Hermite-Hadamard Type Inequalities For The Interval-Valued Harmonically h-Convex Functions Via Fractional Integrals*

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Abstract

In this paper, we first present a new definition of convex interval–valued functions which is called as interval–valued harmonically h–convex functions. Then, we establish some new Hermite–Hadamard type inequalities for interval–valued harmonically h–convex functions by using fractional integrals. We also discussed some special cases of our main results. Finally, a briefly conclusion is given.

1 Introduction

The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard, (see [12], [32, pp. 137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that, if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities in (1) hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied, see [2, 7, 8], [13]–[15], [19, 30, 31], [36]–[43].

On the other hand, interval analysis is a particular case of set–valued analysis which is the study of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real–world phenomena. An old example of interval enclosure is Archimede's method which is related to the computation of the circumference of a circle. In 1966, the first book related to interval analysis was given by Moore who is known as the first user of intervals in computational mathematics, see [25]. After his book, several scientists started to investigate theory and application of interval arithmetic. Nowadays, because of its applications, interval analysis is a useful tool in various areas related to uncertain data. We can see applications in computer graphics, experimental and computational physics, error analysis, robotics and many others.

What's more, several important inequalities (Hermite-Hadamard, Ostrowski, etc.) have been studied for the interval-valued functions in recent years. In [5, 6], C. Cano et al. obtained Ostrowski type inequalities for interval-valued functions by using Hukuhara derivative for interval-valued functions. In [17], R. Flores et al. established Minkowski and Beckenbach's inequalities for interval-valued functions. For the others, please see [9, 10], [16]–[18]. However, inequalities were studied for more general set–valued maps. For example, in [35], Sadowska gave the Hermite-Hadamard inequality. For the other studies, see [24, 28].

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2 Interval Calculus

A real valued interval X is bounded, closed subset of $\mathbb{R}$ and is defined by

$$X = \left[\underline{X}, \overline{X}\right] = \left\{t \in \mathbb{R} : \underline{X} \leq t \leq \overline{X}\right\}$$

where $\underline{X}, \overline{X} \in \mathbb{R}$ and $\underline{X} \leq \overline{X}$. The numbers $\underline{X}$ and $\overline{X}$ are called the left and the right endpoints of interval X, respectively. When $\overline{X} = \underline{X} = a$, the interval X is said to be degenerate and we use the form $X = a = [a, a]$. Also, we call X positive if $\underline{X} > 0$ or negative if $\overline{X} < 0$. The set of all closed intervals of $\mathbb{R}$, the sets of all closed positive intervals of $\mathbb{R}$ and closed negative intervals of $\mathbb{R}$ is denoted by $\mathbb{R}_\mathcal{I}$, $\mathbb{R}_\mathcal{I}^+$ and $\mathbb{R}_\mathcal{I}^-$, respectively. The Pompeiu–Hausdorff distance between the intervals X and Y is defined by

$$d(X, Y) = d\left(\left[\underline{X}, \overline{X}\right], \left[\underline{Y}, \overline{Y}\right]\right) = \max\left\{\left|\underline{X} - \underline{Y}\right|, \left|\overline{X} - \overline{Y}\right|\right\}.$$

It is known that $(\mathbb{R}_\mathcal{I}, d)$ is a complete metric space, see [1].

Now, we give the definitions of basic interval arithmetic operations for the intervals X and Y as follows:

$$X + Y = \left[\underline{X} + \underline{Y}, \overline{X} + \overline{Y}\right],$$

$$X - Y = \left[\underline{X} - \overline{Y}, \overline{X} - \underline{Y}\right],$$

$$X \cdot Y = [\min S, \max S] \text{ where } S = \left\{\underline{X}\,\underline{Y}, \underline{X}\,\overline{Y}, \overline{X}\underline{Y}, \overline{X}\,\overline{Y}\right\},$$

$$X/Y = [\min T, \max T] \text{ where } T = \left\{\underline{X}/\underline{Y}, \underline{X}/\overline{Y}, \overline{X}/\underline{Y}, \overline{X}/\overline{Y}\right\} \text{ and } 0 \notin Y.$$

Scalar multiplication of the interval X is defined by

$$\lambda X = \lambda\left[\underline{X}, \overline{X}\right] = \begin{cases} \left[\lambda\underline{X}, \lambda\overline{X}\right], & \lambda > 0, \\ \{0\}, & \lambda = 0, \\ \left[\lambda\overline{X}, \lambda\underline{X}\right], & \lambda < 0, \end{cases}$$

where $\lambda \in \mathbb{R}$.

The opposite of the interval X is

$$-X := (-1)X = [-\overline{X}, -\underline{X}],$$

where $\lambda = -1$.

The subtraction is given by

$$X - Y = X + (-Y) = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}].$$

In general, $-X$ is not additive inverse for X, i.e. $X - X \neq 0$.

The definitions of operations lead to a number of algebraic properties which allows $\mathbb{R}_\mathcal{I}$ to be quasilinear space, see [22]. They can be listed as follows, (see [21]-[23], [25]):

(1) (Associativity of addition) $(X + Y) + Z = X + (Y + Z)$ for all $X, Y, Z \in \mathbb{R}_\mathcal{I}$,

(2) (Additivity element) $X + 0 = 0 + X = 0$ for all $X \in \mathbb{R}_\mathcal{I}$,

(3) (Commutativity of addition) $X + Y = Y + X$ for all $X, Y \in \mathbb{R}_\mathcal{I}$,

(4) (Cancellation law) $X + Z = Y + Z \Longrightarrow X = Y$ for all $X, Y, Z \in \mathbb{R}_\mathcal{I}$,

(5) (Associativity of multiplication) $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$ for all $X, Y, Z \in \mathbb{R}_\mathcal{I}$,

(6) (Commutativity of multiplication) $X \cdot Y = Y \cdot X$ for all $X, Y \in \mathbb{R}_\mathcal{I}$,

(7) (Unity element) $X \cdot 1 = 1 \cdot X$ for all $X \in \mathbb{R}_\mathcal{I}$,

(8) (Associativity law) $\lambda(\mu X) = (\lambda\mu) X$ for all $X \in \mathbb{R}_\mathcal{I}$ and all $\lambda, \mu \in \mathbb{R}$,

(9) (First distributivity law) $\lambda(X + Y) = \lambda X + \lambda Y$ for all $X, Y \in \mathbb{R}_\mathcal{I}$ and all $\lambda \in \mathbb{R}$,

(10) (Second distributivity law) $(\lambda + \mu)X = \lambda X + \mu X$ for all $X \in \mathbb{R}_\mathcal{I}$ and all $\lambda, \mu \in \mathbb{R}$.

