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A Remark On The Crux Problem 1052*[*]

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Abstract

An interesting family of infinite series is evaluated exactly using standard methods from complex analysis.

1 Introduction and Results

The following problem was given to students in an examination paper at Trinity College, Cambridge, in June 1901: Prove that

$$\frac{1}{1^2 \cdot 3^3 \cdot 5^2} - \frac{1}{3^2 \cdot 5^3 \cdot 7^2} + \frac{1}{5^2 \cdot 7^3 \cdot 9^2} - \cdots = \frac{1}{9} - \frac{\pi}{2^6} - \frac{\pi^3}{2^9}. \tag{1}$$

The problem appeared as Problem 1052* in Crux [3]. Szekeres gave three proofs of the statement in [4]. In this note, we generalize the problem in the following way: For $k \in \mathbb{Z}$, we consider the family of infinite series given by

$$S(k) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-2k+1)^2(2n+1)^3(2n+2k+1)^2}. \tag{2}$$

We derive a closed-form for the series. We recommend the article [2], its content is related to the present problem. It is obvious that $S(1)$ corresponds to the LHS of equation (1). Also, we have the symmetry relation $S(k) = S(-k)$. Finally, we mention that $S(0)$ can be evaluated using Dirichlet's beta function (or L-function) as follows:

$$S(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^7} = 1 - \beta(7) = 1 - \frac{61}{184320}\pi^7,$$

where $\beta(s)$ is given by (see [1] or [5])

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad Re(s) > 1,$$

and where we have used the relation between $\beta(s)$ and Euler numbers

$$\beta(2p+1) = \frac{(-1)^p E_{2p} \pi^{2p+1}}{4^{p+1}(2p)!}, \quad p \in \mathbb{N}.$$

Our main result is the following theorem.

Theorem 1 *Let $k \in \mathbb{Z}$ and let $S(k)$ be defined as in (2). Then,*

$$S(k) = \begin{cases} \frac{1}{(2k-1)^2(2k+1)^2} - \frac{\pi^3}{2^9 k^4}, & k \text{ even} \\ \frac{1}{(2k-1)^2(2k+1)^2} - \frac{\pi}{2^6 k^6} - \frac{\pi^3}{2^9 k^4}, & k \text{ odd.} \end{cases}$$

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The result in (1) is obtained from $S(1)$. We shall state two more special cases:

$$S(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-3)^2(2n+1)^3(2n+5)^2} = \frac{1}{225} - \frac{\pi^3}{2^{13}}$$

and

$$S(3) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-5)^2(2n+1)^3(2n+7)^2} = \frac{1}{1225} - \frac{\pi}{6^6} - \frac{\pi^3}{3^4 2^9}.$$

2 The Proof

We will give a proof mainly based on the theory of residues, extending the arguments of Szekeres. To do so, the following technical lemma will be needed, which seems to be a familiar result. To keep this note self contained, we provide a proof below.

Lemma 1 *Let $z \in \mathbb{C}$. Let further $N \in \mathbb{N}$ and C be the square in the complex plane with corners $(\pm N, \pm N)$. Then, $|\cos(\pi z)| \geq 1$ for all z on the square C.*

Proof. Let $z = x + iy$. Then, from the definition of the complex cosine function,

$$\cos(\pi z) = \cos(\pi x)\cosh(\pi y) - i\sin(\pi x)\sinh(\pi y).$$

Using the half angle formulas

$$\cos(2z) = 2\cos^2(z) - 1 \qquad \text{and} \qquad \cosh(2z) = 2\cosh^2(z) - 1,$$

we get

$$|\cos(\pi z)|^2 = \cos^2(\pi x)\cosh^2(\pi y) + \sin^2(\pi x)\sinh^2(\pi y) = \frac{\cos(2\pi x)}{2} + \frac{\cosh(2\pi y)}{2}.$$

On the vertical sides of C, $z = \pm N + iy$ and

$$|\cos(\pi z)|^2 = \frac{\cos(\pm 2\pi N)}{2} + \frac{\cosh(2\pi y)}{2} = \frac{1}{2} + \frac{\cosh(2\pi y)}{2} \geq 1.$$

Finally, on the horizontal sides of C we have that $z = x \pm iN$ and

$$|\cos(\pi z)|^2 = 1 + \frac{1}{2}\sum_{n=1}^{\infty}((-1)^n x^{2n} + N^{2n})\frac{(2\pi)^{2n}}{(2n)!} \geq 1.$$

■

The proof of Theorem 1 follows.

