the logarithmic derivative of Euler's gamma function,

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad (x>0),$$

and the derivatives of ψ are known as polygamma functions. We have the series and integral representations

$$\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{i=0}^\infty \frac{1}{(x+i)^{n+1}} = (-1)^{n+1}\int_0^\infty e^{-xt}\frac{t^n}{1-e^{-t}}dt \quad (n\in\mathbb{N}, x>0).$$

The main properties of these functions are collected, for instance, in [3, section 6]. In this section, we present two useful inequalities involving ψ', ψ'' and ψ'''.

Lemma 1 *For $x>0$, we have*

$$L(x) = 2\psi'^3(x) - \psi'(x)\psi'''(x) + \psi''^2(x) > 0.$$

Proof. Let $x>0$. Using

$$\psi'(x) = \frac{1}{x^2} + \psi'(x+1), \quad \psi''(x) = -\frac{2}{x^3} + \psi''(x+1), \quad \psi'''(x) = \frac{6}{x^4} + \psi'''(x+1), \tag{5}$$

we obtain

$$L(x) = 2\Big(\frac{1}{x^2} + \psi'(x+1)\Big)^3 - \Big(\frac{1}{x^2} + \psi'(x+1)\Big)\Big(\frac{6}{x^4} + \psi'''(x+1)\Big) + \Big(-\frac{2}{x^3} + \psi''(x+1)\Big)^2$$

and

$$x^3L(x) = (2A^3 - AC + B^2)x^3 + (6A^2 - C)x - 4B \tag{6}$$

with

$$A = \psi'(x+1), \quad B = \psi''(x+1), \quad C = \psi'''(x+1).$$

The following estimates for ψ', ψ'' and ψ''' are given in [4]:

$$\frac{1}{x} + \frac{1}{2x^2} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}, \tag{7}$$

$$\frac{1}{x^2} + \frac{1}{x^3} < -\psi''(x) < \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{2x^4}, \tag{8}$$

$$\frac{2}{x^3} + \frac{3}{x^4} < \psi'''(x) < \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5}. \tag{9}$$

We set $\lambda = 1/(x+1)$. Applying (7), (8) and (9) we conclude from (6) that

$$\begin{aligned} x^3L(x) \quad > \quad & \left[2\left(\lambda + \frac{1}{2}\lambda^2\right)^3 - \left(\lambda + \frac{1}{2}\lambda^2 + \frac{1}{6}\lambda^3\right)\left(2\lambda^3 + 3\lambda^4 + 2\lambda^5\right) + (\lambda^2 + \lambda^3)^2\right] x^3 \\ & + \left[6\left(\lambda + \frac{1}{2}\lambda^2\right)^2 - \left(2\lambda^3 + 3\lambda^4 + 2\lambda^5\right)\right] x + 4(\lambda^2 + \lambda^3) \\ = \quad & \frac{Q(x)}{12(x+1)^8} \end{aligned}$$

with

$$Q(x) = 24x^8 + 216x^7 + 858x^6 + 1973x^5 + 2902x^4 + 2809x^3 + 1728x^2 + 606x + 96.$$

It follows that $L(x) > 0$ for $x > 0$. ■

Lemma 2 *Let*

$$M(x) = -1 + \frac{2}{x} + \frac{\psi''(x)}{\psi'(x)} \quad (x > 0), \quad M(0) = \lim_{x \to 0+} M(x) = -1. \tag{10}$$

For $x \geq 0$, we have $M(x) < 0$.

Proof. From

$$x\psi'(x)M(x) = (2-x)\psi'(x) + x\psi''(x),$$

we conclude that $M(x) < 0$ for $x \geq 2$. Next, let $0 < x < 2$ and $\lambda = 1/(x+1)$. Using (5), (7) and (8) gives

$$\begin{aligned} x\psi'(x)M(x) \quad &= \quad (2-x)\Big(\frac{1}{x^2} + \psi'(x+1)\Big) + x\Big(-\frac{2}{x^3} + \psi''(x+1)\Big) \\ &= \quad -\frac{1}{x} + (2-x)\psi'(x+1) + x\psi''(x+1) \\ &< \quad -\frac{1}{x} + (2-x)\Big(\lambda + \frac{1}{2}\lambda^2 + \frac{1}{6}\lambda^3\Big) - x(\lambda^2 + \lambda^3) = -\frac{R(x)}{6x(x+1)^3} \end{aligned}$$

with

$$R(x) = 6x^4 + 15x^3 + 10x^2 + 2(2-x) + 2 > 0.$$

Thus, $M(x) < 0$ for $x > 0$. ■

3 Main Results

We are now in a position to present a new class of concave functions, defined in terms of the gamma function.