Besides these properties, the distributive law is not always valid for intervals. For example, $X = [1, 2]$, $Y = [2, 3]$ and $Z = [-2, -1]$.

$$X \cdot (Y + Z) = [0, 4]$$

whereas

$$X \cdot Y + X \cdot Z = [-2, 5].$$

But, this law holds in certain cases. If $Y \cdot Z > 0$, then

$$X \cdot (Y + Z) = X \cdot Y + X \cdot Z.$$

What's more, one of the set property is the inclusion $\subseteq$ that is given by

$$X \subseteq Y \Longleftrightarrow \underline{Y} \leq \underline{X} \text{ and } \overline{X} \leq \overline{Y}.$$

Considering together with arithmetic operations and inclusion, one has the following property which is called **inclusion isotone of interval operations**:

Let $\odot$ be the addition, multiplication, subtraction or division. If X, Y, Z and T are intervals such that

$$X \subseteq Y \text{ and } Z \subseteq T,$$

then the following relation is valid

$$X \odot Z \subseteq Y \odot T.$$

The following proposition is about that scalar multiplication preserves the inclusion:

Proposition 1 *Let X, Y be intervals and $\lambda \in \mathbb{R}$. If $X \subseteq Y$, then $\lambda X \subseteq \lambda Y$.*

2.1   Integral of interval-valued Functions

In this section, the notion of integral is mentioned for interval-valued functions. Before the definition of integral, the necessary concepts will be given as the following:

A function F is said to be an interval-valued function of t on $[a, b]$, if it assigns a nonempty interval to each $t \in [a, b]$,

$$F(t) = \left[\underline{F}(t), \overline{F}(t)\right].$$

A partition of $[a, b]$ is any finite ordered subset P having the form:

$$P : a = t_0 < t_1 < \ldots < t_n = b.$$

The mesh of a partition P is defined by

$$mesh(P) = \max\left\{t_i - t_{i-1} : i = 1, 2, \ldots, n\right\}.$$

We denote by $P([a, b])$ the set of all partition of $[a, b]$. Let $P(\delta, [a, b])$ be the set of all $P \in P([a, b])$ such that $mesh(P) < \delta$. Choose an arbitrary point ξ_i in interval $[t_{i-1}, t_i]$, $(i = 1, 2, \ldots, n)$ and let us define the sum

$$S(F, P, \delta) = \sum_{i=1}^{n} F(\xi_i) \left[t_i - t_{i-1}\right],$$

where $F : [a, b] \to \mathbb{R}_\mathcal{I}$. We call $S(F, P, \delta)$ a Riemann sum of F corresponding to $P \in P(\delta, [a, b])$.

Definition 1 ([11, 33, 34]) *A function* $F : [a,b] \rightarrow \mathbb{R}_{\mathcal{I}}$ *is called interval Riemann integrable* ((IR)-*integrable*) *on* $[a,b]$, *if there exists* $A \in \mathbb{R}_I$ *such that, for each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$d\left(S(F,P,\delta),A\right) < \varepsilon$$

for every Riemann sum S *of* F *corresponding to each* $P \in P\left(\delta, [a,b]\right)$ *and independent from choice of* $\xi_i \in [t_{i-1}, t_i]$ *for all* $1 \leq i \leq n$. *In this case,* A *is called the* (IR)-*integral of* F *on* $[a,b]$ *and is denoted by*

$$A = (IR) \int_a^b F(t)dt.$$

The collection of all functions that are (IR)-*integrable on* $[a,b]$ *will be denote by* $\mathcal{IR}_{([a,b])}$.

The following theorem gives relation between (IR)-ntegrable and Riemann integrable (R-integrable) (see [26], pp. 131):

Theorem 1 *Let* $F : [a,b] \rightarrow \mathbb{R}_{\mathcal{I}}$ *be an interval-valued function such that* $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. $F \in \mathcal{IR}_{([a,b])}$ *if and only if* $\underline{F}(t),\ \overline{F}(t) \in \mathcal{R}_{([a,b])}$ *and*

$$(IR) \int_a^b F(t)dt = \left[(R) \int_a^b \underline{F}(t)dt, (R) \int_a^b \overline{F}(t)dt\right],$$

where $\mathcal{R}_{([a,b])}$ *denotes the all R-integrable functions.*

It is seen easily that, if $F(t) \subseteq G(t)$ for all $t \in [a,b]$, then

$$(IR) \int_a^b F(t)dt \subseteq (IR) \int_a^b G(t)dt.$$

In [44, 45], Zhao et al. introduced a kind of convex interval-valued function as follows:

Definition 2 *Let* $h : [c,d] \rightarrow \mathbb{R}$ *be a non-negative function,* $(0,1) \subseteq [c,d]$ *and* $h \neq 0$. *We say that* $F : [a,b] \rightarrow \mathbb{R}_{\mathcal{I}}^+$ *is a h-convex interval-valued function, if for all* $x,y \in [a,b]$ *and* $t \in (0,1)$, *we have*

$$h(t)F(x) + h(1-t)F(y) \subseteq F(tx + (1-t)y). \tag{2}$$

$SX(h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$ *denotes the set of all h-convex interval-valued functions.*

The usual notion of convex interval-valued function corresponds to relation (2) with $h(t) = t$, see [35]. Also, if we take $h(t) = t^s$ in (2), then Definition 2 gives the other convex interval-valued function defined by Breckner, see [3].

Otherwise, Zhao et al. obtained the following Hermite-Hadamard inequality for interval-valued functions:

Theorem 2 ([44]) *Let* $F : [a,b] \rightarrow \mathbb{R}_{\mathcal{I}}^+$ *be an interval-valued function such that* $F(t) = [\underline{F}(t), \overline{F}(t)]$ *and* $F \in \mathcal{IR}_{([a,b])}, h : [0,1] \rightarrow \mathbb{R}$ *be a non-negative function and* $h\left(\frac{1}{2}\right) \neq 0$. *If* $F \in SX(h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, *then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR) \int_a^b F(x)dx \supseteq [F(a) + F(b)] \int_0^1 h(t)dt. \tag{3}$$

Remark 1 *(i) If $h(t) = t$, then (3) reduces to the following result:*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a)+F(b)}{2},\tag{4}$$

which is obtained by [35].

(ii) If $h(t) = t^s$, then (3) reduces to the following result:

$$2^{s-1}F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a)+F(b)}{s+1},$$

which is obtained by [29].

Theorem 3 *Let $F, G : [a,b] \to \mathbb{R}_\mathcal{I}^+$ be two interval-valued functions such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $G(t) = [\underline{G}(t), \overline{G}(t)]$, where $F, G \in \mathcal{IR}_{([a,b])}, h_1, h_2 : [0,1] \to \mathbb{R}$ are two non-negative functions and $h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \neq 0$. If $F, G \in SX(h, [a,b], \mathbb{R}_\mathcal{I}^+)$, then*

$$\frac{1}{2h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)}F\left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right)$$

$$\supseteq \quad \frac{1}{b-a}(IR)\int_a^b F(x)G(x)dx + M(a,b)(IR)\int_0^1 h_1(t)h_2(1-t)dt$$

$$+N(a,b)(IR)\int_0^1 h_1(t)h_2(t)dt\tag{5}$$

and

$$\frac{1}{b-a}(IR)\int_a^b F(x)G(x)dx \supseteq M(a,b)(IR)\int_0^1 h_1(t)h_2(t)dt + N(a,b)(IR)\int_0^1 h_1(t)h_2(1-t)dt,\tag{6}$$

where

$$M(a,b) = F(a)G(a) + F(b)G(b) \text{ and } N(a,b) = F(a)G(b) + F(b)G(a).$$

Remark 2 *If $h(t) = t$, the (5) reduces to the following result:*

$$\frac{1}{b-a}(IR)\int_a^b F(x)G(x)dx \supseteq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b).\tag{7}$$

Remark 3 *If $h(t) = t$, then (6) reduces to the following result:*

$$2F\left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)G(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b).\tag{8}$$

Definition 3 *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$I_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $I_{a+}^0 f(x) = I_{b-}^0 f(x) = f(x).$

Definition 4 *Let $F : [a, b] \to \mathbb{R}_\mathcal{I}$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and let $\alpha > 0$. The interval-valued left-sided and right-sided Riemann-Liouville fractional integral of function F is defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} (IR) \int_a^x (x - s)^{\alpha-1} f(t)dt, \quad x > a,$$

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} (IR) \int_x^b (s - x)^{\alpha-1} f(t)dt, \quad x < b.$$

where Γ is Euler Gamma function.