Proof. We start with the observation that

$$\begin{aligned} S(k) &= \frac{1}{2}\left(S(k) + \sum_{n=-2}^{-\infty} \frac{(-1)^{n+1}}{(2n-2k+1)^2(2n+1)^3(2n+2k+1)^2}\right) \\ &= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(2n-2k+1)^2(2n+1)^3(2n+2k+1)^2} + \frac{2}{(2k-1)^2(2k+1)^2}\right). \end{aligned}$$

To evaluate the sum, we consider for $k \in \mathbb{Z}$ the complex function

$$f(z) = \frac{\pi}{(z-k)^2 z^3 (z+k)^2 \cos(\pi z)}.$$

From Lemma 1, it follows that

$$\lim_{N\to\infty}\int_C f(z)dz = 0$$

on each square C with corners $(\pm N, \pm N)$. This means that

$$\sum_{j\geq 1} Res(f; z_j) = 0,$$

where z_j are the poles of $f(z)$ inside C. The classification of the residues is easy: $f(z)$ has infinitely many poles of order one at $z = n + 1/2$, n an integer, a pole of order two at $z = k$, a pole of order two at $z = -k$, and a pole of order three at $z = 0$. For each n, the residue of $f(z)$ at $z = n + 1/2$ is

$$\begin{aligned} Res(f; z = n + 1/2) &= \lim_{z\to n+1/2} \frac{\pi(z - n - \frac{1}{2})}{(z-k)^2 z^3 (z+k)^2 \cos(\pi z)} \\ &= \frac{128(-1)^{n+1}}{(2n-2k+1)^2(2n+1)^3(2n+2k+1)^2}. \end{aligned}$$

Next, the residue at $z = k$ is

$$Res(f; z = k) = \lim_{z\to k} \frac{d}{dz} \frac{\pi}{z^3(z+k)^2\cos(\pi z)}.$$

Since

$$\frac{d}{dz}\frac{\pi}{z^3(z+k)^2\cos(\pi z)} = \frac{\pi^2 \sin(\pi z)}{z^3(z+k)^2\cos^2(\pi z)} - \frac{3\pi}{z^4(z+k)^2\cos(\pi z)} - \frac{2\pi}{z^3(z+k)^3\cos(\pi z)},$$

we get

$$Res(f; z = k) = (-1)^{k+1}\frac{\pi}{k^6}.$$

In the same manner,

$$Res(f; z = -k) = \lim_{z\to -k} \frac{d}{dz} \frac{\pi}{z^3(z-k)^2\cos(\pi z)} = (-1)^{k+1}\frac{\pi}{k^6}.$$

Finally,

$$Res(f; z = 0) = \lim_{z\to 0} \frac{1}{2}\frac{d^2}{dz^2} \frac{\pi}{(z-k)^2(z+k)^2\cos(\pi z)}.$$

The calculation of the second derivative is straightforward but lengthy. The result is

$$\frac{d^2}{dz^2}\frac{\pi}{(z-k)^2(z+k)^2\cos(\pi z)} = \pi\Big(A(z) + B(z) + C(z) + D(z)\Big),$$

with

$$A(z) = \frac{2\pi\sin(\pi z)((\pi z^2 - \pi k^2)\sin(\pi z) - 4z\cos(\pi z))}{(z-k)^3(z+k)^3\cos^3(\pi z)},$$

$$B(z) = -\frac{3((\pi z^2 - \pi k^2)\sin(\pi z) - 4z\cos(\pi z))}{(z-k)^4(z+k)^3\cos^2(\pi z)},$$

$$C(z) = -\frac{3((\pi z^2 - \pi k^2)\sin(\pi z) - 4z\cos(\pi z))}{(z-k)^3(z+k)^4\cos^2(\pi z)},$$

and

$$D(z) = \frac{6\pi z\sin(\pi z) + \pi(\pi z^2 - \pi k^2)\cos(\pi z) - 4\cos(\pi z))}{(z-k)^3(z+k)^3\cos^2(\pi z)}.$$

Hence,

$$Res(f; z=0) = \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2} \frac{\pi}{(z-k)^2(z+k)^2\cos(\pi z)} = \frac{\pi}{2} \cdot \frac{\pi^2 k^2 + 4}{k^6}.$$

Gathering our results, we end with

$$0 = 128 \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(2n-2k+1)^2(2n+1)^3(2n+2k+1)^2} + (-1)^{k+1}\frac{2\pi}{k^6} + \frac{\pi}{2} \cdot \frac{\pi^2 k^2 + 4}{k^6},$$

or equivalently

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(2n-2k+1)^2(2n+1)^3(2n+2k+1)^2} = \frac{1}{128} \cdot \frac{\pi}{k^6}\Big(2((-1)^k - 1) - \frac{\pi^2 k^2}{2}\Big),$$

and the proof is completed. ■

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