Theorem 1 *Let $0 \le b < a \le 1$ and let $f : (0,1] \to \mathbb{R}$ be a continuous function which is twice differentiable on $(0,1)$. If*

$$f(x) > 0 \quad \text{and} \quad \frac{2}{x}f'(x) + f'^2(x) + f''(x) \le 0 \quad \text{for} \quad x \in (0,1), \tag{11}$$

then

$$F_{a,b,f}(x) = x\frac{\Gamma(a+f(x))}{\Gamma(b+f(x))}$$

is concave on $(0,1]$.

Proof. Since $\Gamma(x+1) = x\Gamma(x)$, we get

$$F_{a,0,f}(x) = xf(x)\frac{\Gamma(a+f(x))}{\Gamma(1+f(x))}.$$

This implies that $F_{a,b,f}$ is continuous on $(0,1]$ not only if $b>0$, but also if $b=0$.

We set $F = F_{a,b,f}$. Let $x \in (0,1)$ and $0 \le b < a \le 1$. By differentiation we obtain

$$\begin{aligned}\frac{1}{F(x)}F''(x) &= f'^2(x)\big([\psi(a+f(x)) - \psi(b+f(x))]^2 + \psi'(a+f(x)) - \psi'(b+f(x))\big) \\ &\quad + \Big(\frac{2}{x}f'(x) + f''(x)\Big)[\psi(a+f(x)) - \psi(b+f(x))].\end{aligned} \tag{12}$$

Since $a > b$ and ψ is increasing, we conclude from (11) and (12) that

$$\frac{1}{F(x)}F''(x) \le f'^2(x)H(a,b;f(x)) \tag{13}$$

with

$$H(a,b;t) = [\psi(a+t) - \psi(b+t)]^2 - [\psi(a+t) - \psi(b+t)] + \psi'(a+t) - \psi'(b+t).$$

Next, we show that $H(a,b;t) < 0$ for $t > 0$. By partial differentiation we find

$$\frac{1}{\psi'(b+t)}\frac{\partial}{\partial b}H(a,b;t) = -2[\psi(a+t) - \psi(b+t)] + 1 - \frac{\psi''(b+t)}{\psi'(b+t)} = P(a,b;t), \quad \text{say.}$$

Using Lemma 1 gives

$$\psi'^2(b+t)\frac{\partial}{\partial b}P(a,b;t) = L(b+t) > 0.$$

Thus,

$$P(a,b;t) > P(a,0;t) = -2[\psi(a+t) - \psi(t)] + 1 - \frac{\psi''(t)}{\psi'(t)} \ge -2[\psi(1+t) - \psi(t)] + 1 - \frac{\psi''(t)}{\psi'(t)} = -M(t),$$

where M is defined in (10). Applying Lemma 2 yields $P(a,b;t) > 0$. It follows that

$$H(a,b;t) < H(a,a;t) = 0. \tag{14}$$

Using (13) and (14) (with $t = f(x)$) leads to $F''(x) \le 0$ for $x \in (0,1)$. This implies that F is concave on $(0,1]$. ■

Remark 1 *The proof of Theorem 1 reveals that if $f'(x) \ne 0$ for $x \in (0,1)$, then $F_{a,b,f}$ is strictly concave on $(0,1]$.*

As a special case of the following corollary we obtain that the function R_α $(0 < \alpha \le 1)$ which is defined in (3) is concave on $(0, 1]$.

Corollary 1 *Let $0 \le b < a \le 1$. The function*

$$G_{a,b}(x) = x\frac{\Gamma(a - \log(x))}{\Gamma(b - \log(x))}$$

is strictly concave on $[0, 1]$.

Proof. Let $f(x) = -\log(x)$. Then

$$f(x) > 0 \quad \text{and} \quad \frac{2}{x}f'(x) + f'^2(x) + f''(x) = 0 \quad \text{for} \quad x \in (0, 1).$$

Applying Theorem 1 and Remark 1 yields that $G_{a,b}$ is strictly concave on $(0, 1]$.

Let $x > 0$ and $t = -\log(x)$. We obtain

$$G_{a,b}(x) = \frac{t^{a-b}}{e^t} \cdot t^{b-a}\frac{\Gamma(a+t)}{\Gamma(b+t)}. \tag{15}$$

Since

$$\lim_{t\to\infty} t^{b-a}\frac{\Gamma(a+t)}{\Gamma(b+t)} = 1,$$

we get from (15)

$$G_{a,b}(0) = \lim_{x\to 0+} G_{a,b}(x) = 0. \tag{16}$$

Thus, $G_{a,b}$ is continuous on $[0, 1]$. It follows that $G_{a,b}$ is strictly concave on $[0, 1]$. ■

Remark 2 *A result of Petrović (see [10, section 1.4.7]) states that if a function g is concave on $[0, \infty)$, then, for $x, y \ge 0$,*

$$g(x + y) + g(0) \le g(x) + g(y).$$

Applying Corollary 1 and (16) yields that if $0 \le b < a \le 1$, then, for $x, y \ge 0$ with $x + y \le 1$,

$$G_{a,b}(x + y) \le G_{a,b}(x) + G_{a,b}(y).$$

This means that $G_{a,b}$ is subadditive on $[0, 1]$. Subadditive functions have interesting applications in various fields, like, for example, in functional analysis and semi-group theory; see [5] for more information on this subject.