Theorem 4 *If $f : [a, b] \to \mathbb{R}_\mathcal{I}$ is an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then we have*

$$J_{a+}^\alpha F(x) = \left[I_{a+}^\alpha \underline{F}(x), I_{a+}^\alpha \overline{F}(x)\right]$$

and

$$J_{b-}^\alpha F(x) = \left[I_{b-}^\alpha \underline{F}(x), I_{b-}^\alpha \overline{F}(x)\right].$$

In [4], Budak et al. obtained the following inequalities of Hermite-Hadamard type for the convex interval-valued functions:

Theorem 5 *If $F : [a, b] \to \mathbb{R}_\mathcal{I}^+$ is a convex interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $\alpha > 0$, then we have*

$$F\left(\frac{a + b}{2}\right) \supseteq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[J_{a+}^\alpha F(b) + J_{b-}^\alpha F(a)\right] \supseteq \frac{F(a) + F(b)}{2}. \tag{9}$$

Theorem 6 *If $F, G : [a, b] \to \mathbb{R}_\mathcal{I}^+$ are two convex interval-valued functions such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $G(t) = \left[\underline{G}(t), \overline{G}(t)\right]$, then for $\alpha > 0$ we have*

$$\frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[J_{a+}^\alpha F(b)G(b) + J_{b-}^\alpha F(a)G(a)\right]$$

$$\supseteq \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)}\right) M(a, b) + \frac{\alpha}{(\alpha + 1)(\alpha + 2)} N(a, b) \tag{10}$$

and

$$2F\left(\frac{a + b}{2}\right) G\left(\frac{a + b}{2}\right)$$

$$\supseteq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[J_{a+}^\alpha F(b)G(b) + J_{b-}^\alpha F(a)G(a)\right]$$

$$+ \frac{\alpha}{(\alpha + 1)(\alpha + 2)} M(a, b) + \left(\frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)}\right) N(a, b), \tag{11}$$

where $M(a, b)$ and $N(a, b)$ are defined in Theorem 3.

For the other fractional inequalities for the convex interval-valued functions, see [20]. Now, we are in position to introduce the new class of convex interval-valued functions as follows:

Definition 5 *Let $h : [c, d] \to \mathbb{R}$ be a non-negative function, $(0, 1) \subseteq [c, d]$ and $h \neq 0$. A function $F : I \to \mathbb{R}_\mathcal{I}^+$ is said to be interval-valued harmonically h-convex function, if*

$$F\left(\frac{xy}{ty + (1 - t)x}\right) \supseteq h(t)F(x) + h(1 - t)F(y), \tag{12}$$

for all $t \in (0, 1)$ and $a, b \in I$.

Motivated by the above literatures, the main objective of this paper is to complete the Riemann–Liouville integrals for interval-valued harmonically h-convex functions and to obtain Hermite-Hadamard inequality via these integrals. We also discuss some new special cases of the main results. At the end, a briefly conclusion is provided as well.

3   Main Results

In this section we prove some inequalities of Hermite-Hadamard type for the interval-valued harmonically h-convex function via fractional integrals. Throughout this section we will take $g(x) = \frac{1}{x}$,

$$M(a,b) = F(a)G(a) + F(b)G(b) \text{ and } N(a,b) = F(a)G(b) + F(b)G(a).$$

Theorem 7 *If $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is interval-valued harmonically h-convex function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then we have the following inequalities for fractional integrals:*

$$\frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right)$$

$$\supseteq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha} \left[J_{(1/b)+}^{\alpha}\left(F \circ g\right)(1/a) + J_{(1/a)-}^{\alpha}\left(F \circ g\right)(1/b)\right]$$

$$\supseteq \alpha\left[\frac{F(a)+F(b)}{2}\right]\int_{0}^{1} t^{\alpha-1}[h(t)+h(1-t)]dt. \tag{13}$$

Proof. Since F is interval-valued harmonically h-convex function, we have

$$F\left(\frac{2xy}{x+y}\right) \supseteq h\left(\frac{1}{2}\right)[F(x)+F(y)]. \tag{14}$$

By setting $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{tb+(1-t)a}$ in (14), we obtain

$$\frac{1}{h\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right) \supseteq F\left(\frac{ab}{ta+(1-t)b}\right) + F\left(\frac{ab}{tb+(1-t)a}\right). \tag{15}$$

Multiplying both sides of (15) by $t^{\alpha-1}$ and integrating the resultant one with respect to t over $[0,1]$, we get

$$\frac{1}{h\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right)\int_{0}^{1} t^{\alpha-1}dt$$

$$\supseteq (IR)\int_{0}^{1} t^{\alpha-1}F\left(\frac{ab}{ta+(1-t)b}\right)dt + (IR)\int_{0}^{1} t^{\alpha-1}F\left(\frac{ab}{tb+(1-t)a}\right)dt. \tag{16}$$

By using Theorem 1, we obtain

$$(IR)\int_{0}^{1} t^{\alpha-1}F\left(\frac{ab}{ta+(1-t)b}\right)dt$$

$$= \left[(R)\int_{0}^{1} t^{\alpha-1}\underline{F}\left(\frac{ab}{ta+(1-t)b}\right)dt, \ (R)\int_{0}^{1} t^{\alpha-1}\overline{F}\left(\frac{ab}{ta+(1-t)b}\right)dt\right]$$

$$= \left[\left(\frac{ab}{b-a}\right)^{\alpha} (R)\int_{1/b}^{1/a} \left(\frac{1}{a}-x\right)\underline{F}\left(\frac{1}{x}\right)dx, \ \left(\frac{ab}{b-a}\right)^{\alpha} (R)\int_{a}^{b} \left(\frac{1}{a}-x\right)\overline{F}\left(\frac{1}{x}\right)dx\right]$$

$$= \left(\frac{ab}{b-a}\right)^{\alpha}\left[\Gamma(\alpha) I_{(1/b)+}^{\alpha}\left(\underline{F} \circ g\right)(1/a), \ \Gamma(\alpha) I_{(1/b)+}^{\alpha}\left(\overline{F} \circ g\right)(1/a)\right]$$

$$= \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha} J_{(1/b)+}^{\alpha}\left(F \circ g\right)(1/a).$$

Similarly, we have

$$(IR) \int_0^1 t^{\alpha-1} F\left(\frac{ab}{tb+(1-t)a}\right) dt$$

$$= \left[(R) \int_0^1 t^{\alpha-1} \underline{F}\left(\frac{ab}{tb+(1-t)a}\right) dt, \ (R) \int_0^1 t^{\alpha-1} \overline{F}\left(\frac{ab}{tb+(1-t)a}\right) dt\right]$$

$$= \Gamma(\alpha) \left(\frac{ab}{b-a}\right)^\alpha J^\alpha_{(1/a)-} (F \circ g)(1/b).$$

Hence, by the inequality (16), we get

$$\frac{1}{\alpha h\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right) \supseteq \Gamma(\alpha) \left(\frac{ab}{b-a}\right)^\alpha \left[J^\alpha_{(1/b)+} (F \circ g)(1/a) + J^\alpha_{(1/a)-} (F \circ g)(1/b)\right]$$

which gives first inequality in (13). To prove the second inequality since F is interval-valued harmonically h-convex function, we get

$$F\left(\frac{ab}{ta+(1-t)b}\right) \supseteq h(t)F(b) + h(1-t)F(a) \tag{17}$$

and

$$F\left(\frac{ab}{tb+(1-t)a}\right) \supseteq h(t)F(a) + h(1-t)F(b). \tag{18}$$

Adding (17) and (18), we have

$$F\left(\frac{ab}{ta+(1-t)b}\right) + F\left(\frac{ab}{tb+(1-t)a}\right) \supseteq [h(t)+h(1-t)][F(a)+F(b)]. \tag{19}$$

Multiplying (19) by $t^{\alpha-1}$ on both sides and integrating the resultant one with respect to t over $[0,1]$, we have

$$(IR) \int_0^1 t^{\alpha-1} F\left(\frac{ab}{ta+(1-t)b}\right) dt + (IR) \int_0^1 t^{\alpha-1} F\left(\frac{ab}{tb+(1-t)a}\right) dt$$

$$\supseteq \ [F(a)+F(b)] \int_0^1 t^{\alpha-1}[h(t)+h(1-t)]dt. \tag{20}$$

This completes the proof. ∎

Theorem 8 *If $F, G : [a,b] \to \mathbb{R}^+_{\mathcal{I}}$ are two interval-valued harmonically h-convex functions such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $G(t) = \left[\underline{G}(t), \overline{G}(t)\right]$, then we have the following inequality for fractional integrals:*

$$\frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left[J^\alpha_{(1/b)+} (F \circ g)(1/a)(G \circ g)(1/a) + J^\alpha_{(1/a)-} (F \circ g)(1/b)(G \circ g)(1/b)\right]$$

$$\supseteq \ \alpha \left[\frac{M(a,b)}{2} \int_0^1 t^{\alpha-1}[h^2(t)+h^2(1-t)]dt + N(a,b) \int_0^1 t^{\alpha-1} h(t)h(1-t)dt\right]. \tag{21}$$

Proof. Since F and G are interval-valued harmonically h-convex functions for $t \in [0,1]$, we have

$$F\left(\frac{ab}{tb+(1-t)a}\right) \supseteq h(t)F(a) + h(1-t)F(b) \tag{22}$$

and

$$G\left(\frac{ab}{tb+(1-t)a}\right) \supseteq h(t)G(a) + h(1-t)G(b). \tag{23}$$

Multiplying (22) and (23), we get

$$F\left(\frac{ab}{tb+(1-t)a}\right)G\left(\frac{ab}{tb+(1-t)a}\right)$$
$$\supseteq \quad h^2(t)F(a)G(a)+h^2(1-t)F(b)G(b)+h(t)h(1-t)\left[F(a)G(b)+F(b)G(a)\right]. \tag{24}$$

Similarly, we obtain

$$F\left(\frac{ab}{ta+(1-t)b}\right)G\left(\frac{ab}{ta+(1-t)b}\right)$$
$$\supseteq \quad h^2(1-t)F(a)G(a)+h^2(t)F(b)G(b)+h(t)h(1-t)\left[F(a)G(b)+F(b)G(a)\right]. \tag{25}$$

Adding (24) and (25), we have the following relation

$$F\left(\frac{ab}{ta+(1-t)b}\right)G\left(\frac{ab}{ta+(1-t)b}\right)+F\left(\frac{ab}{tb+(1-t)a}\right)G\left(\frac{ab}{tb+(1-t)a}\right)$$
$$\supseteq \quad [h^2(t)+h^2(1-t)]M(a,b)+2h(t)h(1-t)N(a,b). \tag{26}$$

Multiplying (26) by $t^{\alpha-1}$ on both sides and integrating the resultant one with respect to t over $[0,1]$, we get

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{ab}{ta+(1-t)b}\right)G\left(\frac{ab}{ta+(1-t)b}\right)dt$$
$$+(IR)\int_0^1 t^{\alpha-1}F\left(\frac{ab}{tb+(1-t)a}\right)G\left(\frac{ab}{tb+(1-t)a}\right)dt$$
$$\supseteq \quad M(a,b)\int_0^1 t^{\alpha-1}[h^2(t)+h^2(1-t)]dt+2N(a,b)\int_0^1 t^{\alpha-1}h(t)h(1-t)dt. \tag{27}$$

Using Theorem 1 in relation (27), we have

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{ab}{ta+(1-t)b}\right)G\left(\frac{ab}{ta+(1-t)b}\right)dt$$
$$= \quad \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}J_{(1/b)+}^{\alpha}(F\circ g)(1/a)(G\circ g)(1/a) \tag{28}$$

and

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{ab}{tb+(1-t)a}\right)G\left(\frac{ab}{tb+(1-t)a}\right)dt$$
$$= \quad \Gamma(\alpha)\left(\frac{ab}{b-a}\right)^{\alpha}J_{(1/a)-}^{\alpha}(F\circ g)(1/b)(G\circ g)(1/b). \tag{29}$$

Substituting (28) and (29) in relation (27), we have our desired result (21). This completes the proof. ∎

Theorem 9 *If $F,G:[a,b]\to\mathbb{R}_{\mathcal{I}}^+$ are two interval-valued harmonically h-convex functions such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$ and $G(t)=\left[\underline{G}(t),\overline{G}(t)\right]$, then we have the following inequality for fractional integrals:*

$$\frac{1}{2h^2\left(\frac{1}{2}\right)}F\left(\frac{2ab}{a+b}\right)G\left(\frac{2ab}{a+b}\right)$$
$$\supseteq \quad \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J_{(1/b)+}^{\alpha}(F\circ g)(1/a)(G\circ g)(1/a)+J_{(1/a)-}^{\alpha}(F\circ g)(1/b)(G\circ g)(1/b)\right]$$
$$+\alpha\left[\frac{N(a,b)}{2}\int_0^1 t^{\alpha-1}[h^2(t)+h^2(1-t)]dt+M(a,b)\int_0^1 t^{\alpha-1}h(t)h(1-t)dt\right]. \tag{30}$$

Proof. For $t \in [0, 1]$, we can write

$$\frac{2ab}{a+b} = \frac{2\frac{ab}{(1-t)a+tb}\frac{ab}{ta+(1-t)b}}{\frac{ab}{(1-t)a+tb} + \frac{ab}{ta+(1-t)b}}.$$