Corollary 2 *Let $0 \le b < a \le 1$. The function*

$$\Phi_{a,b}(x) = \frac{\Gamma(a + \log(x))}{\Gamma(b + \log(x))}$$

is strictly concave on $[1, \infty)$.

Proof. Using Corollary 1 gives for $x > 1$,

$$\Phi_{a,b}(x) = xG_{a,b}(1/x) \quad \text{and} \quad \Phi''_{a,b}(x) = \frac{1}{x^3}G''_{a,b}(1/x) < 0.$$

It follows that $\Phi_{a,b}$ is strictly concave on $[1, \infty)$. ■

Finally, we provide some functions which satisfy (11) but are different from $-\log$.

Corollary 3 *Let*

$$h_{\alpha,\beta}(x) = 1 - \alpha e^{\beta/x} \quad \text{and} \quad S_{a,b;\alpha,\beta}(x) = x\frac{\Gamma(a + h_{\alpha,\beta}(x))}{\Gamma(b + h_{\alpha,\beta}(x))}.$$

If $\beta < 0 < \alpha \leq e^{-\beta}$ *and* $0 \leq b < a \leq 1$, *then* $S_{a,b;\alpha,\beta}$ *is strictly concave on* $(0, 1]$.

Proof. Let $x \in (0, 1)$. Then

$$e^{-\beta/x} h_{\alpha,\beta}(x) = e^{-\beta/x} - \alpha > e^{-\beta} - e^{-\beta} = 0, \quad h'_{\alpha,\beta}(x) = \frac{\alpha\beta}{x^2} e^{\beta/x} < 0$$

and

$$\frac{2}{x} h'_{\alpha,\beta}(x) + h'^2_{\alpha,\beta}(x) + h''_{\alpha,\beta}(x) = -\frac{\alpha\beta^2}{x^4} e^{\beta/x} h_{\alpha,\beta}(x) < 0.$$

Applying Theorem 1 with $f = h_{\alpha,\beta}$ and Remark 1 reveals that $S_{a,b;\alpha,\beta}$ is strictly concave on $(0, 1]$. ■

Remark 3 *We consider the entropy*

$$S(p_1, ..., p_n) = \sum_{i=1}^{n} p_i \frac{\Gamma(a + 1 - \alpha e^{\beta/p_i})}{\Gamma(b + 1 - \alpha e^{\beta/p_i})}. \tag{17}$$

If we set $a = 1$, $b = 0$, $\alpha = e$, $\beta = -1$ *and make use of* $\Gamma(x + 1) = x\Gamma(x)$, *then* (17) *leads to*

$$S^*(p_1, ..., p_n) = \sum_{i=1}^{n} p_i (1 - e^{1-1/p_i}).$$

This entropy function was studied by Tsekouras & Tsallis [11].

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References

[1] S. Abe, A note on the q-deformation-theoretic aspect of the generalized entropies in nonextensive physics, Phys. Lett., A 224(1997), 326–330.

[2] S. Abe, Generalized entropy optimized by a given arbitrary distribution, J. Phys. A, 36(2003), 8733–8738.

[3] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover, New York, 1965.

[4] H. Alzer, On some inequalities for the gamma and psi functions, Math. Comp., 66(1997), 373–389.

[5] H. Alzer, Sub- and superadditive properties of Euler's gamma function, Proc. Amer. Math. Soc., 135(2007), 3641–3648.

[6] J. M. Amigó, S. G. Balogh and S. Hernández, A brief review of generalized entropies, Entropy, 20(2018), Paper No. 813, 21 pp.

[7] R. A. C. Ferreira and J. Tenreiro Machado, An entropy formulation based on the generalized Liouville fractional derivative, Entropy, 21(2019), Paper No. 638, 9 pp.

[8] G. Kaniadakis, Non-linear kinetics underlying generalized statistics, Physica A, 296(2001), 405–425.

[9] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[10] D. S. Mitrinović, Analytic Inequalities, Springer, New York, 1970.

[11] G. A. Tsekouras and C. Tsallis, Generalized entropy arising from a distribution of q indices, Phys. Rev. E, 71(2005), 046144, 8 pp.

[12] M. R. Ubriaco, Entropies based on fractional calculus, Phys. Lett. A, 373(2009), 2516–2519.