Since F and G are two interval-valued harmonically h-convex functions, we have

$$\frac{1}{h^2\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right) G\left(\frac{2ab}{a+b}\right)$$

$$= \frac{1}{h^2\left(\frac{1}{2}\right)} F\left(\frac{2\frac{ab}{(1-t)a+tb}\frac{ab}{ta+(1-t)b}}{\frac{ab}{(1-t)a+tb} + \frac{ab}{ta+(1-t)b}}\right) G\left(\frac{2\frac{ab}{(1-t)a+tb}\frac{ab}{ta+(1-t)b}}{\frac{ab}{(1-t)a+tb} + \frac{ab}{ta+(1-t)b}}\right)$$

$$\supseteq \left[F\left(\frac{ab}{(1-t)a+tb}\right) + F\left(\frac{ab}{ta+(1-t)b}\right)\right] \times \left[G\left(\frac{ab}{(1-t)a+tb}\right) + G\left(\frac{ab}{ta+(1-t)b}\right)\right]$$

$$= F\left(\frac{ab}{(1-t)a+tb}\right) G\left(\frac{ab}{(1-t)a+tb}\right) + F\left(\frac{ab}{ta+(1-t)b}\right) G\left(\frac{ab}{ta+(1-t)b}\right)$$

$$+ F\left(\frac{ab}{(1-t)a+tb}\right) G\left(\frac{ab}{ta+(1-t)b}\right) + F\left(\frac{ab}{ta+(1-t)b}\right) G\left(\frac{ab}{(1-t)a+tb}\right)$$

$$\supseteq F\left(\frac{ab}{(1-t)a+tb}\right) G\left(\frac{ab}{(1-t)a+tb}\right) + F\left(\frac{ab}{ta+(1-t)b}\right) G\left(\frac{ab}{ta+(1-t)b}\right)$$

$$+ [h^2(t) + h^2(1-t)] N(a,b) + 2h(t)h(1-t) M(a,b). \tag{31}$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (31) and integrating the resultant one with respect to t over $[0,1]$, we obtain

$$\frac{1}{h^2\left(\frac{1}{2}\right)} (IR) \int_0^1 t^{\alpha-1} F\left(\frac{2ab}{a+b}\right) G\left(\frac{2ab}{a+b}\right) dt$$

$$\supseteq (IR) \int_0^1 t^{\alpha-1} F\left(\frac{ab}{(1-t)a+tb}\right) G\left(\frac{ab}{(1-t)a+tb}\right) dt$$

$$+ (IR) \int_0^1 t^{\alpha-1} F\left(\frac{ab}{ta+(1-t)b}\right) G\left(\frac{ab}{ta+(1-t)b}\right) dt$$

$$+ N(a,b) \int_0^1 t^{\alpha-1} [h^2(t) + h^2(1-t)] dt$$

$$+ 2M(a,b) \int_0^1 t^{\alpha-1} h(t)h(1-t) dt.$$

By changing the variable of integration we achieved desired inequality (30). ∎

Theorem 10 *If $F : [a,b] \rightarrow \mathbb{R}_{\mathcal{I}}^+$ is interval-valued harmonically h-convex function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then we have the following inequalities for fractional integrals:*

$$\frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{a+b}{2}\right)$$

$$\supseteq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{a+b}\right)^\alpha \left[J^\alpha_{\left(\frac{a+b}{2ab}\right)+} (F \circ g)(1/a) + J^\alpha_{\left(\frac{a+b}{2ab}\right)-} (F \circ g)(1/b)\right]$$

$$\supseteq \alpha \left(\frac{F(a)+F(b)}{2}\right) \int_0^1 t^{\alpha-1} \left[h\left(\frac{2-t}{2}\right) + h\left(\frac{t}{2}\right)\right] dt. \tag{32}$$

Proof. Since F is interval-valued harmonically h-convex function on $[a, b]$, we have

$$F\left(\frac{2xy}{x+y}\right) \supseteq h\left(\frac{1}{2}\right)[F(x) + F(y)]$$

For $x = \frac{2ab}{ta+(2-t)b}$ and $y = \frac{2ab}{(2-t)a+tb}$, we get

$$\frac{1}{h\left(\frac{1}{2}\right)}F\left(\frac{a+b}{2}\right) \supseteq F\left(\frac{2ab}{ta+(2-t)b}\right) + F\left(\frac{2ab}{(2-t)a+tb}\right). \tag{33}$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (33) and integrating the resultant one with respect to t over $[0, 1]$, we obtain

$$\frac{1}{h\left(\frac{1}{2}\right)}F\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}dt$$

$$\supseteq (IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{ta+(2-t)b}\right)dt + (IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(2-t)a+tb}\right)dt. \tag{34}$$

Using Theorem 1 in the relation (34), we have

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{ta+(2-t)b}\right)dt$$

$$= \left[(R)\int_0^1 t^{\alpha-1}\underline{F}\left(\frac{2ab}{ta+(2-t)b}\right)dt, (R)\int_0^1 t^{\alpha-1}\overline{F}\left(\frac{2ab}{ta+(2-t)b}\right)dt\right]$$

$$= \left[\left(\frac{2ab}{a+b}\right)^\alpha (R)\int_{\frac{a+b}{2ab}}^{1/a}\left(\frac{1}{a}-u\right)\underline{F}(1/u)du, \left(\frac{2ab}{a+b}\right)^\alpha (R)\int_{\frac{a+b}{2ab}}^{1/a}\left(\frac{1}{a}-u\right)\overline{F}(1/u)du\right]$$

$$= \left[\left(\frac{2ab}{a+b}\right)^\alpha \Gamma(\alpha) I^\alpha_{\left(\frac{a+b}{2ab}\right)+}(\underline{F}\circ g)(1/a), \left(\frac{2ab}{a+b}\right)^\alpha \Gamma(\alpha) I^\alpha_{\left(\frac{a+b}{2ab}\right)+}(\overline{F}\circ g)(1/a)\right]$$

$$= \Gamma(\alpha)\left(\frac{2ab}{a+b}\right)^\alpha J^\alpha_{\left(\frac{a+b}{2ab}\right)+}(F\circ g)(1/a).$$

Similarly, we get

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(2-t)a+tb}\right)dt$$

$$= \left[\left(\frac{2ab}{a+b}\right)^\alpha \Gamma(\alpha) I^\alpha_{\left(\frac{a+b}{2ab}\right)-}(\underline{F}\circ g)(1/b), \left(\frac{2ab}{a+b}\right)^\alpha \Gamma(\alpha) I^\alpha_{\left(\frac{a+b}{2ab}\right)-}(\overline{F}\circ g)(1/b)\right]$$

$$= \Gamma(\alpha)\left(\frac{2ab}{a+b}\right)^\alpha J^\alpha_{\left(\frac{a+b}{2ab}\right)-}(F\circ g)(1/b).$$

Hence, we proved the first inequality. To prove the second inequality of (32), first we note that since F is interval-valued harmonically h-convex function, we have

$$F\left(\frac{2ab}{ta+(2-t)b}\right) \supseteq h\left(\frac{2-t}{2}\right)F(a) + h\left(\frac{t}{2}\right)F(b) \tag{35}$$

and

$$F\left(\frac{2ab}{tb+(2-t)a}\right) \supseteq h\left(\frac{t}{2}\right)F(a) + h\left(\frac{2-t}{2}\right)F(b). \tag{36}$$

Adding (35) and (36), we get

$$F\left(\frac{2ab}{ta+(2-t)b}\right) + F\left(\frac{2ab}{(2-t)a+tb}\right) \supseteq [F(a) + F(b)]\left[h\left(\frac{2-t}{2}\right) + h\left(\frac{t}{2}\right)\right]. \tag{37}$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (37) and integrating the resultant one with respect to t over $[0, 1]$, we obtain

$$(IR) \int_0^1 t^{\alpha-1} F\left(\frac{2ab}{ta + (2-t)b}\right) dt + (IR) \int_0^1 t^{\alpha-1} F\left(\frac{2ab}{(2-t)a + tb}\right) dt$$

$$\supseteq (IR) \int_0^1 t^{\alpha-1} \left[h\left(\frac{2-t}{2}\right) + h\left(\frac{t}{2}\right)\right] [F(a) + F(b)] dt.$$

By changing the variables of integration we have second inequality of (32). ∎

Theorem 11 *If $F, G : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ are two interval-valued harmonically h-convex functions such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $G(t) = \left[\underline{G}(t), \overline{G}(t)\right]$, then we have the following inequality for fractional integrals:*

$$\frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{a+b}\right)^{\alpha} \left[J^{\alpha}_{\left(\frac{a+b}{2ab}\right)+} (F \circ g)(1/a) + J^{\alpha}_{\left(\frac{a+b}{2ab}\right)-} (F \circ g)(1/b)\right]$$

$$\supseteq \alpha \left[\frac{M(a,b)}{2} \int_0^1 t^{\alpha-1} \left[h^2\left(\frac{2-t}{2}\right) + h^2\left(\frac{t}{2}\right)\right] dt + N(a,b) \int_0^1 t^{\alpha-1} h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) dt\right]. \quad (38)$$

Proof. Since F and G are two interval-valued harmonically h-convex functions, then

$$F\left(\frac{2ab}{ta + (2-t)b}\right) \supseteq h\left(\frac{2-t}{2}\right) F(a) + h\left(\frac{t}{2}\right) F(b) \quad (39)$$

and

$$G\left(\frac{2ab}{ta + (2-t)b}\right) \supseteq h\left(\frac{2-t}{2}\right) G(a) + h\left(\frac{t}{2}\right) G(b). \quad (40)$$

Multiplying (39) and (40), we have

$$F\left(\frac{2ab}{ta + (2-t)b}\right) G\left(\frac{2ab}{ta + (2-t)b}\right)$$

$$\supseteq h^2\left(\frac{2-t}{2}\right) F(a)G(a) + h^2\left(\frac{t}{2}\right) F(b)G(b) + h\left(\frac{2-t}{2}\right) h\left(\frac{t}{2}\right) [F(a)G(b) + F(b)G(a)]. \quad (41)$$

Similarly, we get

$$F\left(\frac{2ab}{(2-t)a + tb}\right) G\left(\frac{2ab}{(2-t)a + tb}\right)$$

$$\supseteq h^2\left(\frac{t}{2}\right) F(a)G(a) + h^2\left(\frac{2-t}{2}\right) F(b)G(b) + h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) [F(a)G(b) + F(b)G(a)]. \quad (42)$$

Adding (41) and (42), we obtain the following relation

$$F\left(\frac{2ab}{(2-t)a + tb}\right) G\left(\frac{2ab}{(2-t)a + tb}\right) + F\left(\frac{2ab}{ta + (2-t)b}\right) G\left(\frac{2ab}{ta + (2-t)b}\right)$$

$$\supseteq h^2\left(\frac{2-t}{2}\right) [F(a)G(a) + F(b)G(b)]$$

$$+ h^2\left(\frac{t}{2}\right) [F(a)G(a) + F(b)G(b)] + 2h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) [F(a)G(b) + F(b)G(a)]$$

$$= \left[h^2\left(\frac{2-t}{2}\right) + h^2\left(\frac{t}{2}\right)\right] M(a,b) + 2h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) N(a,b). \quad (43)$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (43) and integrating the resultant one with respect to t over $[0,1]$, we have

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(2-t)\,a+tb}\right)G\left(\frac{2ab}{(2-t)\,a+tb}\right)dt$$

$$+(IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{ta+(2-t)\,b}\right)G\left(\frac{2ab}{ta+(2-t)\,b}\right)dt$$

$$\supseteq\; M(a,b)\int_0^1 t^{\alpha-1}\left[h^2\left(\frac{2-t}{2}\right)+h^2\left(\frac{t}{2}\right)\right]dt$$

$$+2N(a,b)\int_0^1 t^{\alpha-1}h\left(\frac{t}{2}\right)h\left(\frac{2-t}{2}\right)dt. \tag{44}$$

By using Theorem 1 in relation (44), we obtain our required inequality. $\blacksquare$

Theorem 12 *If $F,G:[a,b]\to\mathbb{R}_{\mathcal{I}}^+$ are two interval-valued harmonically h-convex functions such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$ and $G(t)=\left[\underline{G}(t),\overline{G}(t)\right]$, then we have the following inequality for fractional integrals:*

$$\frac{1}{2h^2\left(\frac{1}{2}\right)}F\left(\frac{2ab}{a+b}\right)G\left(\frac{2ab}{a+b}\right)$$

$$\supseteq\;\frac{\Gamma\left(\alpha+1\right)}{2^{1-\alpha}}\left(\frac{ab}{a+b}\right)^{\alpha}\left[J_{\left(\frac{a+b}{2ab}\right)^+}^{\alpha}\left(F\circ g\right)(1/a)+J_{\left(\frac{a+b}{2ab}\right)^-}^{\alpha}\left(F\circ g\right)(1/b)\right]$$

$$+\alpha\left[M(a,b)\int_0^1 t^{\alpha-1}h\left(\frac{2-t}{2}\right)h\left(\frac{t}{2}\right)dt+\frac{N(a,b)}{2}\int_0^1 t^{\alpha-1}\left[h^2\left(\frac{2-t}{2}\right)+h^2\left(\frac{t}{2}\right)\right]dt\right]. \tag{45}$$

Proof. Since F is an interval-valued harmonically h-convex function on $[a,b]$, we have

$$F\left(\frac{2xy}{x+y}\right)\supseteq h\left(\frac{1}{2}\right)[F(x)+F(y)]. \tag{46}$$

For $x=\frac{2ab}{ta+(2-t)b}$ and $y=\frac{2ab}{(2-t)a+tb}$, we obtain

$$\frac{1}{h\left(\frac{1}{2}\right)}F\left(\frac{2ab}{a+b}\right)\supseteq F\left(\frac{2ab}{ta+(2-t)\,b}\right)+F\left(\frac{2ab}{(2-t)\,a+tb}\right). \tag{47}$$

Similarly, we get

$$\frac{1}{h\left(\frac{1}{2}\right)}G\left(\frac{2ab}{a+b}\right)\supseteq G\left(\frac{2ab}{ta+(2-t)\,b}\right)+G\left(\frac{2ab}{(2-t)\,a+tb}\right). \tag{48}$$

Multiplying the inequalities (47) and (48), we obtain

$$
\frac{1}{h^2\left(\frac{1}{2}\right)} F\left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right)
$$

$$
\supseteq \quad F\left(\frac{2ab}{ta+(2-t)b}\right) G\left(\frac{2ab}{ta+(2-t)b}\right)
$$

$$
+F\left(\frac{2ab}{(2-t)a+tb}\right) G\left(\frac{2ab}{(2-t)a+tb}\right)
$$

$$
+F\left(\frac{2ab}{ta+(2-t)b}\right) G\left(\frac{2ab}{(2-t)a+tb}\right)
$$

$$
+F\left(\frac{2ab}{(2-t)a+tb}\right) G\left(\frac{2ab}{ta+(2-t)b}\right)
$$

$$
\supseteq \quad F\left(\frac{2ab}{ta+(2-t)b}\right) G\left(\frac{2ab}{ta+(2-t)b}\right) + F\left(\frac{2ab}{(2-t)a+tb}\right) G\left(\frac{2ab}{(2-t)a+tb}\right)
$$

$$
+\left[h\left(\frac{2-t}{2}\right)F(a)+h\left(\frac{t}{2}\right)F(b)\right] \times \left[h\left(\frac{t}{2}\right)G(a)+h\left(\frac{2-t}{2}\right)G(b)\right]
$$

$$
+\left[h\left(\frac{t}{2}\right)F(a)+h\left(\frac{2-t}{2}\right)F(b)\right] \times \left[h\left(\frac{2-t}{2}\right)G(a)+h\left(\frac{t}{2}\right)G(b)\right]
$$

$$
= \quad F\left(\frac{2ab}{ta+(2-t)b}\right) G\left(\frac{2ab}{ta+(2-t)b}\right) + F\left(\frac{2ab}{(2-t)a+tb}\right) G\left(\frac{2ab}{(2-t)a+tb}\right)
$$

$$
+2M(a,b)h\left(\frac{2-t}{2}\right)h\left(\frac{t}{2}\right) + \left[h^2\left(\frac{2-t}{2}\right)+h^2\left(\frac{t}{2}\right)\right]N(a,b). \tag{49}
$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (49) and integrating the resultant one with respect to t over $[0,1]$, we obtain our result (45). ∎

Theorem 13 *If $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^+$ is interval-valued harmonically h-convex function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then we have the following inequalities for fractional integrals:*

$$
\frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right)
$$

$$
\supseteq \quad \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \left[J_{(1/a)-}^{\alpha}(F \circ g)\left(\frac{2ab}{a+b}\right) + J_{(1/b)+}^{\alpha}(F \circ g)\left(\frac{2ab}{a+b}\right)\right]
$$

$$
\supseteq \quad \alpha\left(\frac{F(a)+F(b)}{2}\right) \int_0^1 t^{\alpha-1} \left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right] dt. \tag{50}
$$

Proof. Since F is interval-valued harmonically h-convex function on $[a,b]$, we have

$$
F\left(\frac{2xy}{x+y}\right) \supseteq h\left(\frac{1}{2}\right)[F(x)+F(y)].
$$

For $x = \frac{2ab}{(1-t)a+(1+t)b}$ and $y = \frac{2ab}{(1+t)a+(1-t)b}$, we get

$$
\frac{1}{h\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right) \supseteq F\left(\frac{2ab}{(1-t)a+(1+t)b}\right) + F\left(\frac{2ab}{(1+t)a+(1-t)b}\right). \tag{51}
$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (51) and integrating the resultant one with respect to t over $[0,1]$, we obtain

$$\frac{1}{h\left(\frac{1}{2}\right)}F\left(\frac{2ab}{a+b}\right)\int_0^1 t^{\alpha-1}dt$$

$$\supseteq \quad (IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)dt + (IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)dt. \quad (52)$$

By using Theorem 1 in the relation (52), we have

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)dt$$

$$= \quad \left[(R)\int_0^1 t^{\alpha-1}\underline{F}\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)dt,\ (R)\int_0^1 t^{\alpha-1}\overline{F}\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)dt\right]$$

$$= \quad \left[\left(\frac{2ab}{b-a}\right)^{\alpha}(R)\int_{\frac{a+b}{2ab}}^{1/a}\left(u-\frac{a+b}{2ab}\right)\underline{F}(1/u)du,\ \left(\frac{2ab}{b-a}\right)^{\alpha}(R)\int_{\frac{a+b}{2ab}}^{1/a}\left(u-\frac{a+b}{2ab}\right)\overline{F}(1/u)du\right]$$

$$= \quad \left[\Gamma\left(\alpha\right)\left(\frac{2ab}{b-a}\right)^{\alpha}I^{\alpha}_{(1/a)-}\left(\underline{F}\circ g\right)\left(\frac{a+b}{2ab}\right),\ \Gamma\left(\alpha\right)\left(\frac{2ab}{b-a}\right)^{\alpha}I^{\alpha}_{(1/a)-}\left(\overline{F}\circ g\right)\left(\frac{a+b}{2ab}\right)\right]$$

$$= \quad \Gamma\left(\alpha\right)\left(\frac{2ab}{b-a}\right)^{\alpha}J^{\alpha}_{(1/a)-}F\left(\frac{2ab}{a+b}\right).$$

Similarly, we get

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)dt$$

$$= \quad \left[\Gamma\left(\alpha\right)\left(\frac{2ab}{b-a}\right)^{\alpha}I^{\alpha}_{(1/b)+}\left(\underline{F}\circ g\right)\left(\frac{a+b}{2ab}\right),\ \Gamma\left(\alpha\right)\left(\frac{2ab}{b-a}\right)^{\alpha}I^{\alpha}_{(1/b)+}\left(\overline{F}\circ g\right)\left(\frac{a+b}{2ab}\right)\right]$$

$$= \quad \Gamma\left(\alpha\right)\left(\frac{2ab}{b-a}\right)^{\alpha}J^{\alpha}_{(1/b)+}F\left(\frac{2ab}{a+b}\right).$$

Hence, we proved the first inequality. To prove the second inequality of (50), first we note that since F is interval-valued harmonically h-convex function, we have

$$F\left(\frac{2ab}{(1+t)\,b+(1-t)\,a}\right) \supseteq h\left(\frac{1+t}{2}\right)F(a)+h\left(\frac{1-t}{2}\right)F(b) \quad (53)$$

and

$$F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right) \supseteq h\left(\frac{1-t}{2}\right)F(a)+h\left(\frac{1+t}{2}\right)F(b). \quad (54)$$

Adding (53) and (54), we get

$$F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)+F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)$$

$$\supseteq \quad [F(a)+F(b)]\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]. \quad (55)$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (55) and integrating the resultant one with respect to t over $[0,1]$, we obtain

$$(IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)dt + (IR)\int_0^1 t^{\alpha-1}F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)dt$$

$$\supseteq \quad [F(a)+F(b)]\int_0^1 t^{\alpha-1}\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt.$$

This completes the proof. ∎

Theorem 14 *If $F, G : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ are two interval-valued harmonically h-convex functions such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $G(t) = \left[\underline{G}(t), \overline{G}(t)\right]$, then we have the following inequality for fractional integrals:*

$$\frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^\alpha \times \left[J_{(1/a)-}^\alpha (F \circ g)\left(\frac{2ab}{a+b}\right)(G \circ g)\left(\frac{2ab}{a+b}\right)\right.$$

$$\left.+J_{(1/b)+}^\alpha (F \circ g)\left(\frac{2ab}{a+b}\right)(G \circ g)\left(\frac{2ab}{a+b}\right)\right]$$

$$\supseteq \alpha \left[\frac{M(a,b)}{2} \int_0^1 t^{\alpha-1}\left[h^2\left(\frac{1-t}{2}\right) + h^2\left(\frac{1+t}{2}\right)\right] dt\right.$$

$$\left.+N(a,b) \int_0^1 t^{\alpha-1} h\left(\frac{1+t}{2}\right) h\left(\frac{1-t}{2}\right) dt\right]. \tag{56}$$

Proof. Since F and G are two interval-valued harmonically h-convex functions, then

$$F\left(\frac{2ab}{(1-t)b + (1+t)a}\right) \supseteq h\left(\frac{1-t}{2}\right) F(a) + h\left(\frac{1+t}{2}\right) F(b) \tag{57}$$

and

$$G\left(\frac{2ab}{(1-t)b + (1+t)a}\right) \supseteq h\left(\frac{1-t}{2}\right) G(a) + h\left(\frac{1+t}{2}\right) G(b). \tag{58}$$

Multiplying (57) and (58), we have

$$F\left(\frac{2ab}{(1-t)b + (1+t)a}\right) G\left(\frac{2ab}{(1-t)b + (1+t)a}\right)$$

$$\supseteq h^2\left(\frac{1-t}{2}\right) F(a)G(a) + h^2\left(\frac{1+t}{2}\right) F(b)G(b)$$

$$+h\left(\frac{1-t}{2}\right) h\left(\frac{1+t}{2}\right) [F(a)G(b) + F(b)G(a)]. \tag{59}$$

Similarly, we get

$$F\left(\frac{2ab}{(1+t)b + (1-t)a}\right) G\left(\frac{2ab}{(1+t)b + (1-t)a}\right)$$

$$\supseteq h^2\left(\frac{1+t}{2}\right) F(a)G(a) + h^2\left(\frac{1-t}{2}\right) F(b)G(b)$$

$$+h\left(\frac{1+t}{2}\right) h\left(\frac{1-t}{2}\right) [F(a)G(b) + F(b)G(a)]. \tag{60}$$

Adding (59) and (60), we obtain the following relation

$$F\left(\frac{2ab}{(1-t)b + (1+t)a}\right) G\left(\frac{2ab}{(1-t)b + (1+t)a}\right)$$

$$+F\left(\frac{2ab}{(1+t)b + (1-t)a}\right) G\left(\frac{2ab}{(1+t)b + (1-t)a}\right)$$

$$\supseteq h^2\left(\frac{1-t}{2}\right) [F(a)G(a) + F(b)G(b)] + h^2\left(\frac{1+t}{2}\right) [F(a)G(a) + F(b)G(b)]$$

$$+2h\left(\frac{1+t}{2}\right) h\left(\frac{1-t}{2}\right) [F(a)G(b) + F(b)G(a)]. \tag{61}$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (61) and integrating the resultant one with respect to t over $[0, 1]$, we have

$$
\begin{aligned}
(IR) &\int_0^1 t^{\alpha-1} F\left(\frac{2ab}{(1-t)\,b+(1+t)\,a}\right) G\left(\frac{2ab}{(1-t)\,b+(1+t)\,a}\right) dt \\
+(IR) &\int_0^1 t^{\alpha-1} F\left(\frac{2ab}{(1+t)\,b+(1-t)\,a}\right) G\left(\frac{2ab}{(1+t)\,b+(1-t)\,a}\right) dt \\
\supseteq \quad M(a,b) &\int_0^1 t^{\alpha-1}\left[h^2\left(\frac{1-t}{2}\right)+h^2\left(\frac{1+t}{2}\right)\right] dt \\
+2N(a,b) &\int_0^1 t^{\alpha-1} h\left(\frac{1+t}{2}\right) h\left(\frac{1-t}{2}\right) dt.
\end{aligned}
\tag{62}
$$

By using Theorem 1 in relation (62), we obtain our required inequality. ■

Theorem 15 *If $F, G : [a, b] \to \mathbb{R}_{\mathcal{I}}^+$ are two interval-valued harmonically h-convex functions such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $G(t) = \left[\underline{G}(t), \overline{G}(t)\right]$, then we have the following inequality for fractional integrals:*

$$
\begin{aligned}
&\frac{1}{2h^2\left(\frac{1}{2}\right)} F\left(\frac{a+b}{2}\right) G\left(\frac{a+b}{2}\right) \\
\supseteq \quad &\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^\alpha\left[J_{(1/a)-}^\alpha\,(F\circ g)\left(\frac{2ab}{a+b}\right)(G\circ g)\left(\frac{2ab}{a+b}\right)\right. \\
&\left.+J_{(1/b)+}^\alpha\,(F\circ g)\left(\frac{2ab}{a+b}\right)(G\circ g)\left(\frac{2ab}{a+b}\right)\right] \\
&+\alpha\left[M(a,b)\int_0^1 t^{\alpha-1} h\left(\frac{1-t}{2}\right) h\left(\frac{1+t}{2}\right) dt\right. \\
&\left.+\frac{N(a,b)}{2}\int_0^1 t^{\alpha-1}\left[h^2\left(\frac{1-t}{2}\right)+h^2\left(\frac{1+t}{2}\right)\right] dt\right].
\end{aligned}
\tag{63}
$$

Proof. Since F is interval-valued harmonically h-convex function on $[a, b]$, we have

$$
F\left(\frac{2xy}{x+y}\right) \supseteq h\left(\frac{1}{2}\right)[F(x)+F(y)].
\tag{64}
$$

For $x = \frac{2ab}{(1-t)a+(1+t)b}$ and $y = \frac{2ab}{(1+t)a+(1-t)b}$, we obtain

$$
\frac{1}{h\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right) \supseteq F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right) + F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right).
\tag{65}
$$

Similarly, we get

$$
\frac{1}{h\left(\frac{1}{2}\right)} G\left(\frac{2ab}{a+b}\right) \supseteq G\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right) + G\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right).
\tag{66}
$$

Multiplying the inequalities (65) and (66), we obtain

$$
\frac{1}{h^2\left(\frac{1}{2}\right)} F\left(\frac{2ab}{a+b}\right) G\left(\frac{2ab}{a+b}\right)
$$

$$
\supseteq \quad F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right) G\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)
$$

$$
+F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right) G\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)
$$

$$
+F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right) G\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)
$$

$$
+F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right) G\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)
$$

$$
\supseteq \quad F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right) G\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)
$$

$$
+F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right) G\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)
$$

$$
+\left[h\left(\frac{1+t}{2}\right) F(a) + h\left(\frac{1-t}{2}\right) F(b) + H(a,b)\right]
$$

$$
\times\left[h\left(\frac{1-t}{2}\right) G(a) + h\left(\frac{1+t}{2}\right) G(b) + H(a,b)\right]
$$

$$
+\left[h\left(\frac{1-t}{2}\right) F(a) + h\left(\frac{1+t}{2}\right) F(b) + H(a,b)\right]
$$

$$
\times\left[h\left(\frac{1+t}{2}\right) G(a) + h\left(\frac{1-t}{2}\right) G(b) + H(a,b)\right]
$$

$$
= \quad F\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right) G\left(\frac{2ab}{(1-t)\,a+(1+t)\,b}\right)
$$

$$
+F\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right) G\left(\frac{2ab}{(1+t)\,a+(1-t)\,b}\right)
$$

$$
+2M(a,b)h\left(\frac{1-t}{2}\right) h\left(\frac{1+t}{2}\right) + \left[h^2\left(\frac{1-t}{2}\right) + h^2\left(\frac{1+t}{2}\right)\right] N(a,b). \tag{67}
$$

Multiplying by $t^{\alpha-1}$ the both sides of inequality (67) and integrating the resultant one with respect to t over $[0,1]$, we obtain our result (63). ∎

4 Conclusion

It is expected that from the results obtained, and following the methodology applied, additional special functions may also be evaluated. Future works can be developed in the area of numerical analysis and even contributions using the theorems and corollaries presented. Finally, our results can be applied to derive some inequalities using special means. The authors hope that the ideas and techniques of this paper will inspire interested readers working in this fascinating field.

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